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GOVERNMENT OF TAMIL NADU

HIGHER SECONDARY FIRST YEAR

MATHEMATICS

VOLUME - I

A publication under Free Textbook Programme of Government of Tamil Nadu

Department of School Education

Untouchability is Inhuman and a Crime

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HOW TO USE THE BOOK

Scope of Mathematics

- Awareness on the scope of higher educational opportunities; courses, institutions and required competitive examinations.
- Possible financial assistance to help students climb academic ladder.

Learning Objectives

- Overview of the unit
- Give clarity on the intended learning outcomes of the unit.

DO YOU KNOW?

- Additional facts related to the topics have been included to arouse interest for searching of more information for deeper and wider learning.

- Visual representation of concepts with illustrations
- Videos, animations, and tutorials.

ICT

- To increase the span of attention of concepts
- To visualize the concepts for strengthening and understanding
- To link concepts related to one unit with other units.
- To utilize the digital skills in classroom learning and providing students experimental learning.

Summary

- Recapitutation of the salient points of each chapter for recalling the concepts learnt.

Evaluation

- Assessing student's understanding of concepts and get them acquainted with solving exercise problems.

Books for Reference

- List of relevant books for further reading.

Scope for Higher Order Thinking

- To motivate students aspiring to take up competitive examinations such as JEE, KVPY, Math olympiad, etc., the concepts and questions based on Higher Order Thinking are incorporated in the content of this book.

Glossary

- Frequently used Mathematical terms have been given with their Tamil equivalents.

Mathematics Learning

The correct way to learn is to understand the concepts throughly. Each chapter opens with an Introduction, Learning Objectives, Various Definitions, Theorems, Results and Illustrations. These in turn are followed by solved examples and exercise problems which have been classified into various types for quick and effective revision. One can develop the skill of solving mathematical problems only by doing them. So the teacher's role is to teach the basic concepts and problems related to it and to scaffold students to try the other problems on their own. Since the first year of Higher Secondary is considered to be the foundation for learning higher mathematics, the students must be given more attention to each and every concept mentioned in this book.

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Scope for students after completing Higher Secondary

EXAMS

MEDICAL RELATED COURSES:	
JEE (Main & Advanced)	Bio Informatics
NEET, JIPMER, AIIMS	M.B.B.S / B.D.S
ENGINEERING RELATED COURSES:	
JEE (Main & Advanced)	B.E. / B.Tech. / B.Arch in IITs and NITs
B.Tech.	Avionics / Aerospace
B.Tech. & M.Tech	Dual degree programme in IIST Trivandram.
B.E	Agricultural Engineering (TamilNadu Agricultural University Entrance Exam)
CET	Common Entrance Test for Maritime Courses (www.imu.edu.in)
BITS	Birla Institute of Technology and science Admission Test (www.bits – pilani.ac.in)
NATA	National Aptitude Test in Architecture (www.nata.in)
IIIT	Indian Institute of Information Technology (www.iiit.ac.in)
BASIC SCIENCE COURSES:	
TIFR	Tata Institute of Fundamental Research conducts Graduate School admission examination (Univ.tirf.res.in)
NEST, NISER	National Institute of Science Education and Research (www.niser.ac.in)
IISC	Indian Institute of Science (www.iisc.ac.in)
RIE	Regional Institute of Education
CEE	Common Entrance Examination (www.rie ajmer.raj.nic.in)
CUCET	Central Universities Common Entrance Test (http://Ucetexam.in)
GATE	Graduate Aptitude Test in Engineering (gate.iitg.ac.in)
NET	CSIR and UGC - National Eligibility Test (http://cbsenet.nic.in)
JAM	Joint Admission Test for M.Sc (Jam.iitb.ac.in)
ISI	Indian Statistical Institute Admission Test (www.isical.ac.in)
IISER - KVPY / IIT – JEE / NEET	Indian Institute of science Education and Research (www.iiser.ac.in & www.iiseradmission.in)

AFTER COMPLETING +2

UNDER GRADUATE COURSES:	
B.Sc	Mathematics / Applied Mathematics / Statistics
B.Sc	Physics, Chemistry
B.Stat.(Hons.)	Bachelor of Statistics
B.Math (Hons.)	Bachelor of Mathematics
B.S. - M.S Dual Degree	5 years programme
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POST GRADUATE COURSES:	
M.Sc	Mathematics / Applied Mathematics / Statistics
M.Sc	Astronomy
M.Sc	Operational Research
M.Stat.(Hons.)	Master of Statistics
M.Math (Hons.)	Master of Mathematics
M.Sc. Ed.	6 years programme
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Job and Scholarship Opportunities

JOB OPPORTUNITIES

- Indian Forest Services
- Scientist Jobs in ISRO, DRDO, CSIR labs
- Union Public Service Commission
- Staff Selection Commission
- Indian Defense Services
- Public Sector Bank
- Tax Assistant
- Statistical Investigator
- Combined Graduate Level Exam
- Tamil Nadu Public Service Commission
- Teaching Profession

FINANCIAL ASSISTANCE

- Scholarship for Graduate and Post – Graduate courses
- NTSE at the end of X (from class XI to Ph.D)
- International Olympiad: for getting stipend for Higher Education in Science and Mathematics
- DST – INSPIRE Scholarships (for UG and PG)
- DST – INSPIRE Fellowships (for Ph.D)
- UGC National Fellowships (for Ph.D)
- Indira Gandhi Fellowship for Single Girl Child (for UG and PG)
- Moulana Azad Fellowship for minorities (for Ph.D)
- In addition various fellowships for SC / ST / PWD / OBC etc are available. (Visit website of University Grants Commission (UGC) and Department of Science and Technology (DST))
- University Fellowships
- Tamil Nadu Collegiate Education fellowship.

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Institutes in india to pursue research in Mathematics

RESEARCH INSTITUTIONS IN VARIOUS AREAS OF SCIENCE

NAME OF THE INSTITUTION	WEBSITE
Indian Institute of Science (IISc), Bangalore	www.iisc.ac.in
Chennai Mathematical Institute (CMI), Chennai	www.cmi.ac.in
Tata Institue of Fundamental Research (TIFR), Mumbai	www.tifr.res.in
Indian Institute of Space Science and Technology (IIST), Trivandrum	www.iist.ac.in
National Institute of Science Education and Research (NISER)	www.niser.ac.in
Birla Institute of Technology and Science, Pilani	www.bits-pilani.ac.in
Indian Institute of Science Education and Research	www.iiseradmission.in
Anna University	https://www.annauniv.edu/
Indian Institute of Technology in various places (IIT's)	www.iitm.ac.in
National Institute of Technology (NIT's)	www.nitt.edu
Central Universities	www.cucet.ac.in
State Universities	www.ugc.ac.in
Tamil Nadu agricultural University (tnau.ac.in)	tnau.ac.in
International Institute of Information Technology	www.iiit.ac.in
The Institute of Mathematical Sciences (IMSC), Chennai.	www.imsc.res.in
Hyderabad Central university, Hyderabad.	www.uohyd.ac.in
Delhi University, Delhi	www.du.ac.in
Mumbai University, Mumbai	www.mu.ac.in
Savithiri Bai Phule Pune University, Pune	www.unipune.ac.in
Indian Statistical Institute	www.iisical.ac.in
Regional Institute of Education	www.rieajmer.raj.nic.in

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Chapter 1 Sets, Relations and Functions

21RYQC

"a set is many that allows itself to be thought of as a one"

Cantor

1.1 Introduction

The concepts of sets, relations and functions occupy a fundamental place in the mainstream of mathematical thinking. As rightly stated by the Russian mathematician Luzin the concept of functions did not arise suddenly. It underwent profound changes in time. Galileo (1564-1642) explicitly used the dependency of one quantity on another in the study of planetary motions. Descartes (1596-1650) clearly stated that an equation in two variables, geometrically represented by a curve, indicates dependence between variable quantities. Leibnitz (1646-1716) used the word "function", in a 1673 manuscript, to mean any quantity varying from point to point of a curve. Dirichlet (1805-1859), a student of Gauss, was credited with the modern "formal" definition of function with notation $y = f(x)$. In the 20th century, this concept was extended to include all arbitrary correspondence satisfying the uniqueness condition between sets and numerical or non-numerical values.

With the development of set theory, initiated by Cantor (1845-1918), the notion of function continued to evolve. From the notion of correspondence, mathematicians moved to the notion of relation. However even now in the theory of computation, a function is not viewed as a relation but as a computational rule. The modern definition of a function is given in terms of relation so as to suit to develop artificial intelligence.

Cantor
1845 - 1918

In the previous classes, we have studied and are well versed with the real numbers and arithmetic operations on them. We also learnt about sets of real numbers, Venn diagrams, Cartesian product of sets, basic definitions of relations and functions. For better understanding, we recall more about sets and Cartesian products of sets. In this chapter, we see a new facelift to the mathematical notions of "Relations" and "Functions".

Learning Objectives

On completion of this chapter, the students are expected to

- list and work with many properties of sets and Cartesian product;
- know the concepts of constants, variables, intervals and neighbourhoods;
- understand about various types of relations; create relations of any required type;
- represent functions in different ways;
- work with elementary functions, types of functions, operations on functions including inverse of a bijective function;
- identify the graphs of some special functions;
- visualize and sketch the graphs of some relatively complicated functions.

1.2 Sets

In the earlier classes, we have seen that a ***set*** is a collection of well-defined objects. As the theory of sets is the building blocks of modern mathematics, one has to learn the concepts of sets carefully and deeply. Now we look at the term "well-defined" a little more deeply. Consider the two statements:

(i) The collection of all beautiful flowers in Ooty Rose Garden.
(ii) The collection of all old men in Tamilnadu.

The terms "beautiful flowers" and "old men" are not well-defined. We cannot define the term "beautiful flower" in a sharp way as there is no concrete definition for beauty because the concept of beauty varies from person to person, content to content and object to object. We should not consider statements like "the collection of all beautiful flowers in Ooty Rose Garden" as a set. Now, can we say "the collection of all red flowers in Ooty Rose Garden" a set? The answer is "yes".

One may consider a person of age 60 as old and others may not agree. There is no specific and concrete definition for "old men". The second statement can be made more sharply as

"the collection of all men in Tamilnadu of age greater than 70".

Now, the above collection becomes a set because of definiteness in the age. Thus, the description of a set should enable us to concretely decide whether a given particular object (element) is available in the collection or not. So set is a distinguishable collection of objects.

We have also seen and learnt to use symbols like $\in$, $\subset$, $\subseteq$, $\cup$ and $\cap$. Let us start with the question:

"If A and B are two sets, is it meaningful to write $A \in B$?".

At the first sight one may hurry to say that this is always meaningless by telling, "the symbol $\in$ should be used between an element and a set and it should not be used between two sets". The first part of the statement is true whereas the second part is not true. For example, if $A = \{1, 2\}$ and $B = \{1, \{1, 2\}, 3, 4\}$, then $A \in B$. In this section we shall discuss the meaning of such symbols more deeply.

As we learnt in the earlier classes the set containing no elements is called an ***empty set*** or a ***void set***. It is usually denoted by $\emptyset$ or $\{\ \}$. By $A \subseteq B$, we mean every element of the set A is an element of the set B. In this case, we say A is a ***subset*** of B and B is a ***super set*** of A. For any two sets A and B, if $A \subseteq B$ and $B \subseteq A$, then the two sets are ***equal***. For any set A, the empty set $\emptyset$ and the set A are always subsets of A. These two subsets are called ***trivial subsets***. Further, we say A is a ***proper subset*** of B if A is a subset of B and $A \neq B$. That is, B contains all elements of A and at least one element which is not in A. Note that, as every element of A is an element of A, we have $A \subseteq A$. Thus, any set is a subset of itself. This subset is called an ***improper subset***. In other words, for any set A, A is the improper subset of A. It is known that, $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, where $\mathbb{N}$ denotes the set of all natural numbers or positive integers, $\mathbb{W}$ denotes the set of all non-negative integers, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{Q}$ denotes the set of all rational numbers and $\mathbb{R}$ denotes the set of all real numbers. Note that, the set of all irrational numbers is a subset of $\mathbb{R}$ but not a subset of any other set mentioned above.

We learnt that the ***union*** of two sets A and B is denoted by $A \cup B$ and is defined as

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

and the ***intersection*** as

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Two sets A and B are ***disjoint*** if they do not have any common element. That is, A and B are disjoint if $A \cap B = \emptyset$.

Let us see some more notations. We are familiar with notations like $\sum_{i=1}^{n} a_i$. This in fact stands for $a_1 + a_2 + \cdots + a_n$. Similarly we can use the notations $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$ to denote $A_1 \cup A_2 \cup \cdots \cup A_n$ and $A_1 \cap A_2 \cap \cdots \cap A_n$ respectively.

Thus, $\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } i\}$ and $\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for each } i\}$. These notations are useful when we discuss more number of sets.

If A is a set, then the set of all subsets of A is called the ***power set*** of A and is usually denoted as $\mathscr{P}(A)$. That is, $\mathscr{P}(A) = \{B : B \subseteq A\}$. The number of elements in $\mathscr{P}(A)$ is 2^n, where n is the number of elements in A.

Now, to define the complement of a set, it is necessary to know about the concept of universal set. Usually all sets under consideration in a mathematical process are assumed to be subsets of some fixed set. This basic set is called the ***universal set***. For example, depending on the situation, for the set of prime numbers, the universal set can be any one of the sets containing the set of prime numbers. Thus, one of the sets $\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ may be taken as a universal set for the set of prime numbers, depending on the requirement. Universal set is usually denoted by U.

To define the complement of a set, we have to fix the universal set. Let A be a subset of the universal set U. The ***complement*** of A with respect to U is denoted as A' or A^c and defined as $A' = \{x : x \in U \text{ and } x \notin A\}$.

The ***set difference*** of the set A to the set B is denoted by either $A - B$ or $A \backslash B$ and is defined as

$$A - B = \{a : a \in A \text{ and } a \notin B\}.$$

Note that,

(i) $U - A = A'$ (ii) $A - A = \emptyset$ (iii) $\emptyset - A = \emptyset$ (iv) $A - \emptyset = A$ (v) $A - U = \emptyset$.

The ***symmetric difference*** between two sets A and B is denoted by $A \Delta B$ and is defined as $A \Delta B = (A - B) \cup (B - A)$. Actually the elements of $A \Delta B$ are the elements of $A \cup B$ which are not in $A \cap B$. Thus $A \Delta B = (A \cup B) - (A \cap B)$.

A set X is said to be a ***finite set*** if it has k elements for some $k \in \mathbb{W}$. In this case, we say the finite set X is of ***cardinality*** k and is denoted by $n(X)$. A set is an ***infinite set*** if it is not finite. For an infinite set A, the cardinality is infinity. If $n(A) = 1$, then it is called a ***singleton set***. Note that $n(\emptyset) = 0$ and $n(\{\emptyset\}) = 1$.

1.2.1 Properties of Set Operations

We now list out some of the properties.

Commutative

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$.

Associative

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $(A \cap B) \cap C = A \cap (B \cap C)$.

Distributive

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Identity

(i) $A \cup \emptyset = A$ (ii) $A \cap U = A$.

Idempotent

(i) $A \cup A = A$ (ii) $A \cap A = A$.

Absorption

(i) $A \cup (A \cap B) = A$ (ii) $A \cap (A \cup B) = A$.

De Morgan Laws

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
(iii) $A - (B \cup C) = (A - B) \cap (A - C)$ (iv) $A - (B \cap C) = (A - B) \cup (A - C)$.

On Symmetric Difference

(i) $A \Delta B = B \Delta A$ (ii) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$
(iii) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

On Empty Set and Universal Set

(i) $\emptyset' = U$ (ii) $U' = \emptyset$
(iii) $A \cup A' = U$ (iv) $A \cap A' = \emptyset$
(v) $A \cup U = U$ (vi) $A \cap U = A$.

On Cardinality

(i) For any two finite sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
(ii) If A and B are disjoint finite sets, then $n(A \cup B) = n(A) + n(B)$.
(iii) For any three finite sets A, B and C,
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

1.3 Cartesian Product

We know that the Cartesian product of sets is nothing but a set of ordered elements. In particular, Cartesian product of two sets is a set of ordered pairs, while the Cartesian product of three sets is a set of ordered triplets. Precisely, let A, B and C be three sets. Then the ***Cartesian product*** of A with B is denoted by $A \times B$. It is defined by

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Similarly, the Cartesian product $A \times B \times C$ is defined by

$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}.$$

Thus $A \times A = \{(a, b) : a, b \in A\}$.

Is it correct to say $A \times A = \{(a, a) : a \in A\}$?

It is important that the elements of the Cartesian product are ordered and hence, for non-empty sets,

$$A \times B \neq B \times A, \text{ unless } A = B.$$

That is, for non-empty sets $A \times B = B \times A$ if and only if $A = B$. We know that $\mathbb{R}$ denotes the set of real numbers and

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &= \{(x, y) : x, y \in \mathbb{R}\}. \\ \mathbb{R} \times \mathbb{R} \times \mathbb{R} &= \{(x, y, z) : x, y, z \in \mathbb{R}\}. \end{aligned}$$

Symbolically, $\mathbb{R} \times \mathbb{R}$ can be represented as $\mathbb{R}^2$ and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ as $\mathbb{R}^3$. Note that $\mathbb{R} \times \mathbb{R}$ is a set of ordered pairs and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is a set of ordered triplets.

If $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$ then

$$A \times B = \{(1, 2), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 2), (3, 4), (3, 6)\}.$$

Here $A \times B$ is a subset of $\mathbb{R} \times \mathbb{R}$. The number of elements in $A \times B$ is the product of the number of elements in A and the number of elements in B, that is, $n(A \times B) = n(A)n(B)$, if A and B are finite. Further $n(A \times B \times C) = n(A)n(B)n(C)$, if A, B and C are finite.

It is easy to see that the following are the subsets of $\mathbb{R} \times \mathbb{R}$.

(i) $\{(x, 2x) : x \in \mathbb{R}\}$ (ii) $\{(x, x^2) : x \in \mathbb{R}\}$
(iii) $\{(x, \sqrt{x}) : x$ is a non-negative real number$\}$ (iv) $\{(x^2, x) : x \in \mathbb{R}\}$.
(v) $\{(x, -\sqrt{x}) : x$ is a non-negative real number$\}$

Example 1.1 Find the number of subsets of A if $A = \{x : x = 4n + 1,\ 2 \leq n \leq 5, n \in \mathbb{N}\}$.

Solution:
Clearly $A = \{x : x = 4n + 1,\ n = 2, 3, 4, 5\} = \{9, 13, 17, 21\}$.
Hence $n(A) = 4$. This implies that $n(\mathscr{P}(A)) = 2^4 = 16$.

Example 1.2 In a survey of 5000 persons in a town, it was found that 45% of the persons know Language A, 25% know Language B, 10% know Language C, 5% know Languages A and B, 4% know Languages B and C, and 4% know Languages A and C. If 3% of the persons know all the three Languages, find the number of persons who knows only Language A.

Solution:
This problem can be solved either by property of cardinality or by Venn diagram.

Cardinality: Given that $n(A) = 45\%$ of $5000 = 2250$

Similarly, $n(B) = 1250, n(C) = 500, n(A \cap B) = 250, n(B \cap C) = 200, n(C \cap A) = 200$ and $n(A \cap B \cap C) = 150$.

The number of persons who knows only Language A is

$$\begin{aligned} n(A \cap B' \cap C') &= n\{A \cap (B \cup C)'\} = n(A) - n\{A \cap (B \cup C)\}. \\ &= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C). \\ &= 2250 - 250 - 200 + 150 = 1950. \end{aligned}$$

Thus the required number of persons is 1950.

Venn diagram: We draw the Venn Diagram using percentage.

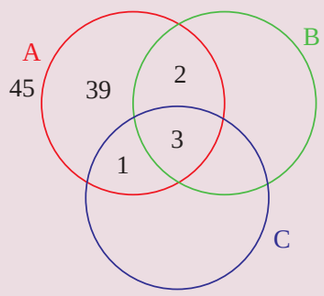


Figure 1.1

From Figure 1.1, the percentage of persons who knows only Language A is 39. Therefore, the required number of persons is $5000 \times \frac{39}{100} = 1950$.

Example 1.3 Prove that

$$((A \cup B' \cup C) \cap (A \cap B' \cap C')) \cup ((A \cup B \cup C') \cap (B' \cap C')) = B' \cap C'.$$

Solution:
We have $A \cap B' \cap C' \subseteq A \subseteq A \cup B' \cup C$ and hence $(A \cup B' \cup C) \cap (A \cap B' \cap C') = A \cap B' \cap C'$.
Also, $B' \cap C' \subseteq C' \subseteq A \cup B \cup C'$ and hence $(A \cup B \cup C') \cap (B' \cap C') = B' \cap C'$.
Now as $A \cap B' \cap C' \subseteq B' \cap C'$, we have

$$((A \cup B' \cup C) \cap (A \cap B' \cap C')) \cup ((A \cup B \cup C') \cap (B' \cap C')) = B' \cap C'.$$

Try to simplify the above expression using Venn diagram.

Example 1.4 If $X = \{1, 2, 3, \ldots 10\}$ and $A = \{1, 2, 3, 4, 5\}$, find the number of sets $B \subseteq X$ such that $A - B = \{4\}$

Solution:
For every subset C of $\{6, 7, 8, 9, 10\}$, let $B = C \cup \{1, 2, 3, 5\}$. Then $A - B = \{4\}$. In other words, for every subset C of $\{6, 7, 8, 9, 10\}$, we have a unique set B so that $A - B = \{4\}$. So number of sets $B \subseteq X$ such that $A - B = \{4\}$ and the number of subsets of $\{6, 7, 8, 9, 10\}$ are the same. So the number of sets $B \subseteq X$ such that $A - B = \{4\}$ is $2^5 = 32$.

Example 1.5 If A and B are two sets so that $n(B - A) = 2n(A - B) = 4n(A \cap B)$ and if $n(A \cup B) = 14$, then find $n(\mathscr{P}(A))$.

Solution:
To find $n(\mathscr{P}(A))$, we need $n(A)$.
Let $n(A \cap B) = k$. Then $n(A - B) = 2k$ and $n(B - A) = 4k$.
Now $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B) = 7k$.
It is given that $n(A \cup B) = 14$. Thus $7k = 14$ and hence $k = 2$.
So $n(A - B) = 4$ and $n(B - A) = 8$. As $n(A) = n(A - B) + n(A \cap B)$, we get $n(A) = 6$ and hence $n(\mathscr{P}(A)) = 2^6 = 64$.

Example 1.6 Two sets have m and k elements. If the total number of subsets of the first set is 112 more than that of the second set, find the values of m and k.

Solution:
Let A and B be the two sets with $n(A) = m$ and $n(B) = k$. Since A contains more elements than B, we have $m > k$. From the given conditions we see that $2^m - 2^k = 112$. Thus we get, $2^k(2^{m-k} - 1) = 2^4 \times 7$.

Then the only possibility is $k = 4$ and $2^{m-k} - 1 = 7$. So $m - k = 3$ and hence $m = 7$.

Example 1.7 If $n(A) = 10$ and $n(A \cap B) = 3$, find $n((A \cap B)' \cap A)$.

Solution:
$(A \cap B)' \cap A = (A' \cup B') \cap A = (A' \cap A) \cup (B' \cap A) = \emptyset \cup (B' \cap A) = (B' \cap A) = A - B$.
So $n((A \cap B)' \cap A) = n(A - B) = n(A) - n(A \cap B) = 7$.

Example 1.8 If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, find $n((A \cup B) \times (A \cap B) \times (A \Delta B))$.

Solution:
We have $n(A \cup B) = 6, n(A \cap B) = 2$ and $n(A \Delta B) = 4$.
So, $n((A \cup B) \times (A \cap B) \times (A \Delta B)) = n(A \cup B) \times n(A \cap B) \times n(A \Delta B) = 6 \times 2 \times 4 = 48$.

Example 1.9 If $\mathscr{P}(A)$ denotes the power set of A, then find $n(\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset))))$.

Solution:
Since $\mathscr{P}(\emptyset)$ contains 1 element, $\mathscr{P}(\mathscr{P}(\emptyset))$ contains 2^1 elements and hence $\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset)))$ contains 2^2 elements. That is, 4 elements.

Exercise - 1.1

1. Write the following in roster form.
   (i) $\{x \in \mathbb{N} : x^2 < 121$ and x is a prime$\}$.
   (ii) the set of all positive roots of the equation $(x - 1)(x + 1)(x^2 - 1) = 0$.
   (iii) $\{x \in \mathbb{N} : 4x + 9 < 52\}$.
   (iv) $\{x : \frac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\}$.
2. Write the set $\{-1, 1\}$ in set builder form.
3. State whether the following sets are finite or infinite.
   (i) $\{x \in \mathbb{N} : x$ is an even prime number$\}$.
   (ii) $\{x \in \mathbb{N} : x$ is an odd prime number$\}$.
   (iii) $\{x \in \mathbb{Z} : x$ is even and less than $10\}$.
   (iv) $\{x \in \mathbb{R} : x$ is a rational number$\}$.
   (v) $\{x \in \mathbb{N} : x$ is a rational number$\}$.
4. By taking suitable sets A, B, C, verify the following results:
   (i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
   (ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
   (iii) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$.
   (iv) $C - (B - A) = (C \cap A) \cup (C \cap B')$.
   (v) $(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$.
   (vi) $(B - A) \cup C = (B \cup C) - (A - C)$.
5. Justify the trueness of the statement:

   "An element of a set can never be a subset of itself."
6. If $n(\mathscr{P}(A)) = 1024, n(A \cup B) = 15$ and $n(\mathscr{P}(B)) = 32$, then find $n(A \cap B)$.
7. If $n(A \cap B) = 3$ and $n(A \cup B) = 10$, then find $n(\mathscr{P}(A \Delta B))$.
8. For a set $A, A \times A$ contains 16 elements and two of its elements are $(1, 3)$ and $(0, 2)$. Find the elements of A.

9. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1), (y, 2), (z, 1)$ are in $A \times B$, find A and B, where x, y, z are distinct elements.
10. If $A \times A$ has 16 elements, $S = \{(a, b) \in A \times A : a < b\}$; $(-1, 2)$ and $(0, 1)$ are two elements of S, then find the remaining elements of S.

1.4 Constants and Variables, Intervals and Neighbourhoods

To continue our discussion, we need certain prerequisites namely, constants, variables, independent variables, dependent variables, intervals and neighbourhoods.

1.4.1 Constants and Variables

A quantity that remains unaltered throughout a mathematical process is called a ***constant***. A quantity that varies in a mathematical process is called a ***variable***. A variable is an ***independent variable*** when it takes any arbitrary (independent) value not depending on any other variables, whereas if its value depends on other variables, then it is called a ***dependent variable***.

We know the area A of a triangle is given by $A = \frac{1}{2}bh$. Here $\frac{1}{2}$ is a constant and A, b, h are variables. Moreover b and h are independent variables and A is a dependent variable. We ought to note that the terms *dependent* and *independent* are relative terms. For example in the equation $x + y = 1$, x, y are variables and 1 is a constant. Which of x and y is dependent and which one is independent? If we consider x as an independent variable, then y becomes dependent whereas if we consider y as an independent variable, then x becomes dependent.

Further consider the following examples:

(i) area of a rectangle $A = \ell b$.
(ii) area of a circle $A = \pi r^2$.
(iii) volume of a cuboid $V = \ell bh$.

From the above examples we can directly infer that b, h, ℓ, r are independent variables; A and V are dependent variables and π is a constant.

1.4.2 Intervals and Neighbourhoods

The system $\mathbb{R}$ of real numbers can be represented by the points on a line and a point on the line can be related to a unique real number as in Figure 1.2. By this, we mean that any real number can be identified as a point on the line. With this identification we call the line as the ***real line***.

Figure 1.2

The value increases as we go right and decreases as we go left. If x lies to the left of y on the real line then $x < y$. As there is no gap in a line, we have infinitely many real numbers between any two real numbers.

Definition 1.1

A subset I of $\mathbb{R}$ is said to be an ***interval*** if

(i) I contains at least two elements and
(ii) $a, b \in I$ and $a < c < b$ then $c \in I$.

Geometrically, intervals correspond to rays and line segments on the real line.

Note that the set of all natural numbers, the set of all non-negative integers, set of all odd integers, set of all even integers, set of all prime numbers are not intervals. Further observe that, between any two real numbers there are infinitely many real numbers and hence the above sets are not intervals.

Consider the following sets:

(i) The set of all real numbers greater than 0.
(ii) The set of all real numbers greater than 5 and less than 7.
(iii) The set of all real numbers x such that $1 \leq x \leq 3$.
(iv) The set of all real numbers x such that $1 < x \leq 2$.

The above four sets are intervals. In particular (i) is an infinite interval and (ii), (iii) and (iv) are finite intervals. The term "finite interval" does not mean that the interval contains only finitely many real numbers, however both ends are finite numbers. Both finite and infinite intervals are infinite sets. The intervals correspond to line segments are finite intervals whereas the intervals that correspond to rays and the entire real line are infinite intervals.

A finite interval is said to be ***closed*** if it contains both of its end points and ***open*** if it contains neither of its end points. Symbolically the above four intervals can be written as $(0, \infty), (5, 7), [1, 3], (1, 2]$. Note that for symbolic form we used parentheses and square brackets to denote intervals. ( ) parentheses indicate open interval and [ ] square brackets indicate closed interval. The first two sets are open intervals, third one is a closed interval. Note that fourth set is neither open nor closed, that is, one end open and other end closed.

In particular $[1, 3]$ contains both 1 and 3 and in between real numbers. The interval $(1, 3)$ does not contain 1 and 3 but contains all in between the numbers. The interval $(1, 2]$ does not contain 1 but contains 2 and all in between numbers.

Note that ∞ is not a number. The symbols $-\infty$ and ∞ are used to indicate the ends of real line. Further, the intervals (a, b) and $[a, b]$ are subsets of $\mathbb{R}$.

Type of Intervals

There are many types of intervals. Let $a, b \in \mathbb{R}$ such that $a < b$. The following table describes various types of intervals. It is not possible to draw a line if a point is removed. So we use an unfilled circle "$\circ$" to indicate that the point is removed and use a filled circle "$\bullet$" to indicate that the point is included.

Interval	*Notation*	*Set*	*Diagramatic Representation*
finite	(a, b)	$\{x : a < x < b\}$	$\circ\, a$ —— $\circ\, b$
	$[a, b]$	$\{x : a \leq x \leq b\}$	$\bullet\, a$ —— $\bullet\, b$
	$(a, b]$	$\{x : a < x \leq b\}$	$\circ\, a$ —— $\bullet\, b$
	$[a, b)$	$\{x : a \leq x < b\}$	$\bullet\, a$ —— $\circ\, b$
infinite	(a, ∞)	$\{x : a < x < \infty\}$	$\circ\, a$ ——→
	$[a, \infty)$	$\{x : a \leq x < \infty\}$	$\bullet\, a$ ——→
	$(-\infty, b)$	$\{x : -\infty < x < b\}$	←—— $\circ\, b$
	$(-\infty, b]$	$\{x : -\infty < x \leq b\}$	←—— $\bullet\, b$
	$(-\infty, \infty)$ or $\mathbb{R}$	$\{x : -\infty < x < \infty\}$ or the set of real numbers	←——→

Write the following intervals in symbolic form.

(i) $\{x : x \in \mathbb{R}, -2 \leq x \leq 0\}$ (ii) $\{x : x \in \mathbb{R}, 0 < x < 8\}$

(iii) $\{x : x \in \mathbb{R}, -8 < x \leq -2\}$ (iv) $\{x : x \in \mathbb{R}, -5 \leq x \leq 9\}$.

Neighbourhood

Neighbourhood of a point 'a' is any open interval containing 'a'. In particular, if ϵ is a positive number, usually very small, then the ϵ-neighbourhood of 'a' is the open interval $(a - \epsilon, a + \epsilon)$. The set $(a - \epsilon, a + \epsilon) - \{a\}$ is called ***deleted neighbourhood*** of 'a' and it is denoted as $0 < |x - a| < \epsilon$ (See Figure 1.3).

Figure 1.3

1.5 Relations

We approach the concept of relations in different aspects using real life sense, Cryptography and Geometry through Cartesian product of sets.

In our day to day life very often we come across questions like, "*How is he related to you?*". Some probable answers are,

(i) He is my father.
(ii) He is my teacher.
(iii) He is not related to me.

From this we see that the word relation connects a person with another person. Extending this idea, in mathematics we consider relations as one which connects mathematical objects. Examples,

(i) A number m is related to a number n if m divides n in $\mathbb{N}$.
(ii) A real number x is related to a real number y if $x \leq y$.
(iii) A point p is related to a line L if p lies on L.
(iv) A student X is related to a school S if X is a student of S.

Illustration 1.1 (Cryptography) For centuries, people have used ciphers or codes, to keep confidential information secure. Effective ciphers are essential to the military, to financial institutions and to computer programmers. The study of the techniques used in creating coding and decoding these ciphers is called cryptography.

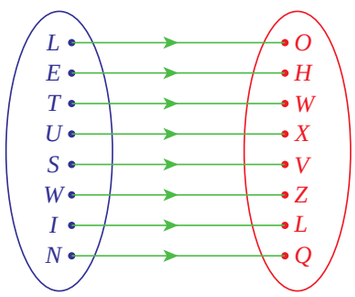


Figure 1.4

One of the earliest methods of coding a message was a simple substitution. For example, each letter in a message might be replaced by the letter that appears three places later in the alphabet.

Using this coding scheme, “LET US WIN” becomes “OHW XVZ LQ”. This scheme was used by Julius Caesar and is called the Caesars cipher. To decode, replace each letter by the letter three places before it. This concept is used often in Mental Ability Tests. The above can be represented as an arrow diagram as given in Figure 1.4.

This can be viewed as the set of ordered pairs

$$\{(L,O),(E,H),(T,W),(U,X),(S,V),(W,Z),(I,L),(N,Q)\}$$

which is a subset of the Cartesian product $C \times D$ where $C = \{L,E,T,U,S,W,I,N\}$ and $D = \{O,H,W,X,V,Z,L,Q\}$.

DO YOU KNOW?

If “KDUGZRUN” means “HARDWORK”, then “DFKLHYHPHQW” becomes “ACHIEVEMENT”

“Is it $f(x) = x - 3$?”.

Illustration 1.2 (Geometry) Consider the following three equations

$$\text{(i) } 2x - y = 0 \quad \text{(ii) } x^2 - y = 0 \quad \text{(iii) } x - y^2 = 0$$

(i) $\boldsymbol{2x - y = 0}$

The equation $2x - y = 0$ represents a straight line. Clearly the points, $(1,2),(3,6)$ lie on it whereas $(1,1),(3,5),(4,5)$ are not lying on the straight line. The analytical relation between x and y is given by $y = 2x$. The set of all points that lie on the straight line is given as $\{(x,2x) : x \in \mathbb{R}\}$. Clearly this is a subset of $\mathbb{R} \times \mathbb{R}$. (See Figure 1.5.)

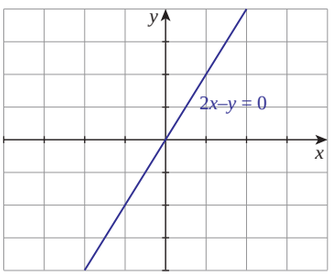


Figure 1.5

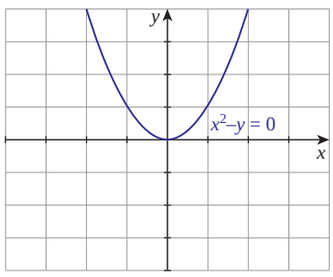


Figure 1.6

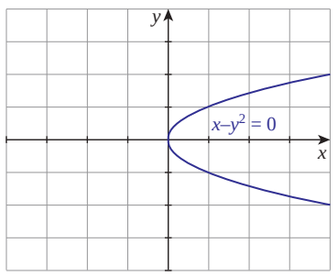


Figure 1.7

(ii) $\boldsymbol{x^2 - y = 0.}$

As we discussed earlier, the relation between x and y is $y = x^2$. The set of all points on the curve is $\{(x,x^2) : x \in \mathbb{R}\}$ (See Figure 1.6). Again this is a subset of the Cartesian product $\mathbb{R} \times \mathbb{R}$.

(iii) $\boldsymbol{x - y^2 = 0}$

As above, the relation between x and y is $y^2 = x$ or $y = \pm\sqrt{x},\ x \geq 0$. The equation can also be re-written as $y = +\sqrt{x}$ and $y = -\sqrt{x}$. The set of all points on the curve is the union of the sets $\{(x,\sqrt{x})\}$ and $\{(x,-\sqrt{x})\}$, where x is a non-negative real number, are the subsets of the Cartesian product $\mathbb{R} \times \mathbb{R}$. (See Figure 1.7).

From the above examples we intuitively understand what a relation is. But in mathematics, we have to give a rigorous definition for each and every technical term we are using. Now let us start defining the term “relation” mathematically.

Definition of Relation

Let $A = \{p, q, r, s, t, u\}$ be a set of students and let $B = \{X, Y, Z, W\}$ be a set of schools. Let us consider the following "relation".

A student $a \in A$ is related to a school $S \in B$ if "a" is studying or studied in the school S.

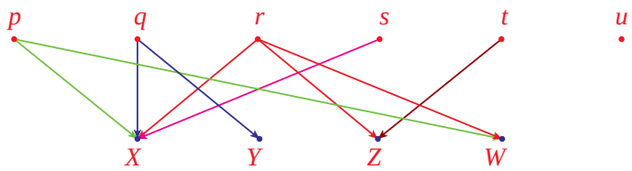
Let us assume that p studied in X and now studying in W, q studied in X and now studying in Y, r studied in X and W, and now studying in Z, s has been studying in X from the beginning, t studied in Z and now studying in no school, and u never studied in any of these four schools.

Though the relations are given explicitly, it is not possible to give a relation always in this way. So let us try some other representations for expressing the same relation:

(i)

p	p	q	q	r	r	r	s	t
X	W	X	Y	X	Z	W	X	Z

(ii)



(iii) $\{\ (p, X),\ (p, W),\ (q, X),\ (q, Y),\ (r, X),\ (r, Z),\ (r, W),\ (s, X),\ (t, Z)\ \}$

(iv) $pRX, pRW, qRX, qRY, rRX, rRZ, rRW, sRX, tRZ.$

Among these four representations of the relation, the third one seems to be more convenient and comfortable to deal with a relation in terms of sets.

The set given in the third representation is a subset of the Cartesian product $A \times B$. In Illustrations 1.1 and 1.2 also, we arrived at subsets of a Cartesian product.

Definition 1.2

Let A and B be any two non-empty sets. A ***relation*** R from A to B is defined as a subset of the Cartesian product of A and B. Symbolically $R \subseteq A \times B$.

A relation from A to B is different from a relation from B to A.

The set $\{a \in A : (a, b) \in R$ for some $b \in B\}$ is called the ***domain*** of the relation.

The set $\{b \in B : (a, b) \in R$ for some $a \in A\}$ is called the ***range*** of the relation.

Thus the domain of the relation R is the set of all first coordinates of the ordered pairs and the range of the relation R is the set of all second coordinates of the ordered pairs.

Illustration 1.3 Consider the diagram in Figure 1.8. Here the alphabets are mapped onto the natural numbers. A simple cipher is to assign a natural number to each alphabet. Here a is represented by 1, b is represented by 2, $\ldots, z$ is represented by 26. This correspondence can be written as the set

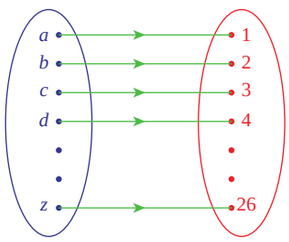


Figure 1.8

of ordered pairs $\{(a,1),(b,2),\ldots,(z,26)\}$. This set of ordered pairs is a relation. The domain of the relation is $\{a,b,\ldots z\}$ and the range is $\{1,2,\ldots,26\}$.

Now we recall that the relations discussed in Illustrations 1.1 and 1.2 also end up with subsets of the cartesian product of two sets. So the term *relation* used in all discussions we had so far, fits with the mathematical term *relation* defined in Definition 1.2.

The domain of the relation discussed in Illustration 1.1 is the set $\{L,E,T,U,S,W,I,N\}$ and the range is $\{O,H,W,X,V,Z,L,Q\}$. In Illustration 1.2, the domain and range of the relation discussed for the equation $2x-y=0$ are $\mathbb{R}$ and $\mathbb{R}$ (See Figure 1.9); for the equation $x^2-y=0$, the domain is $\mathbb{R}$ and the range is $[0,\infty)$ (See Figure 1.10); and in the case of the third equation $x-y^2=0$, the domain is $[0,\infty)$ and the range is $\mathbb{R}$ (See Figures 1.11 and 1.12).

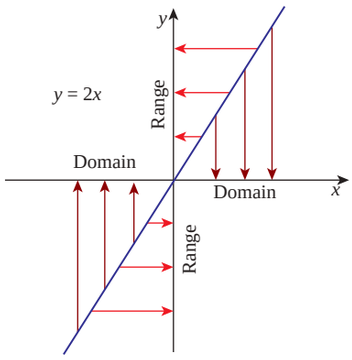


Figure 1.9

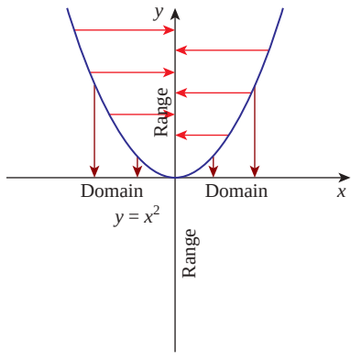


Figure 1.10

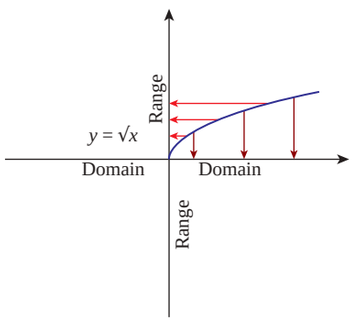


Figure 1.11

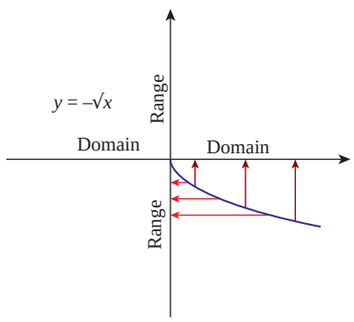


Figure 1.12

Note that, the domain of a relation is a subset of the first set in the Cartesian product and the range is a subset of second set. Usually we call the second set as ***co-domain*** of the relation. Thus, the range of a relation is the collection of all elements in the co-domain which are related to some element in the domain. Let us note that the range of a relation is a subset of the co-domain.

For any set A, $\emptyset$ and $A\times A$ are subsets of $A\times A$. These two relations are called ***extreme relations***. The former relation is an ***empty relation*** and the later is an ***universal relation***.

We will discuss more about domain, co-domain and the range in the next section namely, "Functions".

If R is a relation from A to B and if $(x,y)\in R$, then sometimes we write xRy (read this as "x is related to y") and if $(x,y)\notin R$, then sometimes we write $x\not R y$ (read this as "x is not related to y").

Though the general definition of a relation is defined from one set to another set, relations defined **on** a set are of more interest in mathematical point of view. So let us concentrate on relations defined on a set.

1.5.1 Type of Relations

Consider the following examples:

(i) Let $S = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 3), (2, 3)\}$ on S.
(ii) Let $S = \{1, 2, 3, \ldots 10\}$ and define "m is related to n, if m divides n".
(iii) Let $\mathscr{C}$ be the set of all circles in a plane and define "a circle C is related to a circle C', if the radius of C is equal to the radius of C'".
(iv) In the set S of all people define "a is related to b, if a is a brother of b".
(v) Let S be the set of all people. Define the relation on S by the rule "mother of".

In the second example, as every number divides itself, "a is related a for all $a \in S$"; the same is true in the third relation also. In the first example "a is related a for all $a \in S$" is not true as 2 is not related to 2.

It is easy to see that the property "if a is related to b, then b is related to a" is true in the third but not in the second.

It is easy to see that the property "if a is related to b and b is related to c, then a is related to c" is true in the second and third relations but not in the fifth.

These properties, together with some more properties are very much studied in mathematical structures. Let us define them now.

Definition 1.3

Let S be any non-empty set. Let R be a relation on S. Then

- R is said to be ***reflexive*** if a is related to a for all $a \in S$.
- R is said to be ***symmetric*** if a is related to b implies that b is related to a.
- R is said to be ***transitive*** if "a is related to b and b is related to c" implies that a is related to c.

These three relations are called ***basic relations***.

Let us rewrite the definitions of these basic relations in a different form:

Let S be any non-empty set. Let R be a relation on S. Then R is

- reflexive if "$(a, a) \in R$ for all $a \in S$".
- symmetric if "$(a, b) \in R \Rightarrow (b, a) \in R$".
- transitive if "$(a, b), (b, c) \in R \Rightarrow (a, c) \in R$".

Definition 1.4

Let S be any set. A relation on S is said to be an ***equivalence relation*** if it is reflexive, symmetric and transitive.

Let us consider the following two relations.

(1) In the set S_1 of all people, define a relation R_1 by the rule: "a is related to b, if a is a brother of b".
(2) In the set S_2 of all males, define a relation R_2 by the rule: "a is related to b, if a is a brother of b".

The rules that define the relations on S_1 and S_2 are the same. But the sets are not same. R_1 is not a symmetric relation on S_1 whereas R_2 is a symmetric relation on S_2. This shows that not only the rule defining the relation is important, the set on which the relation is defined, is also important. So whenever one considers a relation, both the relation as well as the set on which the relation is defined have to be given explicitly. Note that the relation $\{(1, 1), (2, 2), (3, 3), (1, 2)\}$ is reflexive if it is defined on the set $\{1, 2, 3\}$; it is not reflexive if it is defined on the set $\{1, 2, 3, 4\}$.

Illustration 1.4

1. Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 1), (2, 2), (3, 3), (1, 3), (4, 4), (1, 2), (3, 1)\}$. As $(1, 1), (2, 2), (3, 3)$ and $(4, 4)$ are all in R, it is reflexive. Also for each pair $(a, b) \in R$ the pair (b, a) is also in R. So R is symmetric. As $(2, 1), (1, 3) \in R$ and $(2, 3) \notin R$, we see that R is not transitive. Thus R is not an equivalence relation.
2. Let P denote the set of all straight lines in a plane. Let R be the relation defined on P as $\ell R m$ if ℓ is parallel to m.

   This relation is reflexive, symmetric and transitive. Thus it is an equivalence relation.
3. Let A be the set consist of parents with their children , two girls and a boy. Let R be the relation defined by aRb if a is a sister of b.

   This relation is to be looked into carefully. A woman is not a sister of herself. So it is not reflexive. It is not symmetric also. Clearly it is not transitive. So it is not an equivalence relation. (If we consider the same relation on the subset of females, then it becomes symmetric; even in this case it is not transitive).
4. On the set of natural numbers let R be the relation defined by xRy if $x + 2y = 21$. It is better to write the relation explicitly. The relation R is the set

$$\{(1, 10), (3, 9), (5, 8), (7, 7), (9, 6), (11, 5), (13, 4), (15, 3), (17, 2), (19, 1)\}.$$

   As $(1, 1) \notin R$ it is not reflexive; as $(1, 10) \in R$ and $(10, 1) \notin R$ it is not symmetric.
   As $(3, 9) \in R, (9, 6) \in R$ but $(3, 6) \notin R$, the relation is not transitive.
5. Let $X = \{1, 2, 3, 4\}$ and $R = \emptyset$, where $\emptyset$ is the empty set.

   As $(1, 1) \notin R$ it is not reflexive. As we cannot find a pair (x, y) in R such that $(y, x) \notin R$, the relation is not 'not symmetric'; so it is symmetric. Similarly it is transitive.
6. The universal relation is always an equivalence relation.
7. An empty relation can be considered as symmetric and transitive.
8. If a relation contains a single element, then the relation is transitive.

Let us discuss some more special relations now.

Example 1.10 Check the relation $R = \{(1, 1), (2, 2), (3, 3), \ldots, (n, n)\}$ defined on the set $S = \{1, 2, 3, \ldots, n\}$ for the three basic relations.

Solution:
As $(a, a) \in R$ for all $a \in S$, R is reflexive.
There is no pair (a, b) in R such that $(b, a) \notin R$. In other words, for every pair $(a, b) \in R$, (b, a) is also in R. Thus R is symmetric.
We cannot find two pairs (a, b) and (b, c) in R, such that $(a, c) \notin R$. Thus the statement "R is not transitive" is not true; therefore, the statement "R is transitive" is true; hence R is transitive.

Since R is reflexive, symmetric and transitive, this relation is an equivalence relation.

From the very beginning we have denoted all the relations by the same letter R. It is not necessary to do so. We may use the Greek letter ρ (Read as rho) to denote relations. Equivalence relations are mostly denoted by "$\sim$".

If a relation is not of required type, then by inserting or deleting some pairs we can make it of the required type. We do this in the following problem.

Example 1.11 Let $S = \{1, 2, 3\}$ and $\rho = \{(1, 1), (1, 2), (2, 2), (1, 3), (3, 1)\}$.

(i) Is ρ reflexive? If not, state the reason and write the minimum set of ordered pairs to be included to ρ so as to make it reflexive.

(ii) Is ρ symmetric? If not, state the reason, write minimum number of ordered pairs to be included to ρ so as to make it symmetric and write minimum number of ordered pairs to be deleted from ρ so as to make it symmetric.

(iii) Is ρ transitive? If not, state the reason, write minimum number of ordered pairs to be included to ρ so as to make it transitive and write minimum number of ordered pairs to be deleted from ρ so as to make it transitive.

(iv) Is ρ an equivalence relation? If not, write the minimum ordered pairs to be included to ρ so as to make it an equivalence relation.

Solution:

(i) ρ is not reflexive because $(3,3)$ is not in ρ. As $(1,1)$ and $(2,2)$ are in ρ, it is enough to include the pair $(3,3)$ to ρ so as to make it reflexive.

(ii) ρ is not symmetric because $(1,2)$ is in ρ, but $(2,1)$ is not in ρ. It is enough to include the pair $(2,1)$ to ρ so as to make it symmetric.

It is enough to remove the pair $(1,2)$ from ρ so as to make it symmetric

(iii) ρ is not transitive because $(3,1)$ and $(1,3)$ are in ρ, but $(3,3)$ is not in ρ. To make it transitive we have to include $(3,3)$ in ρ. Even after including $(3,3)$, the relation is not transitive because $(3,1)$ and $(1,2)$ are in ρ, but $(3,2)$ is not in ρ. To make it transitive we have to include $(3,2)$ also in ρ. Now it becomes transitive. So $(3,3)$ and $(3,2)$ are to be included so as to make ρ transitive.

But if we remove $(3,1)$ from ρ, then it becomes transitive.

(iv) We have seen that

- to make ρ reflexive, we have to include $(3,3)$;
- to make ρ symmetric, we have to include $(2,1)$;
- and to make ρ transitive, we have to include $(3,3)$ and $(3,2)$.

To make ρ as an equivalence relation we have to include all these pairs. So after including the pairs the relation becomes $\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(3,2)\}$

But this relation is not symmetric because $(3,2)$ is in the relation and $(2,3)$ is not in the relation. So we have to include $(2,3)$ also. Now the new relation becomes

$$\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(3,2),(2,3)\}$$

It can be seen that this relation is reflexive, symmetric and transitive, and hence it is an equivalence relation. Thus we have to include $(3,3)$, $(2,1)$, $(3,2)$ and $(2,3)$ to ρ so as to make it an equivalence relation.

Now let us learn how to create relations having certain properties through the following example.

Example 1.12 Let $A=\{0,1,2,3\}$. Construct relations on A of the following types:

(i) not reflexive, not symmetric, not transitive.
(ii) not reflexive, not symmetric, transitive.
(iii) not reflexive, symmetric, not transitive.
(iv) not reflexive, symmetric, transitive.
(v) reflexive, not symmetric, not transitive.
(vi) reflexive, not symmetric, transitive.
(vii) reflexive, symmetric, not transitive.
(viii) reflexive, symmetric, transitive.

Solution:

(i) Let us use the pair $(1,2)$ to make the relation "not symmetric" and consider the relation $\{(1,2)\}$. It is transitive. If we include $(2,3)$ and not include $(1,3)$, then the relation is not transitive. So the relation $\{(1,2),(2,3)\}$ is not reflexive, not symmetric and not transitive. Similarly we can construct more examples.

(ii) Just now we have seen that the relation $\{(1,2)\}$ is transitive, not reflexive and not symmetric.

(iii) Let us start with the pair $(1,2)$. Since we need symmetricity, we have to include the pair $(2,1)$. At this stage as $(1,1),(2,2)$ are not here, the relation is not transitive. Thus $\{(1,2),(2,1)\}$ is not reflexive; it is symmetric; and it is not transitive.

(iv) If we include the pairs $(1,1)$ and $(2,2)$ to the relation discussed in (iii), it will become transitive. Thus $\{(1,2),(2,1),(1,1),(2,2)\}$ is not reflexive; it is symmetric and it is transitive.

(v) For a relation on $\{0,1,2,3\}$ to be reflexive, it must have the pairs $(0,0),(1,1),(2,2),(3,3)$. Fortunately, it becomes symmetric and transitive. Therefore, as in (i) if we insert $(1,2)$ and $(2,3)$ we get the required one. Thus $\{(0,0),(1,1),(2,2),(3,3),(1,2),(2,3)\}$ is reflexive; it is not symmetric and it is not transitive.

(vi) Proceeding like this we get the relation $\{(0,0),(1,1),(2,2),(3,3),(1,2)\}$ that is reflexive, transitive and not symmetric.

(vii) As above we get the relation $\{(0,0),(1,1),(2,2),(3,3),(1,2),(2,3),(2,1),(3,2)\}$ that is reflexive, symmetric and not transitive.

(viii) We have the relation $\{(0,0),(1,1),(2,2),(3,3)\}$ which is reflexive, symmetric and transitive.

Example 1.13 In the set $\mathbb{Z}$ of integers, define mRn if $m-n$ is a multiple of 12. Prove that R is an equivalence relation.

Solution:
As $m-m=0$ and $0=0\times 12$, hence mRm proving that R is reflexive.

Let mRn. Then $m-n=12k$ for some integer k; thus $n-m=12(-k)$ and hence nRm. This shows that R is symmetric.

Let mRn and nRp; then $m-n=12k$ and $n-p=12\ell$ for some integers k and ℓ.
So $m-p=12(k+\ell)$ and hence mRp. This shows that R is transitive.

Thus R is an equivalence relation.

Theorem 1.1: The number of relations from a set containing m elements to a set containing n elements is 2^{mn}. In particular the number of relations on a set containing n elements is 2^{n^2}.

Proof. Let A and B be sets containing m and n elements respectively. Then $A\times B$ contains mn elements and $A\times B$ has 2^{mn} subsets. Since every subset of $A\times B$ is a relation from A to B, there are 2^{mn} relations from a set containing m elements to a set containing n elements.

Taking $A=B$, we see that the number of relations on a set containing n elements is 2^{n^2}. ■

(i) The number of reflexive relations on a set containing n elements is 2^{n^2-n}.

(ii) The number of symmetric relations on a set containing n elements is $2^{\frac{(n^2+n)}{2}}$.

Definition 1.5

If R is a relation from A to B, then the relation R^{-1} defined from B to A by

$$R^{-1}=\{(b,a):(a,b)\in R\}$$

is called the ***inverse*** of the relation R.

For example, if $R = \{(1, a), (2, b), (2, c), (3, a)\}$, then

$$R^{-1} = \{(a, 1), (b, 2), (c, 2), (a, 3)\}.$$

It is easy to see that the domain of R becomes the range of R^{-1} and the range of R becomes the domain of R^{-1}.

DO YOU KNOW?

An equivalence relation on a set decomposes it into a disjoint union of its subsets (equivalence classes). Such a decomposition is called a partition. This is explained in the following example.

For $a, b \in \mathbb{Z}$, aRb if and only if $a - b = 3k$, $k \in \mathbb{Z}$ is an equivalence relation on $\mathbb{Z}$.

$$\begin{aligned} Z_0 &= \{x \in \mathbb{Z} : xR0\} &= \{\ldots, -6, -3, 0, 3, 6, \ldots\} \\ Z_1 &= \{x \in \mathbb{Z} : xR1\} &= \{\ldots, -5, -2, 1, 4, 7, \ldots\} \\ Z_2 &= \{x \in \mathbb{Z} : xR2\} &= \{\ldots, -4, -1, 2, 5, 8, \ldots\} \end{aligned}$$

Thus $\mathbb{Z} = Z_0 \cup Z_1 \cup Z_2$ and all are disjoint subsets.

For a given partition $S_1 \cup S_2 \cup \cdots \cup S_n$ of a set S into disjoint subsets, one can construct an equivalence relation R on S by xRy if $x, y \in S_i$ for some i.

Equivalence relation is used in almost all branches of higher mathematics.

Exercise - 1.2

1. Discuss the following relations for reflexivity, symmetricity and transitivity:
   (i) The relation R defined on the set of all positive integers by "mRn if m divides n".
   (ii) Let P denote the set of all straight lines in a plane. The relation R defined by "ℓRm if ℓ is perpendicular to m".
   (iii) Let A be the set consisting of all the members of a family. The relation R defined by "aRb if a is not a sister of b".
   (iv) Let A be the set consisting of all the female members of a family. The relation R defined by "aRb if a is not a sister of b".
   (v) On the set of natural numbers the relation R defined by "xRy if $x + 2y = 1$".
2. Let $X = \{a, b, c, d\}$ and $R = \{(a, a), (b, b), (a, c)\}$. Write down the minimum number of ordered pairs to be included to R to make it
   (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence
3. Let $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (a, c)\}$. Write down the minimum number of ordered pairs to be included to R to make it
   (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence
4. Let P be the set of all triangles in a plane and R be the relation defined on P as aRb if a is similar to b. Prove that R is an equivalence relation.
5. On the set of natural numbers let R be the relation defined by aRb if $2a + 3b = 30$. Write down the relation by listing all the pairs. Check whether it is
   (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence
6. Prove that the relation "friendship" is not an equivalence relation on the set of all people in Chennai.
7. On the set of natural numbers let R be the relation defined by aRb if $a + b \leq 6$. Write down the relation by listing all the pairs. Check whether it is
   (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence

8. Let $A = \{a, b, c\}$. What is the equivalence relation of smallest cardinality on A? What is the equivalence relation of largest cardinality on A?
9. In the set Z of integers, define mRn if $m - n$ is divisible by 7. Prove that R is an equivalence relation.

1.6 Functions

Suppose that a particle is moving in the space. We assume the physical particle as a point. As time varies, the particle changes its position. Mathematically at any time the point occupies a position in the three dimensional space $\mathbb{R}^3$. Let us assume that the time varies from 0 to 1. So the movement or *functioning* of the particle decides the position of the particle at any given time t between 0 and 1. In other words, for each $t \in [0, 1]$, the functioning of the particle gives a point in $\mathbb{R}^3$. Let us denote the position of the particle at time t as $f(t)$.

Let us see another simple example. We know that the equation $2x - y = 0$ describes a straight line. Here whenever x assumes a value, y assumes some value accordingly. The movement or *functioning* of y is decided by that of x. Let us denote y by $f(x)$. We may see many situation like this in nature. In the study of natural phenomena, we find that it is necessary to consider the variation of one quantity depending on the variation of another.

The relation of the time and the position of the particle, the relation of a point in the x-axis to a point in the y-axis and many more such relations are studied for a very long period in the name *function*. Before Cantor, the term function is defined as a rule which associates a variable with another variable. After the development of the concept of sets, a function is defined as a rule that associates for every element in a set A, a unique element in a set B. However the terms *rule* and *associate* are not properly defined mathematical terminologies. In modern mathematics every term we use has to be defined properly. So a definition for function is given using relations.

Suppose that we want to discuss a test written by a set of students. We shall see this as a relation.

Let A be the set of students appeared for an examination and let $B = \{0, 1, 2, 3, \ldots 100\}$ be the set of possible marks. We define a relation R as follows:

A student a is related to a mark b if a got b marks in the test.

We observe the following from this example:

- Every student got a mark. In other words, for every $a \in A$, there is an element $b \in B$ such that $(a, b) \in R$.
- A student cannot get two different marks in any test. In other words, for every $a \in A$, there is definitely only one $b \in B$ such that $(a, b) \in R$. This can be restated in a different way: If $(a, b), (a, c) \in R$ then $b = c$.

Relations having the above two properties form a very important class of relations, called functions. Let us now have a rigorous definition of a function through relations.

Definition 1.6

Let A and B be two sets. A relation f from A to B, a subset of $A \times B$, is called a **function** from A to B if it satisfies the following:

(i) for all $a \in A$, there is an element $b \in B$ such that $(a, b) \in f$.
(ii) if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.

That is, a function is a relation in which each element in the domain is mapped to exactly one element in the range.

A is called the ***domain*** of f and B is called the ***co-domain*** of f. If (a, b) is in f, then we write $f(a) = b$; the element b is called the ***image*** of a and the element a is called a ***pre-image*** of b and $f(a)$ is known as the value of f at a. The set $\{b : (a, b) \in f$ for some $a \in A\}$ is called the ***range*** of the function. If B is a subset of $\mathbb{R}$, then we say that the function is a ***real-valued function***.

Two functions f and g are said to be ***equal functions*** if their domains are same and $f(a) = g(a)$ for all a in the domain.

If f is a function with domain A and co-domain B, we write $f : A \to B$ (Read this as f is from A to B or f be a function from A to B). We also say that f maps A into B. If $f(a) = b$, then we say f maps a to b or a is mapped onto b by f, and so on.

The range of a function is the collection of all elements in the co-domain which have pre-images. Clearly the range of a function is a subset of the co-domain. Further the first condition says that every element in the domain must have an image; this is the reason for defining the domain of a relation R from a set A to a set B as the set of all elements of A having images and not as A. The second condition says that an element in the domain cannot have two or more images.

Naturally one may have the following doubts:

- In the definition, why we use the definite article "the" for image of a and the indefinite article "a" for pre-image of b?
- We have a condition stating that every element in the domain must have an image; is there any condition like "every element in the co-domain must have a pre-image"? If not, why?
- We have a condition stating that an element in the domain cannot have two or more images; is there any condition like "an element in the co-domain cannot have two or more pre-images"? If not, why?

As an element in the domain has exactly one image and an element in the co-domain can have more than one pre-image according to the definition, we use the definite article "the" for image of a and the indefinite article "a" for pre-image of b. There are no conditions as asked in the other two questions; the reason behind it can be understood from the problem of students' mark we considered above.

We observe that every function is a relation but a relation need not be a function.

Let $f = \{(a, 1), (b, 2), (c, 2), (d, 4)\}$.

Is f a function? This is a function from the set $\{a, b, c, d\}$ to $\{1, 2, 4\}$. This is not a function from $\{a, b, c, d, e\}$ to $\{1, 2, 3, 4\}$ because e has no image. This is not a function from $\{a, b, c, d\}$ to $\{1, 2, 3, 5\}$ because the image of d is not in the co-domain; f is not a subset of $\{a, b, c, d\} \times \{1, 2, 3, 5\}$. So whenever we consider a function the domain and the co-domain must be stated explicitly.

The relation discussed in Illustration 1.1 is a function with domain $\{L, E, T, U, S, W, I, N\}$ and co-domain $\{O, H, W, X, V, Z, L, Q\}$. The relation discussed in Illustration 1.3 is again a function with domain $\{a, b, \ldots, z\}$ and the co-domain $\{1, 2, 3, \ldots, 26\}$.

In Illustration 1.2, we discussed three relations, namely

$$\text{(i) } y = 2x \quad \text{(ii) } y = x^2 \quad \text{(iii) } y^2 = x.$$

Clearly (i) and (ii) are functions whereas (iii) is not a function, if the domain and the co-domain are $\mathbb{R}$. In (iii) for the same x, we have two y values which contradict the definition of the function. But if we split into two relations, that is, $y = \sqrt{x}$ and $y = -\sqrt{x}$ then both become functions with same domain non-negative real numbers and the co-domains $[0, \infty)$ and $(-\infty, 0]$ respectively.

1.6.1 Ways of Representing Functions

(a) Tabular Representation of a Function

When the elements of the domain are listed like $x_1, x_2, x_3 \ldots x_n$, we can use this tabular form.Here, the values of the arguments $x_1, x_2, x_3 \ldots x_n$ and the corresponding values of the function $y_1, y_2, y_3 \ldots y_n$ are written out in a definite order.

x	x_1	x_2	$\ldots$	x_n
y	y_1	y_2	$\ldots$	y_n

(b) Graphical Representation of a Function

When the domain and the co-domain are subsets of $\mathbb{R}$, many functions can be represented using a graph with x-axis representing the domain and y-axis representing the co-domain in the (x, y)-plane.

We note that the first and second figures in Illustration 1.2 represent the functions $f(x) = 2x$ and $f(x) = x^2$ respectively. Usually the variable x is treated as independent variable and y as a dependent variable. The variable x is called the ***argument*** and $f(x)$ is called the value.

(c) Analytical Representation of a Function

If the functional relation $y = f(x)$ is such that f denotes an analytical expression, we say that the function y of x is represented or defined analytically. Some examples of analytical expressions are

$$x^3 + 5, \ \frac{\sin x + \cos x}{x^2 + 1}, \ \log x + 5\sqrt{x}.$$

That is, a series of symbols denoting certain mathematical operations that are performed in a definite sequence on numbers, letters which designate constants or variable quantities.

Examples of functions defined analytically are

$$\text{(i)} \ \ y = \frac{x-1}{x+1} \quad \text{(ii)} \ \ y = \sqrt{9 - x^2} \quad (iii) \ \ y = \sin x + \cos x \quad \text{(iv)} \ \ A = \pi r^2.$$

One of the usages of writing functions analytically is finding domains naturally. That is, the set of values of x for which the analytical expressions on the right-hand side has a definite value is the natural domain of definition of a function represented analytically.

Thus, the natural domain of the function,

(i)	$y = x^3 + 3$	is	$(-\infty, \infty)$	(ii)	$y = x^4 - 2$	is	$(-\infty, \infty)$
(iii)	$y = \frac{x-1}{x+1}$	is	$\mathbb{R} - \{-1\}$	(iv)	$y = +\sqrt{4 - x^2}$	is	$-2 \le x \le 2.$

Now recall the domain of the functions (i) $y = 2x$, (ii) $y = x^2$, (iii) $y = +\sqrt{x}$, (iv) $y = -\sqrt{x}$ which are analytical in nature described earlier.

Sometimes we may come across piece-wise defined functions. For example, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & \text{if} \quad -\infty < x \le -2 \\ 2x & \text{if} \quad -2 < x \le 3 \\ x^2 & \text{if} \quad 3 < x < \infty \end{cases}$$

Depending upon the value of x, we have to select the formula to be used to find the value of f at any point x. To find the value of f at any real number, first we have to find to which interval x belongs to; then using the corresponding formula we can find the value of f at that point. To find $f(6)$ we know $3 \le 6 < \infty$ (or $6 \in [3, \infty)$); so we use the formula $f(x) = x^2$ and find $f(6) = 36$. Similarly $f(-1) = -2, f(-5) = 0$ and so on.

If the function is defined from $\mathbb{R}$ or a subset of $\mathbb{R}$ then we can draw the graph of the function. For example, if $f : [0, 4] \to \mathbb{R}$ is defined by $f(x) = \frac{x}{2} + 1$, then we can plot the points $(x, \frac{x}{2} + 1)$ for all $x \in [0, 4]$. Then we will get a straight line segment joining $(0, 1)$ and $(4, 3)$. (See Figure 1.13)

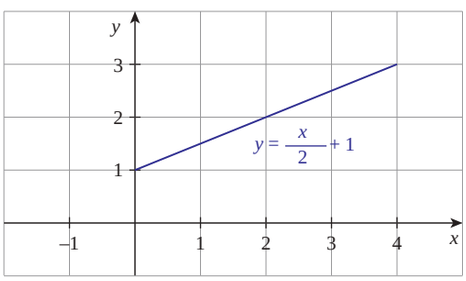


Figure 1.13

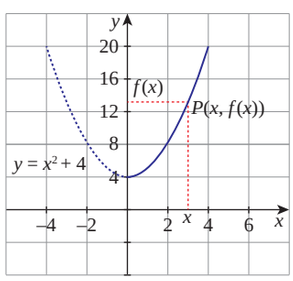


Figure 1.14

Consider another function $f(x) = x^2 + 4, x \geq 0$. The function will be given by its graph. (See Figure 1.14)

Let x be a point in the domain. Let us draw a vertical line through the point x. Let it meet the curve at P. The point at which the horizontal line drawn through P meets the y-axis is $f(x)$. Similarly using horizontal lines through a point y in the co-domain, we can find the pre-images of y.

Can we say that any curve drawn on the plane be considered as a function from a subset of $\mathbb{R}$ to $\mathbb{R}$? No, we cannot. There is a simple test to find this.

Vertical Line Test

As we noted earlier, the vertical line through any point x in the domain meets the curve at some point, then the y-coordinate of the point is $f(x)$. If the vertical line through a point x in the domain meets the curve at more than one point, we will get more than one value for $f(x)$ for one x. This is not allowed in a function. Further, if the vertical line through a point x in the domain does not meet the curve, then there will be no image for x; this is also not possible in a function. So we can say,

> "if the vertical line through a point x in the domain meets the curve at more than one point or does not meet the curve, then the curve will not represent a function".

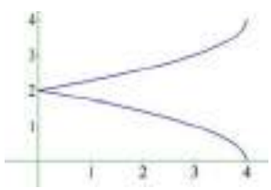


Figure 1.15

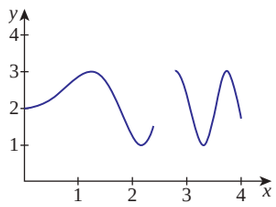


Figure 1.16

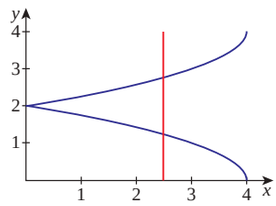


Figure 1.17

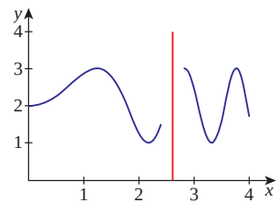


Figure 1.18

The curve indicated in Figure 1.15 does not represent a function from $[0, 4]$ to $\mathbb{R}$ because a vertical line meets the curve at more than one point (See Figure 1.17). The curve indicated in Figure 1.16 does not represent a function from $[0, 4]$ to $\mathbb{R}$ because a vertical line drawn through $x = 2.5$ in $[0, 4]$ does not meet the curve (See Figure 1.18).

Testing whether a given curve represents a function or not by drawing vertical lines is called ***vertical line test*** or simply vertical test.

The third curve $y^2 = x$ in Illustration 1.2 fails in the vertical line test and hence it is not a function from $\mathbb{R}$ to $\mathbb{R}$.

1.6.2 Some Elementary Functions

Some frequently used functions are known by names. Let us list some of them.

(i) Let X be any non-empty set. The function $f : X \to X$ defined by $f(x) = x$ for all $x \in X$ is called the ***identity function*** on X (See Figure 1.19). It is denoted by I_X or I.

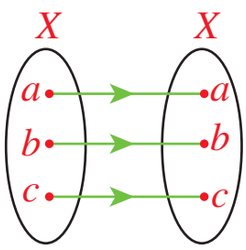


Figure 1.19

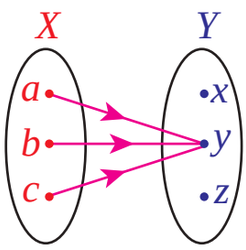


Figure 1.20

(ii) Let X and Y be two sets. Let c be a fixed element of Y. The function $f : X \to Y$ defined by $f(x) = c$ for all $x \in X$ is called a ***constant function*** (See Figure 1.20).

The value of a constant function is same for all values of x throughout the domain.

If X and Y are $\mathbb{R}$, then the graph of the identity function and a constant function are as in Figures 1.21 and 1.22. If X is any set, then the constant function defined by $f(x) = 0$ for all x is called the ***zero function***. So zero function is a particular case of constant function.

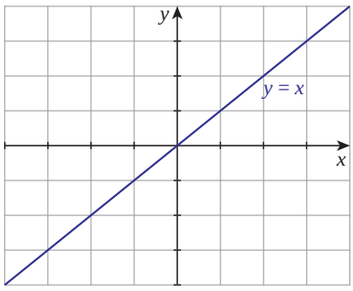


Figure 1.21

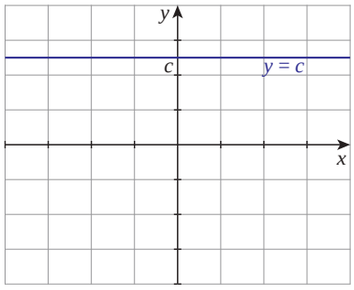


Figure 1.22

(iii) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$, where $|x|$ is the modulus or absolute value of x, is called the ***modulus function*** or ***absolute value function***. (See Figure 1.23.)

Let us recall that $|x|$ is defined as

$$|x| = \begin{cases} -x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ x & \text{if} \quad x > 0 \end{cases} \quad \text{or } |x| = \begin{cases} -x & \text{if} \quad x \leq 0 \\ x & \text{if} \quad x > 0 \end{cases} \quad \text{or } |x| = \begin{cases} -x & \text{if} \quad x < 0 \\ x & \text{if} \quad x \geq 0 \end{cases}$$

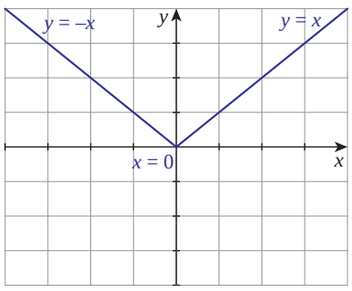


Figure 1.23

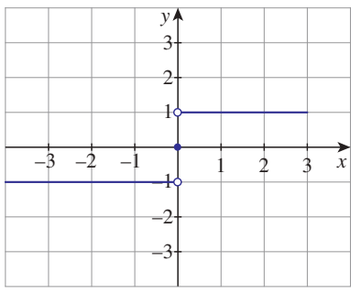


Figure 1.24

(iv) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \frac{x}{|x|} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ is called the ***signum function***. This function is denoted by sgn. (See Figure 1.24)

(v) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x)$ is the greatest integer less than or equal to x is called the integral part function or the ***greatest integer function*** or the ***floor function***. This function is denoted by $\lfloor x \rfloor$. (See Figure 1.25.)

(vi) The function $f : \mathbb{R} \to \mathbb{R}$defined by $f(x)$ is the smallest integer greater than or equal to x is called the ***smallest integer function*** or the ***ceil function*** (See Figure 1.26.). This function is denoted by $\lceil \cdot \rceil$; that is $f(x)$ is denoted by $\lceil x \rceil$.

The functions (v) and (vi) are also called ***step functions***.

Let us note that $\lfloor 1\frac{1}{5} \rfloor = 1$, $\lfloor 7.23 \rfloor = 7$, $\lfloor -2\frac{1}{2} \rfloor = -3$ (not -2), $\lfloor 6 \rfloor = 6$ and $\lfloor -4 \rfloor = -4$.

Let us note that $\lceil 1\frac{1}{5} \rceil = 2$, $\lceil 7.23 \rceil = 8$, $\lceil -2\frac{1}{2} \rceil = -2$ (not -3), $\lceil 6 \rceil = 6$ and $\lceil -4 \rceil = -4$.

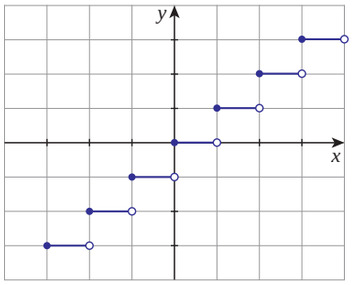


Figure 1.25

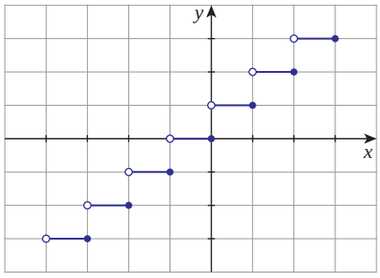


Figure 1.26

One may note that the relations among the names of these functions, the symbols denoting the functions are the commonly used words ceiling and floor of a room. Their graphs are given in Figures 1.25 and 1.26.

1.6.3 Types of Functions

Though functions can be classified into various types according to the need, we are going to concentrate on two basic types: *one-to-one functions and onto functions*.

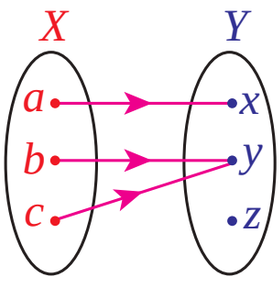


Figure 1.27

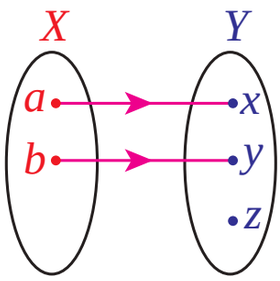


Figure 1.28

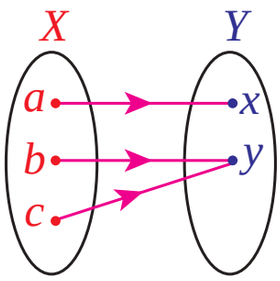


Figure 1.29

Let us look at the two simple functions given in Figure 1.27 and Figure 1.28. In the first function two elements of the domain, b and c, are mapped into the same element y, whereas it is not the case in the Figure 1.28. Functions like the second one are examples of one-to-one functions.

Let us look at the two functions given in Figures 1.28 and 1.29. In Figure 1.28 the element z in the co-domain has no pre-image, whereas it is not the case in Figure 1.29. Functions like in Figure 1.29 are examples of onto functions. Now we define one-to-one and onto functions.

Definition 1.7

A function $f : A \to B$ is said to be ***one-to-one*** if $x, y \in A, x \neq y \Rightarrow f(x) \neq f(y)$ [or equivalently $f(x) = f(y) \Rightarrow x = y$]. A function $f : A \to B$ is said to be ***onto***, if for each $b \in B$ there exists at least one element $a \in A$ such that $f(a) = b$. That is, the range of f is B.

Another name for one-to-one function is ***injective function***; onto function is ***surjective function***. A function $f : A \to B$ is said to be ***bijective*** if it is both one-to-one and onto.

To prove a function $f : A \to B$ to be one-to-one, it is enough to prove any one of the following:

if $x \neq y$, then $f(x) \neq f(y)$, or equivalently if $f(x) = f(y)$, then $x = y$.

It is easy to observe that every identity function is one-to-one function as well as onto. A constant function is not onto unless the co-domain contains only one element. The following statements are some important simple results.

Let A and B be two sets with m and n elements.

(i) There is no one-to-one function from A to B if $m > n$.
(ii) If there is an one-to-one function from A to B, then $m \leq n$.
(iii) There is no onto function from A to B if $m < n$.
(iv) If there is an onto function from A to B, then $m \geq n$.
(v) There is a bijection from A to B, if and only if, $m = n$.
(vi) There is no bijection from A to B if and only if, $m \neq n$.

A function which is not onto is called an into function. That is, the range of the function is a proper subset of its co-domain. Let us see some illustrations.

(1) $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d, e\}$and $f = \{(1, a), (2, c), (3, e), (4, b)\}$.
This function is one-to-one but not onto.

(2) $X = \{1, 2, 3, 4\}, Y = \{a, b\}$ and $f = \{(1, a), (2, a), (3, a), (4, a)\}$.
This function is not one-to-one; it is not onto.

(3) $X = \{1, 2, 3, 4\}, Y = \{a\}$ and $f = \{(1, a), (2, a), (3, a), (4, a)\}$.
This function is not one-to-one but it is onto. It seems that this function is same as the previous one. The co-domain of the function is very important when deciding whether the function is onto or not.

(4) $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d, e\}$ and $f = \{(1, a), (2, c), (3, b), (4, b)\}$.
This function is neither one-to-one nor onto.

(5) $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d\}$ and $f = \{(1, a), (2, c), (3, d), (4, b)\}$.
This function is both one-to-one and onto.

(6) $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d, e\}$ and $f = \{(1, a), (2, c), (3, e)\}$.
This is not at all a function, only a relation.

(7) Let X be a finite set with k elements. Then, we have a bijection from X to $\{1, 2, \ldots k\}$.

Let us consider functions defined on some known sets through a formula rule.

Example 1.14 Check whether the following functions are one-to-one and onto.

(i) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 2$.
(ii) $f : \mathbb{N} \cup \{-1, 0\} \to \mathbb{N}$ defined by $f(n) = n + 2$.

Solution:

(i) If $f(n) = f(m)$, then $n + 2 = m + 2$ and hence $m = n$. Thus f is one-to-one. As 1 has no pre-image, this function is not onto. (See Figure 1.30)
(ii) As above, this function is one-to-one. If m is in the co-domain, then $m - 2$ is in the domain and $f(m - 2) = (m - 2) + 2 = m$; thus m has a pre-image and hence this function is onto. (See Figure 1.31)

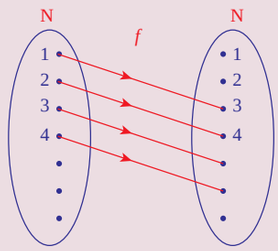


Figure 1.30

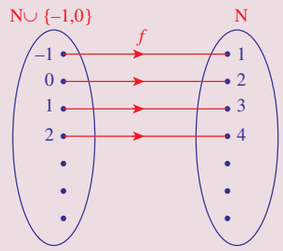


Figure 1.31

It seems that the second function (ii) is same as the first function (i). But the domains are different. From this we see that the domain of the function is also important in deciding whether the function is onto or not. The co-domain has no role in deciding whether the function is one-to-one or not. But it is important to decide whether the function is onto or not.

Example 1.15 Check the following functions for one-to-oneness and ontoness.

(i) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^2$.
(ii) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(n) = n^2$.

Solution:

(i) $f(m) = f(n) \Rightarrow m^2 = n^2 \Rightarrow m = n$ since $m, n \in \mathbb{N}$. Thus f is one-to-one. But, non-perfect square elements in the co-domain do not have pre-images and hence not onto.
(ii) Two different elements in the domain have same images and hence f is not one-to-one. Clearly the range of f is a proper subset of $\mathbb{R}$. Thus it is not onto.

Now, we recall Illustration 1.1. In this illustration the function $f : C \to D$ is defined by

$$f(L) = O, f(E) = H, f(T) = W, f(U) = X, f(S) = V, f(W) = Z, f(I) = L, f(N) = Q$$

where $C = \{L, E, T, U, S, W, I, N\}$ and $D = \{O, H, W, X, V, Z, L, Q\}$, is an one-to-one and onto function.

In Illustration 1.3, the function $f : A \to \mathbb{N}$ is defined by $f(a) = 1, f(b) = 2, \ldots, f(z) = 26$, where $A = \{a, b, \ldots z\}$. This function is one-to-one. If we take $\mathbb{N}$ as co-domain, the function is not onto. Instead of $\mathbb{N}$ if we take the co-domain as $\{1, 2, 3 \ldots, 26\}$ then it becomes onto.

Example 1.16 Check whether the following for one-to-oneness and ontoness.

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$.
(ii) $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$.

Solution:

(i) This is not at all a function because $f(x)$ is not defined for $x = 0$.
(ii) This function is one-to-one (verify) but not onto because 0 has no pre-image.

If we consider $\mathbb{R} - \{0\}$ as the co-domain for the second, then f will become a bijection.

Example 1.17 If $f : \mathbb{R} - \{-1, 1\} \to \mathbb{R}$ is defined by $f(x) = \frac{x}{x^2-1}$, verify whether f is one-to-one or not.

Solution:
We start with the assumption $f(x) = f(y)$. Then,

$$\begin{array}{rrcl} & \frac{x}{x^2-1} & = & \frac{y}{y^2-1} \\ \Rightarrow & x(y^2-1) & = & y(x^2-1) \\ \Rightarrow & xy^2 - x - yx^2 + y & = & 0 \\ \Rightarrow & (y-x)(xy+1) & = & 0 \end{array}$$

This implies that $x = y$ or $xy = -1$. So if we select two numbers x and y so that $xy = -1$, then $f(x) = f(y)$. $(2, -\frac{1}{2}), (7, -\frac{1}{7}), (-2, \frac{1}{2})$ are some among the infinitely many possible pairs. Thus $f(2) = f(\frac{-1}{2}) = \frac{2}{3}$. That is, $f(x) = f(y)$ does not imply $x = y$. Hence it is not one-to-one.

Example 1.18 If $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = 2x^2 - 1$, find the pre-images of $17, 4$ and -2.

Solution:
To find the pre-image of 17, we solve the equation $2x^2 - 1 = 17$. The two solutions of this equation, 3 and -3 are the pre-images of 17 under f. The equation $2x^2 - 1 = 4$ yields $\sqrt{\frac{5}{2}}$ and $-\sqrt{\frac{5}{2}}$ as the two pre-images of 4. To find the pre-image of -2, we solve the equation $2x^2 - 1 = -2$. This shows that $x^2 = -\frac{1}{2}$ which has no solution in $\mathbb{R}$ because square of a number cannot be negative and hence -2 has no pre-image under f.

Example 1.19 If $f : [-2, 2] \to B$ is given by $f(x) = 2x^3$, then find B so that f is onto.

Solution:
The minimum value is $f(-2)$ and its maximum value is $f(2)$ which are equal to -16 and 16 respectively. So B is $[-16, 16]$.

As $f(x) = 2x^3$ is an increasing function on $[-2, 2]$, the minimum value is attained at the left end and the maximum value is attained at the right end. (For more about increasing / decreasing functions one may refer later chapters.)

Example 1.20 Check whether the function $f(x) = x|x|$ defined on $[-2, 2]$ is one-to-one or not. If it is one-to-one, find a suitable co-domain so that the function becomes a bijection.

Solution:
Let $x, y \in [-2, 2]$ such that $f(x) = f(y)$. If $y = 0$, then $x = 0$. So let $y \neq 0$ and hence $x \neq 0$. Now $x|x| = y|y|$ since $f(x) = f(y)$. This implies that $\frac{x}{y} = \frac{|y|}{|x|}$. Since $\frac{|y|}{|x|} > 0$, $\frac{x}{y} > 0$; thus x and y are either both positive or both negative and hence $x^2 = y^2$.

So if $f(x) = f(y)$, we must have $x^2 = y^2$. Also x and y are either both negative or both positive. This is possible only if $x = y$. Thus f is one-to-one. When $x < 0$, $f(x) = -x^2$ and when $x \geq 0$, $f(x) = x^2$. So the range is $[-4, 4]$. So f becomes a bijection from $[-2, 2]$ to $[-4, 4]$.

$f(x) = x|x|$ is also an increasing function.

Horizontal Test

Similar to the vertical line test we have a test called horizontal test to check whether a function is one-to-one, onto or not. Let a function be given as a curve in the plane. If the horizontal line through a point y in the co-domain meets the curve at some points, then the x-coordinate of all the points give pre-images for y.

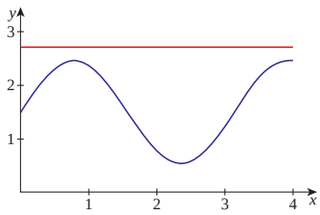


Figure 1.32

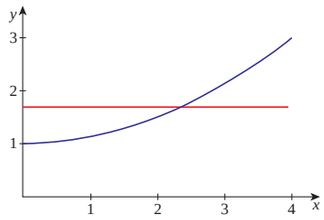


Figure 1.33

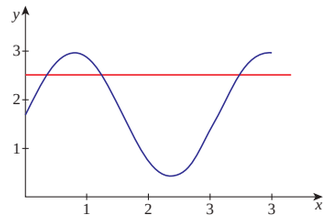


Figure 1.34

(i) If the horizontal line through a point y in the co-domain does not meet the curve, then there will be no pre-image for y and hence the function is not onto.
(ii) If the horizontal line through atleast one point of the codomain meets the curve at more than one point, then the function is not one-to-one.
(iii) If for all y in the range the horizontal line through y meets the curve at only one point, then the function is one-to-one.

So we may say, the function represented by a curve is one-to-one if and only if for all y in the range, the horizontal line through the point y meets the curve at exactly one point.

The function represented by a curve is onto if and only if for all y in the co-domain, the horizontal line through the point y meets the curve atleast one point.

The curve given in Figure 1.32 represents a function from $[0, 4]$ which is not onto if the co-domain contains $[1, 3]$. The curve given in Figure 1.33 represents a one-to-one function from $[0, 4]$ to $\mathbb{R}$ and the curve given in Figure 1.34 represents a function from $[0, 4]$ to $\mathbb{R}$ which is not one-to-one.

Testing whether a given curve represents a one-to-one function, onto function or not by drawing horizontal lines is called ***horizontal line test*** or simply horizontal test.

Further by seeing the diagrams in Illustration 1.2 and Figures 1.5 to Figure 1.7, the function

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ is an one-to-one and onto function.
(ii) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is neither one-to-one nor onto.
(iii) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = +\sqrt{x}$ is an one-to-one but not onto function.
(iv) $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = +\sqrt{x}$ is an one-to-one and onto function.
(v) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = -\sqrt{x}$ is one-to-one but not onto function.
(vi) $f : [0, \infty) \to (-\infty, 0]$ defined by $f(x) = -\sqrt{x}$ is one-to-one and onto function.

Example 1.21 Find the largest possible domain for the real valued function f defined by $f(x) = \sqrt{x^2 - 5x + 6}$.

Solution:

As we are finding the square root of $x^2 - 5x + 6$, we must have $x^2 - 5x + 6 \geq 0$ for all x in the domain. For this, follow the given procedure.

Solving $x^2 - 5x + 6 = 0$, we get $x = 2$ and 3. Now draw the number line as in Figure 1.35.

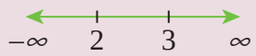


Figure 1.35

Now we have three intervals. $(-\infty, 2), (2, 3)$ and $(3, \infty)$

(i) Take any point in $(-\infty, 2)$, say $x = 1$. Clearly $x^2 - 5x + 6$ is positive.
(ii) Take any point in $(2, 3)$, say $x = 2.5$. Clearly $x^2 - 5x + 6$ is negative.
(iii) Take any point in $(3, \infty)$, say $x = 4$. Clearly $x^2 - 5x + 6$ is positive.

For all x, in the intervals $(-\infty, 2)$ and $(3, \infty)$, $x^2 - 5x + 6$ is positive. At $x = 2, 3$ the value of $x^2 - 5x + 6$ is zero. Thus, $\sqrt{x^2 - 5x + 6}$ is defined for all x in $(-\infty, 2] \cup [3, \infty)$.
Hence the domain of $\sqrt{x^2 - 5x + 6}$ is $(-\infty, 2] \cup [3, \infty)$.

Example 1.22 Find the domain of $f(x) = \frac{1}{1-2\cos x}$.

Solution:
The function is defined for all $x \in \mathbb{R}$ except $1 - 2\cos x = 0$. That is, except $\cos x = \frac{1}{2}$. That is except $x = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$. Hence the domain is $\mathbb{R} - \{2n\pi \pm \frac{\pi}{3}\}, n \in \mathbb{Z}$

Example 1.23 Find the range of the function $f(x) = \frac{1}{1-3\cos x}$.

Solution:
Clearly,

$$\begin{array}{rrcccl} & -1 & \leq & \cos x & \leq & 1 \\ \Rightarrow & 3 & \geq & -3\cos x & \geq & -3 \\ \Rightarrow & -3 & \leq & -3\cos x & \leq & 3 \\ \Rightarrow & 1-3 & \leq & 1-3\cos x & \leq & 1+3 \end{array}$$

Thus we get $-2 \leq 1 - 3\cos x$ and $1 - 3\cos x \leq 4$.
By taking reciprocals, we get $\frac{1}{1-3\cos x} \leq -\frac{1}{2}$ and $\frac{1}{1-3\cos x} \geq \frac{1}{4}$.
Hence the range of f is $(-\infty, -\frac{1}{2}] \cup [\frac{1}{4}, \infty)$.

Example 1.24 Find the largest possible domain for the real valued function given by $f(x) = \frac{\sqrt{9-x^2}}{\sqrt{x^2-1}}$.

Solution:
If $x < -3$ or $x > 3$, then x^2 will be greater than 9 and hence $9 - x^2$ will become negative which has no square root in $\mathbb{R}$. So x must lie on the interval $[-3, 3]$.

Also if $x \geq -1$ and $x \leq 1$, then $x^2 - 1$ will become negative or zero. If it is negative, $x^2 - 1$ has no square root in $\mathbb{R}$. If it is zero, f is not defined. So x must lie outside $[-1, 1]$. That is, x must lie on $(-\infty, -1) \cup (1, \infty)$. Combining these two conditions, the largest possible domain for$f$ is $[-3, 3] \cap ((-\infty, -1) \cup (1, \infty))$. That is, $[-3, -1) \cup (1, 3]$.

Draw the number line and plot the intervals to get the required domain interval.

1.6.4 Operations on Functions

Composition

Let there be two functions f and g as given in the Figure 1.36 and Figure 1.37. Let us note that the co-domain of f and the domain of g are the same. Let us cut off Figure 1.37 of g and paste it on the Figure 1.36 of f so that the domain Y of g is pasted on co-domain Y of f. (See Figure 1.38.)

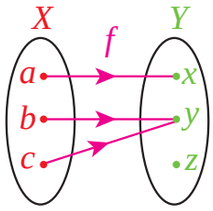


Figure 1.36

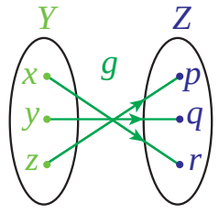


Figure 1.37

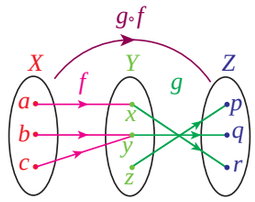


Figure 1.38

Now we can define a function $h : X \to Z$ in a natural way. To find the image of a under h, we first see the image of a under f; it is x; then we see the image of this x under g; this is r. That is, $h(a) = r$. Similarly, we declare $h(b) = q$ and $h(c) = q$. In this way we can define a new function h. This h is called the composition of f with g.

Definition 1.8

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then the function $h : X \to Z$ defined as $h(x) = g(f(x))$ for every $x \in X$ is called the ***composition*** of f with g. It is denoted by $g \circ f$ (Read this as f composite with g). (See Figures 1.38 and 1.39.)

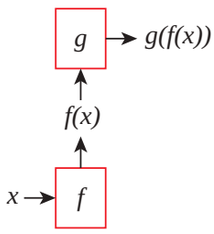


Figure 1.39

We can note that the range of f need not be Y. If $f : X \to Y_1, g : Y_2 \to Z$ and $Y_1 \subseteq Y_2$, then also we can define $g \circ f$; we can take Y_2 as the co-domain of f and use the same definition. So we can define $g \circ f$ if and only if the range of f is contained in the domain of g.

Example 1.25 Let $f = \{(1, 2), (3, 4), (2, 2)\}$ and $g = \{(2, 1), (3, 1), (4, 2)\}$. Find $g \circ f$ and $f \circ g$.

Solution:

To check whether compositions can be defined, let us find the domain and range of these functions.

Domain of $f = \{1, 2, 3\}$, Range of $f = \{2, 4\}$, Domain of $g = \{2, 3, 4\}$ and Range of $g = \{1, 2\}$. Since the range of f is contained in the domain of g we can define $g \circ f$; so as to find the image of 1 under $g \circ f$, we first find the image of 1 under f and then its image under g. The image of 1 under f is 2 and its image under g is 1. So $(g \circ f)(1) = g(f(1)) = g(2) = 1$.

Similarly we find that $(g \circ f)(2) = 1$ and $(g \circ f)(3) = 2$. So $g \circ f = \{(1,1),(2,1),(3,2)\}$. Similarly $f \circ g = \{(2,2),(3,2),(4,2)\}$.

Example 1.26 Let $f = \{(1,4),(2,5),(3,5)\}$ and $g = \{(4,1),(5,2),(6,4)\}$. Find $g \circ f$. Can you find $f \circ g$?

Solution:
Clearly, $g \circ f = \{(1,1),(2,2),(3,2)\}$. But $f \circ g$ is not defined because the range of $g = \{1,2,4\}$ is not contained in the domain of $f = \{1,2,3\}$.

Example 1.27 Let f and g be the two functions from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 3x - 4$ and $g(x) = x^2 + 3$. Find $g \circ f$ and $f \circ g$.

Solution:
We have,
$(g \circ f)(x) = g(f(x)) = g(3x-4) = (3x-4)^2 + 3 = 9x^2 - 24x + 19.$
$(f \circ g)(x) = f(g(x)) = f(x^2+3) = 3(x^2+3) - 4 = 3x^2 + 5.$

Here we have $f \circ g \neq g \circ f$. Thus the operation "composition of functions" is in general not commutative.

Theorem 1.2: Let $f : A \to B$ and $g : B \to C$ be two functions. If f and g are one-to-one, then $g \circ f$ is one-to-one.

Proof. Let $x \neq y$ in A. Since f is one-to-one, $f(x) \neq f(y)$. Since g is one-to-one, $g(f(x)) \neq g(f(y))$. That is, $x \neq y \Rightarrow (g \circ f)(x) \neq (f \circ g)(y)$. Hence $g \circ f$ is one-to-one. ■

Example 1.28 Show that the statement,

"**if f and $g \circ f$ are one-to-one, then g is one-to-one**" is not true.

Solution:
To claim a statement is not true we have to prove by giving one counter example. Consider the diagram given in Figure 1.40.

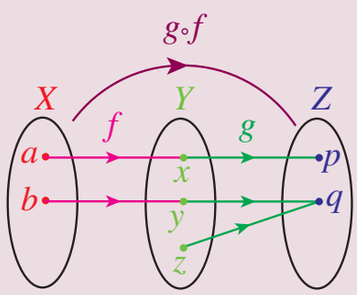


Figure 1.40

Clearly f and $g \circ f$ are one-to-one. But g is not one-to-one. Thus from the above diagram it shows that the statement is not true.

Example 1.29 Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x - |x|$ and $g(x) = 2x + |x|$. Find $f \circ g$.

Solution:
We know

$$|x| = \begin{cases} -x & \text{if} \quad x \leq 0 \\ x & \text{if} \quad x > 0 \end{cases}$$

So

$$f(x) = \begin{cases} 2x - (-x) & \text{if} \quad x \leq 0 \\ 2x - x & \text{if} \quad x > 0 \end{cases}$$

Thus

$$f(x) = \begin{cases} 3x & \text{if} \quad x \leq 0 \\ x & \text{if} \quad x > 0 \end{cases}$$

Also

$$g(x) = \begin{cases} 2x + (-x) & \text{if} \quad x \leq 0 \\ 2x + x & \text{if} \quad x > 0 \end{cases}$$

Thus

$$g(x) = \begin{cases} x & \text{if} \quad x \leq 0 \\ 3x & \text{if} \quad x > 0 \end{cases}$$

Let $x \leq 0$. Then

$$(f \circ g)(x) = f(g(x)) = f(x) = 3x.$$

The last equality is taken because $3x \leq 0$ whenever $x \leq 0$.

Let $x > 0$. Then

$$(f \circ g)(x) = f(g(x)) = f(3x) = 3x.$$

Thus $(f \circ g)(x) = 3x$ for all x.

1.6.5 Inverse of a Function

Let there be a bijection $f : X \to Y$ as given in the Figure 1.41.

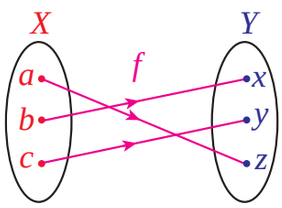


Figure 1.41

If we look this function in a mirror, we get a function from Y to X. Let us call that function as g. Then g is a function from Y to X defined by $g(x) = b, g(y) = c, g(z) = a$.

This function g is an example for the inverse of f. Now we define the inverse of a function.

Definition 1.9

Let $f : X \to Y$ be a bijection. The function $g : Y \to X$ defined by $g(y) = x$ if $f(x) = y$, is called the ***inverse*** of f and is denoted by f^{-1}.

If a function f has an inverse, then we say that f is ***invertible***. There is a nice relationship between composition of functions and inverse.

Let $f : X \to Y$ be a bijection and $g : Y \to X$ be its inverse. Then $g \circ f = I_X$ and $f \circ g = I_Y$ where I_X and I_Y are identity functions on X and Y respectively. Moreover, if $f : X \to Y$ and $g : Y \to X$ are functions such that $g \circ f = I_X$ and $f \circ g = I_Y$, then both f and g are bijections and they are inverses to each other; that is $f^{-1} = g$ and $g^{-1} = f$.

Using the discussions above, the terms invertible and inverse can be defined in some other way as follows:

Definition 1.10

A function $f : X \to Y$ is said to be ***invertible*** if there exists a function $g : Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$ where I_X and I_Y are identity functions on X and Y respectively. In this case, g is called the inverse of f and g is denoted by f^{-1}.

We may use this concept to prove some functions are bijective.

If f is a bijection, then $f^{-1}(y)$ is nothing but the pre-image of y under f. Let us note that the inverses are defined only for bijections. If f is not one-to-one , then there exists a and b such that $a \neq b$ and $f(a) = f(b)$. Let this value be y. Then we cannot define $f^{-1}(y)$ because both a and b are pre-images of y under f, as f^{-1} cannot assume two different values for y. If f is not onto, then there will be a y in Y without a pre-image. In this case also we cannot assign any value to $f^{-1}(y)$.

For example, if $A = \{1, 2, 3, 4\}$ and $f = \{(1, 2), (2, 4), (3, 1), (4, 3)\}$. Then the range of f is $\{1, 2, 3, 4\}$; the inverse of f is $\{(1, 3), (2, 1), (3, 4), (4, 2)\}$.

Working Rule to Find the Inverse of Functions from $\mathbb{R}$ to $\mathbb{R}$:

Let $f : \mathbb{R} \to \mathbb{R}$ be the given function.

i. write $y = f(x)$;
ii. write x in terms of y;
iii. write $f^{-1}(y) =$ the expression in y.
iv. replace y as x.

Example 1.30 If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 2x - 3$ prove that f is a bijection and find its inverse.

Solution:
Method 1:

One-to-one : Let $f(x) = f(y)$. Then $2x - 3 = 2y - 3$; this implies that $x = y$. That is, $f(x) = f(y)$ implies that $x = y$. Thus f is one-to-one.

Onto : Let $y \in \mathbb{R}$. Let $x = \frac{y+3}{2}$. Then $f(x) = 2(\frac{y+3}{2}) - 3 = y$. Thus f is onto. This also can be proved by saying the following statement. The range of f is $\mathbb{R}$ (how?) which is equal to the co-domain and hence f is onto.

Inverse Let $y = 2x - 3$. Then $y + 3 = 2x$ and hence $x = \dfrac{y+3}{2}$. Thus $f^{-1}(y) = \dfrac{y+3}{2}$. By replacing y as x, we get $f^{-1}(x) = \dfrac{x+3}{2}$.

Method 2:
Let $y = 2x - 3$. Then $x = \dfrac{y+3}{2}$. Let $g(y) = \dfrac{y+3}{2}$.

Now

$$(g \circ f)(x) = g(f(x)) = g(2x-3) = \frac{(2x-3)+3}{2} = x.$$

$$(f \circ g)(y) = f(g(y)) = f\left(\frac{y+3}{2}\right) = 2\left(\frac{y+3}{2}\right) - 3 = y.$$

Thus, $g \circ f = I_X$ and$f \circ g = I_Y$

This implies that f and g are bijections and inverses to each other. Hence f is a bijection and $f^{-1}(y) = \dfrac{y+3}{2}$. Replacing y by x we get, $f^{-1}(x) = \dfrac{x+3}{2}$.

1.6.6 Algebra of Functions

A function whose co-domain is $\mathbb{R}$ or a subset of $\mathbb{R}$ is called a real valued function. We can discuss many more operations on functions if it is real valued.

Let f and g be two real valued functions. Can we define addition of f and g? Naturally we expect the sum of two functions to be a function. The value of $f + g$ at a point x should be related to the values of f and g at x. So to define $f + g$ at a point x, we must know both $f(x)$ and $g(x)$. In other words x must be in the domain of f as well as in the domain of g. And the natural way of defining $f + g$ at x is $f(x) + g(x)$. So if we impose a condition that the domains of f and g to be the same, then we can define $f + g$. In the same way we can define subtraction, multiplication and many more algebraic operations available on the set $\mathbb{R}$ of the real numbers.

Definition 1.11

Let X be any set. Let f and g be real valued functions defined on X. Define, for all $x \in X$

- $(f+g)(x) = f(x) + g(x)$.
- $(f-g)(x) = f(x) - g(x)$.
- $(fg)(x) = f(x)g(x)$.
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$.
- $(cf)(x) = cf(x)$, where c is a real constant.
- $(-f)(x) = -f(x)$.

Note that the domain may be any set, not necessarily a set of numbers. For example if X is a set of students of a class, f and g functions representing the marks obtained by the students in two tests, then the function $f + g$ represent the total marks of the students in the two tests. It is easy to see that the operations addition, subtraction, multiplication and division defined above satisfy the following properties.

- $(f+g)+h = f+(g+h)$
- $f+g = g+f$
- $\mathbf{0} + f = f + \mathbf{0}$, where $\mathbf{0}$ is the zero function defined by $\mathbf{0}(x) = 0$ for all x.
- $f + (-f) = (-f) + f = \mathbf{0}$
- $f(g+h) = fg + fh$
- $(c_1 + c_2)f = c_1 f + c_2 f$ where c_1 and c_2 are real constants.

We can list many more properties of these operations. The proofs are simple; however let us prove only one to show a way in which these properties can be proved.

Let us prove $f(g+h) = fg + fh$. To prove $f(g+h) = fg + fh$ we have to prove that $(f(g+h))(x) = (fg + fh)(x)$ for all x in the domain.

Theorem 1.3: If f and g are real-valued functions, then $f(g+h) = fg + fh$.

Proof. Let X be any set and f and g be real-valued functions defined on X. Let $x \in X$.

$$\begin{array}{rcll} (f(g+h))(x) & = & f(x)(g+h)(x) & \text{(by the definition of product)} \\ & = & f(x)[g(x)+h(x)] & \text{(by the definition of addition)} \\ & = & f(x)g(x) + f(x)h(x) & \text{(by the distributivity of reals)} \\ & = & (fg)(x) + (fh)(x) & \text{(by the definition of product)} \\ & = & (fg+fh)(x) & \text{(by the definition of addition)} \end{array}$$

Thus $(f(g+h))(x) = (fg+fh)(x)$ for all $x \in X$; hence $f(g+h) = fg + fh$. ■

1.6.7 Some Special Functions

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Now let us see some special functions.

(i) The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n,$$

where a_i are constants, is called a ***polynomial function***. Since the right hand side of the equality defining the function is a polynomial, this function is called a polynomial function.

(ii) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$ where $a \neq 0$ and b are constants, is called a ***linear function***. A function which is not linear is called a ***non-linear function***.

Clearly a linear function is a polynomial function. The graph of this function is a straight line; a straight line is called a linear curve; so this function is called a linear function. (one may come across different definitions for linear functions in higher study of mathematics.)

(iii) Let a be a non-negative constant. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a^x$. If $a = 0, x \neq 0$ then the function becomes the zero function and if $a = 1$, then function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a^x$ is the constant function $f(x) = 1$. [See, Figures 1.42 and 1.43].

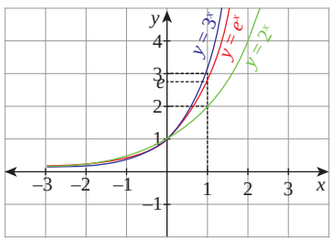


Figure 1.42

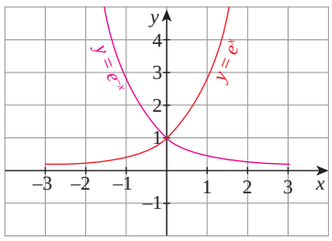


Figure 1.43

When $a > 1$, the function $f(x) = a^x$ is called an ***exponential function***. Moreover, any function having x in the "power" is called as an exponential function.

e is a special irrational number lies between 2 and 3. We will study more about e in the subsequent chapters.

(iv) Let $a > 1$ be a constant. The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \log_a x$ is called a ***logarithmic function***. In fact, the inverse of an exponential function $f(x) = a^x$ on a suitable domain is called a logarithmic function. [See, Figure 1.44].

(v) The real valued function f defined by $f(x) = \frac{p(x)}{q(x)}$ on a suitable domain, where $p(x)$ and $q(x)$ are polynomials, $q(x) \neq 0$, is called a ***rational function***. In fact, the domain of this function is the set obtained from $\mathbb{R}$ by removing the real numbers at which $q(x) = 0$.

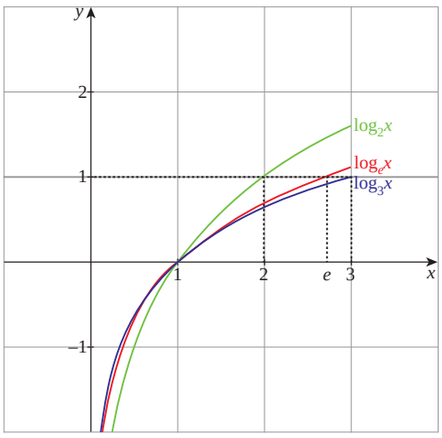


Figure 1.44

(vi) If f is a real valued function such that $f(x) \neq 0$, then the real valued function g defined by $g(x) = \frac{1}{f(x)}$ on a suitable domain is called the ***reciprocal function*** of f. The domain of g is the set obtained from $\mathbb{R}$ by removing the real numbers at which $f(x) = 0$. For example, the largest possible domain of $f(x) = \frac{1}{x-1}$ is $\mathbb{R} - \{1\}$.

Let us see two more categories of functions.

Definition 1.12

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be an ***odd function*** if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. It is said to be an ***even function*** if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. [See, Figures 1.45 and 1.46].

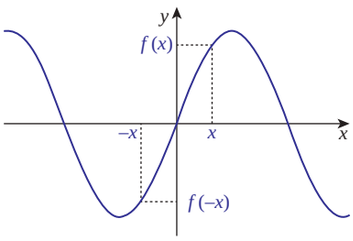


Figure 1.45

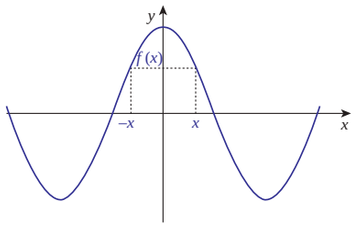


Figure 1.46

The function defined by $f(x) = x$, $f(x) = 2x$ and $f(x) = x^3 + 2x$ are some examples for odd functions. The functions defined by $f(x) = x^2$, $f(x) = 3$, $f(x) = x^4 + x^2$ and $f(x) = |x|$ are some examples for even functions. Note that the function $f(x) = x + x^2$ is neither even nor odd.

We can prove the following results.

(i) The sum of two odd functions is an odd function.
(ii) The sum of two even functions is an even function
(iii) The product of two odd functions is an even function.
(iv) The product of two even functions is an even function.
(v) The product of an odd function and an even function is an odd function.
(vi) The only function which is both odd and even function is the zero function.
(vii) The product of a positive constant and an even function is an even function.
(viii) The product of a negative constant and an even function is also an even function.
(ix) The product of a constant and an odd function is an odd function.
(x) There are functions which are neither odd nor even.

Let us prove one of the above properties. The other properties can be proved similarly.

Theorem 1.4: The product of an odd function and an even function is an odd function.

Proof. Let f be an odd function and g be an even function. Let $h = fg$. Now
$h(-x) = (fg)(-x) = f(-x)g(-x) = -f(x)g(x)$ (as f is odd and g is even)
$= -h(x)$

Thus h is an odd function. This shows that fg is an odd function. ■

If one function is not odd then don't think that the function is an even function. There are plenty of functions which are neither even nor odd.

Exercise - 1.3

1. Suppose that 120 students are studying in 4 sections of eleventh standard in a school. Let A denote the set of students and B denote the set of the sections. Define a relation from A to B as "*x related to y if the student x belongs to the section y*". Is this relation a function? What can you say about the inverse relation? Explain your answer.
2. Write the values of f at $-4, 1, -2, 7, 0$ if

$$f(x) = \begin{cases} -x+4 & \text{if } -\infty < x \leq -3 \\ x+4 & \text{if } -3 < x < -2 \\ x^2 - x & \text{if } -2 \leq x < 1 \\ x - x^2 & \text{if } 1 \leq x < 7 \\ 0 & \text{otherwise} \end{cases}$$

3. Write the values of f at $-3, 5, 2, -1, 0$ if

$$f(x) = \begin{cases} x^2 + x - 5 & \text{if } x \in (-\infty, 0) \\ x^2 + 3x - 2 & \text{if } x \in (3, \infty) \\ x^2 & \text{if } x \in (0, 2) \\ x^2 - 3 & \text{otherwise} \end{cases}$$

4. State whether the following relations are functions or not. If it is a function check for one-to-oneness and ontoness. If it is not a function, state why?
   (i) If $A = \{a, b, c\}$ and $f = \{(a, c), (b, c), (c, b)\}; (f : A \to A)$.
   (ii) If $X = \{x, y, z\}$ and $f = \{(x, y), (x, z), (z, x)\}; (f : X \to X)$.
5. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. Give a function from $A \to B$ for each of the following:
   (i) neither one-to-one nor onto. (ii) not one-to-one but onto.
   (iii) one-to-one but not onto. (iv) one-to-one and onto.
6. Find the domain of $\dfrac{1}{1 - 2\sin x}$.
7. Find the largest possible domain of the real valued function $f(x) = \dfrac{\sqrt{4 - x^2}}{\sqrt{x^2 - 9}}$.
8. Find the range of the function $\dfrac{1}{2\cos x - 1}$.
9. Show that the relation $xy = -2$ is a function for a suitable domain. Find the domain and the range of the function.
10. If $f, g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = |x| + x$ and $g(x) = |x| - x$, find $g \circ f$ and $f \circ g$.

11. If f, g, h are real valued functions defined on $\mathbb{R}$, then prove that $(f+g)\circ h = f\circ h + g\circ h$. What can you say about $f\circ(g+h)$? Justify your answer.
12. If$f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x)=3x-5$, prove that f is a bijection and find its inverse.
13. The weight of the muscles of a man is a function of his body weight x and can be expressed as $W(x)=0.35x$. Determine the domain of this function.
14. The distance of an object falling is a function of time t and can be expressed as $s(t)=-16t^2$. Graph the function and determine if it is one-to-one.
15. The total cost of airfare on a given route is comprised of the base cost C and the fuel surcharge S in rupee. Both C and S are functions of the mileage m; $C(m)=0.4m+50$ and $S(m)=0.03m$. Determine a function for the total cost of a ticket in terms of the mileage and find the airfare for flying 1600 miles.
16. A salesperson whose annual earnings can be represented by the function $A(x)=30,000+0.04x$, where x is the rupee value of the merchandise he sells. His son is also in sales and his earnings are represented by the function $S(x)=25,000+0.05x$. Find $(A+S)(x)$ and determine the total family income if they each sell Rupees $1,50,00,000$ worth of merchandise.
17. The function for exchanging American dollars for Singapore Dollar on a given day is $f(x)=1.23x$, where x represents the number of American dollars. On the same day the function for exchanging Singapore Dollar to Indian Rupee is $g(y)=50.50y$, where y represents the number of Singapore dollars. Write a function which will give the exchange rate of American dollars in terms of Indian rupee.
18. The owner of a small restaurant can prepare a particular meal at a cost of Rupees 100. He estimates that if the menu price of the meal is x rupees, then the number of customers who will order that meal at that price in an evening is given by the function $D(x)=200-x$. Express his day revenue, total cost and profit on this meal as functions of x.
19. The formula for converting from Fahrenheit to Celsius temperatures is $y=\dfrac{5x}{9}-\dfrac{160}{9}$. Find the inverse of this function and determine whether the inverse is also a function.
20. A simple cipher takes a number and codes it, using the function $f(x)=3x-4$. Find the inverse of this function, determine whether the inverse is also a function and verify the symmetrical property about the line $y=x$ (by drawing the lines).

1.7 Graphing Functions using Transformations

"*A picture is worth a thousand words*" is a well known proverb. To know about a function well, its graph will help us more than its analytical expression. To draw graphs quickly without plotting many points is an invaluable skill. Familiarity with shapes of some basic functions will help to graph other complicated functions. Understanding and usage of symmetry and transformations will then enable to strengthen graphing abilities. This section is not simply a data base of graphs, we learn some methods to graph certain functions.

Suppose that we want to draw or sketch the curve of the function $y=2\sin(x-1)+3$. At the very first sight it looks that it is very difficult to draw the curve representing this function. But it will be very easy to draw after understanding the content of this section.

If we know a half of a graph is the mirror image of the other half with respective to a line, or a graph can be obtained just by moving a known graph in some direction, then we can draw the new one using the known one. Moreover if we know that a graph can be obtained by enlarging or shrinking a known one, then also we can draw the new one using the known ones.

The following type of transformations play very important roles in graphing.

(i) Reflection
(ii) Translation
(iii) Dilations.

In the case of reflections and translations, they produce graphs congruent to the original graph; that is, the size and the shape of the graph does not change, but in dilation, it produce graphs with shapes related to those of the original graph.

Reflection

The ***reflection*** of the graph of a function with respect to a line ℓ is the graph that is symmetric to it with respect to ℓ. A reflection is the mirror image of the graph where line ℓ is the mirror of the reflection. (See Figure 1.47.)

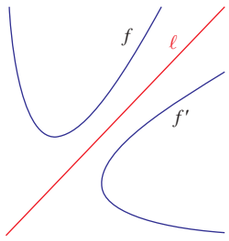


Figure 1.47

Here f' is the mirror image of f with respect to ℓ. Every point of f has a corresponding image in f'. Some useful reflections of $y = f(x)$ are

(i) The graph $y = -f(x)$ is the reflection of the graph of f about the x-axis.
(ii) The graph $y = f(-x)$ is the reflection of the graph of f about the y-axis.
(iii) The graph of $y = f^{-1}(x)$ is the reflection of the graph of f in $y = x$.

Illustration 1.5 Consider the functions:

$$(i)\ \ y = x^2 \quad (ii)\ \ y = -x^2.$$

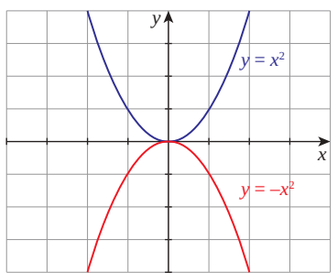


Figure 1.48

For the curve $f(x) = x^2$, $-f(x) = -x^2$. Hence, $y = -x^2$ is the reflection of $y = x^2$ about x-axis. (See Figure 1.48.)

Illustration 1.6 Consider the positive branches of

$$y^2 = x \text{ and } y^2 = -x.$$

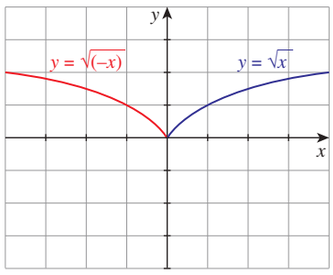


Figure 1.49

For the curve $f(x) = \sqrt{x}$, we have $f(-x) = \sqrt{-x}$ and hence $f(-x) = \sqrt{-x}$ where $x < 0$, is the reflection of $f(x) = \sqrt{x}$ about y-axis. (See Figure 1.49.)

Illustration 1.7 Consider the functions:

$$\text{(i) } y = e^x \quad \text{(ii) } y = \log_e x.$$

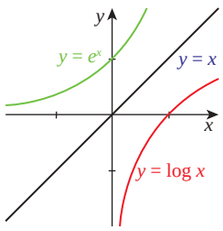


Figure 1.50

We know that, $y = e^x$ is the inverse function of $y = \log_e x$ and hence $y = e^x$ is the reflection of $y = \log_e x$ about $y = x$. (See Figure 1.50.)

Translation

A ***translation*** of a graph is a vertical or horizontal shift of the graph that produces congruent graphs.

The graph of	$y = f(x + c), c > 0$	causes the shift to the left.
	$y = f(x - c), c > 0$	causes the shift to the right.
	$y = f(x) + d, d > 0$	causes the shift to the upward.
	$y = f(x) - d, d > 0$	causes the shift to the downward.

Illustration 1.8 Consider the functions:

$$\text{(i) } f(x) = |x| \text{ (ii) } f(x) = |x - 1| \text{ (iii) } f(x) = |x + 1|$$

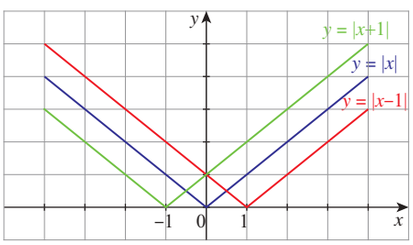


Figure 1.51

$f(x) = |x - 1|$ causes the graph of the function $f(x) = |x|$ shifts to the right for one unit.

$f(x) = |x + 1|$ causes the graph of the function $f(x) = |x|$ shifts to the left for one unit. (See Figure 1.51.)

Illustration 1.9 Consider the functions:

$$\text{(i) } f(x) = |x| \text{ (ii) } f(x) = |x| - 1 \text{ (iii) } f(x) = |x| + 1$$

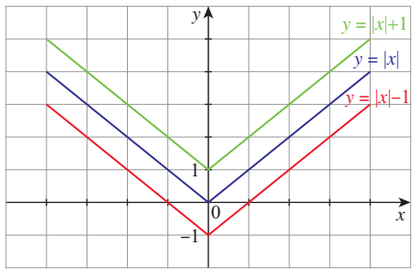


Figure 1.52

$f(x) = |x| - 1$ causes the graph of the function $f(x) = |x|$ shifts to the downward for one unit.

$f(x) = |x| + 1$ causes the graph of the function $f(x) = |x|$ shifts to the upward for one unit. (See Figure 1.52.)

Dilation

Dilation is also a transformation which causes the curve stretches (expands) or compresses (contracts). Multiplying a function by a positive constant vertically stretches or compresses its graph; that is, the graph moves away from x-axis or towards x-axis.

If the positive constant is greater than one, the graph moves away from the x-axis. If the positive constant is less than one, the graph moves towards the x-axis.

Illustration 1.10 Consider the functions:

$$\text{(i) } f(x) = x^2 \text{ (ii) } f(x) = \tfrac{1}{2}x^2 \text{ (iii) } f(x) = 2x^2$$

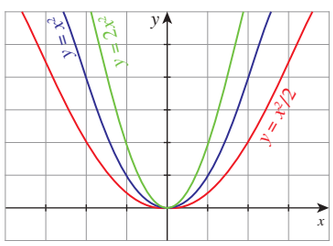


Figure 1.53

$f(x) = \frac{1}{2}x^2$ causes the graph of the function $f(x) = x^2$ stretches towards the x-axis since the multiplying factor is $\frac{1}{2}$ which is less than one.

$f(x) = 2x^2$ causes the graph of the function $f(x) = x^2$ compresses towards the $y-$ axis that is, moves away from the $x-$axis since the multiplying factor is 2 which is greater than one. (See Figure 1.53.)

Illustration 1.11 Consider the functions:

(i) $f(x) = x^2$ (ii) $f(x) = x^2 + 1$ (iii) $f(x) = (x+1)^2$.

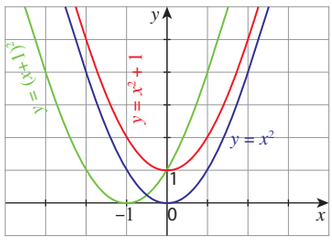


Figure 1.54

$f(x) = x^2 + 1$ causes the graph of the function $f(x) = x^2$ shifts to the upward for one unit.

$f(x) = (x+1)^2$ causes the graph of the function $f(x) = x^2$ shifts to the left for one unit. (See Figure 1.54.)

Illustration 1.12 Compare and contrast the graphs $y = x^2 - 1, y = 4(x^2 - 1)$ and $y = (4x)^2 - 1$

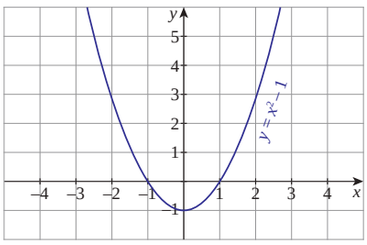


Figure 1.55

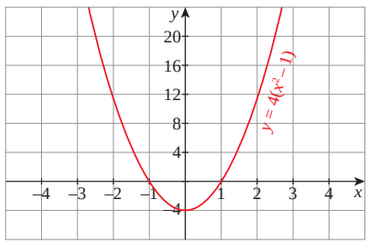


Figure 1.56

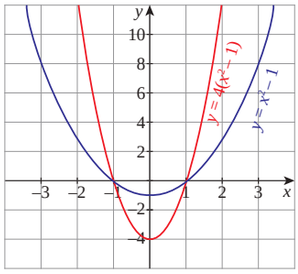


Figure 1.57

The graphs Figures 1.55 and 1.56 look identical until we compare the scales on the y-axis. The scale in Figure 1.56 is four times as large, reflecting the multiplication of the original function by 4 (Figure 1.55). The effect looks different when the functions are plotted on the same scale as in Figure 1.57.

The graph of $y = (4x)^2 - 1$ is shown in Figure 1.58. Can you spot the difference between Figure 1.55 and Figure 1.58? In this case, x-scale has now changed, by the same factor of 4 as in the function (Figure 1.58). To see this, note that substituting $x = \frac{1}{4}$ into $(4x)^2 - 1$ produces $1^2 - 1$, exactly the same as substituting $x = 1$ into the original function (Figure 1.55). When plotted on the same set of axes (as in Figure 1.59) the parabola $y = (4x)^2 - 1$ looks thinner. Here, the x-intercepts are different, but y-intercepts are the same.

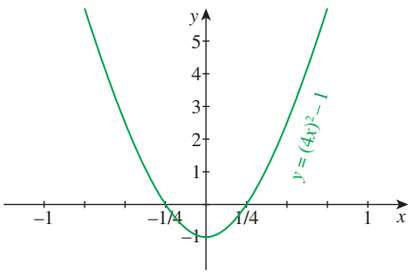


Figure 1.58

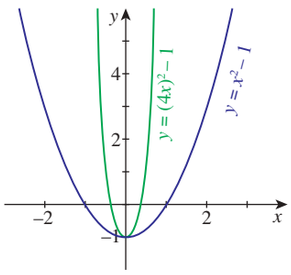


Figure 1.59

Illustration 1.13 By using the same concept applied in Illustration 1.12, graphs of $y = \sin x$ and $y = \sin 2x$, and also their combined graphs are given Figures 1.60, 1.61 and 1.62. The minimum and maximum values of $\sin x$ and $\sin 2x$ are the same. But they have different x-intercepts. The x-intercepts for $y = \sin x$ are $\pm n\pi$ and for $y = \sin 2x$ are $\pm\frac{1}{2}n\pi, n \in \mathbb{Z}$.

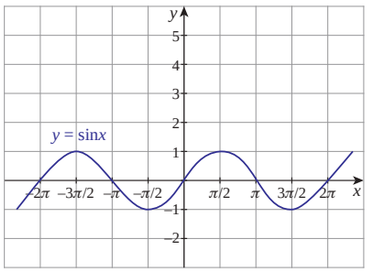


Figure 1.60

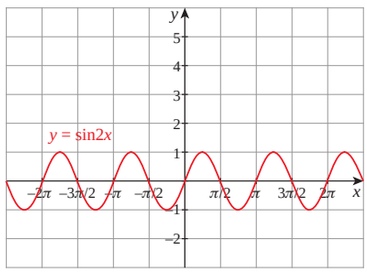


Figure 1.61

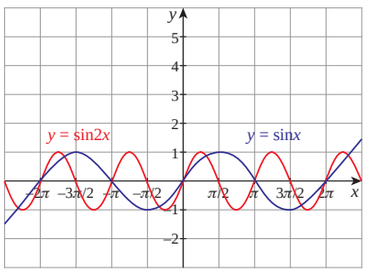


Figure 1.62

In the beginning of the section we talked about drawing the graph of $y = 2\sin(x-1)+3$. Now we are well equipped to draw the curve and even we can draw more complicated curve.

Illustration 1.14 Let us now draw the graph of $y = 2\sin(x-1)+3$.

It is clear that the curve can be obtained from that of $y = \sin x$ using translation and dilation.

So first we draw $y = \sin x$. From that it is easy to draw the curve $y = \sin(x-1)$; then draw $y = 2\sin(x-1)$ and finally $y = 2\sin(x-1)+3$. (See Figures 1.63 to 1.66.)

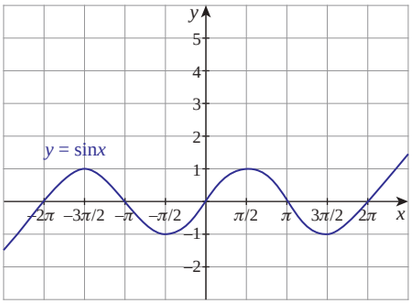


Figure 1.63

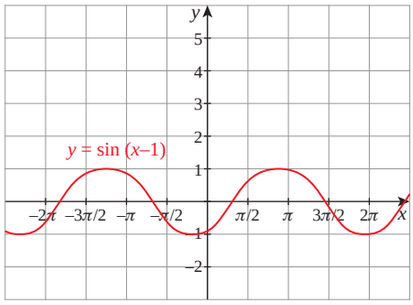


Figure 1.64

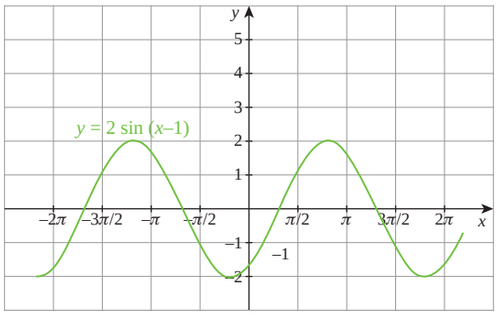


Figure 1.65

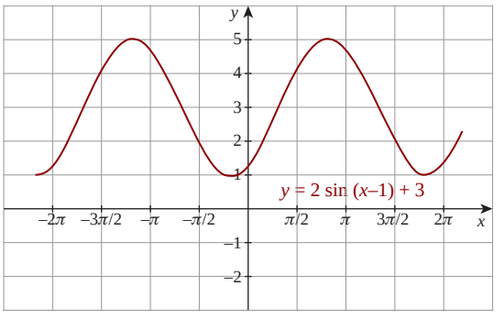


Figure 1.66

Exercise - 1.4

1. For the curve $y = x^3$ given in Figure 1.67, draw

   (i) $y = -x^3$ (ii) $y = x^3 + 1$ (iii) $y = x^3 - 1$ (iv) $y = (x+1)^3$ with the same scale.

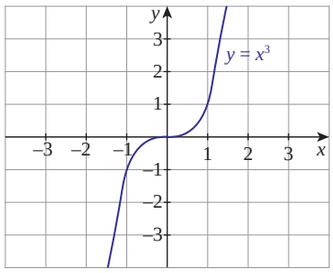


Figure 1.67

2. For the curve $y = x^{(\frac{1}{3})}$ given in Figure 1.68, draw

   (i) $y = -x^{(\frac{1}{3})}$ (ii) $y = x^{(\frac{1}{3})} + 1$ (iii) $y = x^{(\frac{1}{3})} - 1$ (iv) $y = (x+1)^{(\frac{1}{3})}$

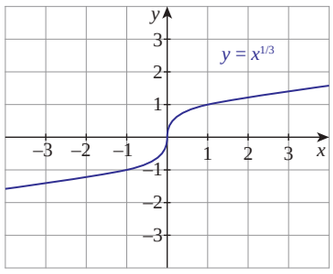


Figure 1.68

3. Graph the functions $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ on the same coordinate plane. Find $f \circ g$ and graph it on the plane as well. Explain your results.
4. Write the steps to obtain the graph of the function $y = 3(x-1)^2 + 5$ from the graph $y = x^2$.
5. From the curve $y = \sin x$, graph the functions

   (i) $y = \sin(-x)$ (ii) $y = -\sin(-x)$ (iii) $y = \sin\left(\frac{\pi}{2} + x\right)$ which is $\cos x$

   (iv) $y = \sin\left(\frac{\pi}{2} - x\right)$ which is also $\cos x$ (refer trigonometry)
6. From the curve $y = x$, draw

   (i) $y = -x$ (ii) $y = 2x$ (iii) $y = x + 1$ (iv) $y = \frac{1}{2}x + 1$ (v) $2x + y + 3 = 0$.
7. From the curve $y = |x|$, draw (i) $y = |x-1| + 1$ (ii) $y = |x+1| - 1$ (iii) $y = |x+2| - 3$.
8. From the curve $y = \sin x$, draw $y = \sin|x|$ (Hint: $\sin(-x) = -\sin x$.)

Activities

Balls and Runs

What a Celebration! What a Relation!! What a Function!!!

BATTING BLASTERCLASS

Rohit's Joint Fastest T20 Ton Off 35 Balls Sinks Lanka

91.52

10

Source: Times of India

1. Interpret the above data as,
   (i) a relation
   (ii) a function
   (iii) an onto function
   (iv) can you make the data as an one-to-one function? If not, why?
2. Identify the curves in Figure 1.69 and the corresponding equations for the base curve $y = x^2$ (graph with dotted line) by seeing the scale.

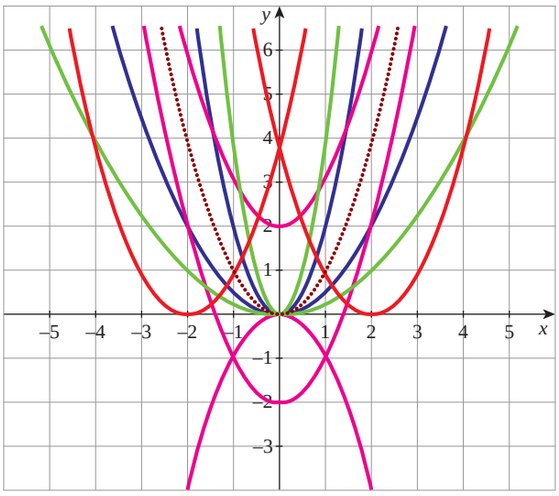


Figure 1.69

Exercise - 1.5

Choose the correct or the most suitable answer.

1. If $A = \{(x, y) : y = e^x, x \in R\}$ and $B = \{(x, y) : y = e^{-x}, x \in R\}$ then $n(A \cap B)$ is

   (1) Infinity (2) 0 (3) 1 (4) 2

2. If $A = \{(x, y) : y = \sin x, x \in R\}$ and $B = \{(x, y) : y = \cos x, x \in R\}$ then $A \cap B$ contains

   (1) no element (2) infinitely many elements
   (3) only one element (4) cannot be determined.

3. The relation R defined on a set $A = \{0, -1, 1, 2\}$ by xRy if $|x^2 + y^2| \leq 2$, then which one of the following is true?

   (1) $R = \{(0, 0), (0, -1), (0, 1), (-1, 0), (-1, 1), (1, 2), (1, 0)\}$
   (2) $R^{-1} = \{(0, 0), (0, -1), (0, 1), (-1, 0), (1, 0)\}$
   (3) Domain of R is $\{0, -1, 1, 2\}$
   (4) Range of R is $\{0, -1, 1\}$

4. If $f(x) = |x - 2| + |x + 2|, x \in \mathbb{R}$, then

   (1) $f(x) = \begin{cases} -2x & \text{if} \quad x \in (-\infty, -2] \\ 4 & \text{if} \quad x \in (-2, 2] \\ 2x & \text{if} \quad x \in (2, \infty) \end{cases}$

   (2) $f(x) = \begin{cases} 2x & \text{if} \quad x \in (-\infty, -2] \\ 4x & \text{if} \quad x \in (-2, 2] \\ -2x & \text{if} \quad x \in (2, \infty) \end{cases}$

   (3) $f(x) = \begin{cases} -2x & \text{if} \quad x \in (-\infty, -2] \\ -4x & \text{if} \quad x \in (-2, 2] \\ 2x & \text{if} \quad x \in (2, \infty) \end{cases}$

   (4) $f(x) = \begin{cases} -2x & \text{if} \quad x \in (-\infty, -2] \\ 2x & \text{if} \quad x \in (-2, 2] \\ 2x & \text{if} \quad x \in (2, \infty) \end{cases}$

5. Let $\mathbb{R}$ be the set of all real numbers. Consider the following subsets of the plane $\mathbb{R} \times \mathbb{R}$:

   $S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$ and $T = \{(x, y) : x - y \text{ is an integer }\}$

   Then which of the following is true?

   (1) T is an equivalence relation but S is not an equivalence relation.
   (2) Neither S nor T is an equivalence relation
   (3) Both S and T are equivalence relation
   (4) S is an equivalence relation but T is not an equivalence relation.

6. Let A and B be subsets of the universal set $\mathbb{N}$, the set of natural numbers. Then $A' \cup [(A \cap B) \cup B']$ is

   (1) A (2) A' (3) B (4) $\mathbb{N}$

7. The number of students who take both the subjects Mathematics and Chemistry is 70. This represents 10% of the enrollment in Mathematics and 14% of the enrollment in Chemistry. The number of students take at least one of these two subjects, is

   (1) 1120 (2) 1130 (3) 1100 (4) insufficient data

8. If $n((A \times B) \cap (A \times C)) = 8$ and $n(B \cap C) = 2$, then $n(A)$ is
   (1) 6 (2) 4 (3) 8 (4) 16

9. If $n(A) = 2$ and $n(B \cup C) = 3$, then $n[(A \times B) \cup (A \times C)]$ is
   (1) 2^3 (2) 3^2 (3) 6 (4) 5

10. If two sets A and B have 17 elements in common, then the number of elements common to the set $A \times B$ and $B \times A$ is
   (1) 2^{17} (2) 17^2 (3) 34 (4) insufficient data

11. For non-empty sets A and B, if $A \subset B$ then $(A \times B) \cap (B \times A)$ is equal to
   (1) $A \cap B$ (2) $A \times A$ (3) $B \times B$ (4) none of these.

12. The number of relations on a set containing 3 elements is
   (1) 9 (2) 81 (3) 512 (4) 1024

13. Let R be the universal relation on a set X with more than one element. Then R is
   (1) not reflexive (2) not symmetric (3) transitive (4) none of the above

14. Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (2, 1), (3, 1), (1, 4), (4, 1)\}$. Then R is
   (1) reflexive (2) symmetric (3) transitive (4) equivalence

15. The range of the function $\frac{1}{1-2\sin x}$ is
   (1) $(-\infty, -1) \cup (\frac{1}{3}, \infty)$ (2) $(-1, \frac{1}{3})$ (3) $[-1, \frac{1}{3}]$ (4) $(-\infty, -1] \cup [\frac{1}{3}, \infty)$.

16. The range of the function $f(x) = |\lfloor x \rfloor - x|, x \in \mathbb{R}$ is
   (1) $[0, 1]$ (2) $[0, \infty)$ (3) $[0, 1)$ (4) $(0, 1)$

17. The rule $f(x) = x^2$ is a bijection if the domain and the co-domain are given by
   (1) $\mathbb{R}, \mathbb{R}$ (2) $\mathbb{R}, (0, \infty)$ (3) $(0, \infty), \mathbb{R}$ (4) $[0, \infty), [0, \infty)$

18. The number of constant functions from a set containing m elements to a set containing n elements is
   (1) mn (2) m (3) n (4) $m + n$

19. The function $f : [0, 2\pi] \to [-1, 1]$ defined by $f(x) = \sin x$ is
   (1) one-to-one (2) onto (3) bijection (4) cannot be defined

20. If the function $f : [-3, 3] \to S$ defined by $f(x) = x^2$ is onto, then S is
   (1) $[-9, 9]$ (2) $\mathbb{R}$ (3) $[-3, 3]$ (4) $[0, 9]$

21. Let $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d\}$ and $f = \{(1, a), (4, b), (2, c), (3, d), (2, d)\}$. Then f is
   (1) an one-to-one function (2) an onto function
   (3) a function which is not one-to-one (4) not a function

22. The inverse of $f(x) = \begin{cases} x & \text{if} \quad x < 1 \\ x^2 & \text{if} \quad 1 \le x \le 4 \\ 8\sqrt{x} & \text{if} \quad x > 4 \end{cases}$ is

(1) $f^{-1}(x) = \begin{cases} x & \text{if} \quad x < 1 \\ \sqrt{x} & \text{if} \quad 1 \le x \le 16 \\ \frac{x^2}{64} & \text{if} \quad x > 16 \end{cases}$

(2) $f^{-1}(x) = \begin{cases} -x & \text{if} \quad x < 1 \\ \sqrt{x} & \text{if} \quad 1 \le x \le 16 \\ \frac{x^2}{64} & \text{if} \quad x > 16 \end{cases}$

(3) $f^{-1}(x) = \begin{cases} x^2 & \text{if} \quad x < 1 \\ \sqrt{x} & \text{if} \quad 1 \le x \le 16 \\ \frac{x^2}{64} & \text{if} \quad x > 16 \end{cases}$

(4) $f^{-1}(x) = \begin{cases} 2x & \text{if} \quad x < 1 \\ \sqrt{x} & \text{if} \quad 1 \le x \le 16 \\ \frac{x^2}{8} & \text{if} \quad x > 16 \end{cases}$

23. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 - |x|$. Then the range of f is

(1) $\mathbb{R}$ (2) $(1, \infty)$ (3) $(-1, \infty)$ (4) $(-\infty, 1]$

24. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \sin x + \cos x$ is

(1) an odd function
(2) neither an odd function nor an even function
(3) an even function
(4) both odd function and even function.

25. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \frac{(x^2 + \cos x)(1 + x^4)}{(x - \sin x)(2x - x^3)} + e^{-|x|}$$

is (1) an odd function (2) neither an odd function nor an even function
(3) an even function (4) both odd function and even function.

Summary

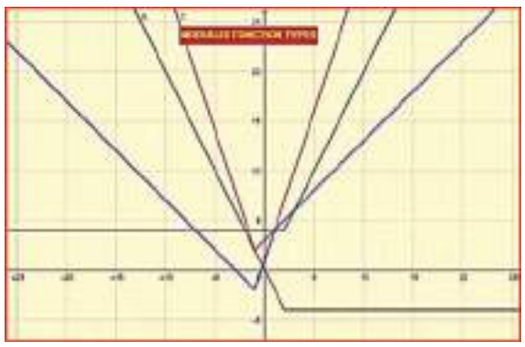
In this chapter we have acquired the knowledge of

- Set
  - Subset, super set, trivial subset, proper subset, improper subset
  - Empty set, power set, universal set, singleton set, finite set, infinite set
  - Cardinality of a set
  - Union, Intersection, Complement, Set Difference, Symmetric Difference
  - Properties and De Morgan Laws
  - Cartesian Product
- Intervals
  - Constants, dependent and independent variables
  - Open, Closed, finite and infinite intervals and neighbourhoods;

- Relations
  - Domain and range of relation
  - Extreme relations (empty and universal)
  - Inverse of a relation
  - Reflexive, Symmetric, Transitive, Equivalence Relations
- Functions
  - Definition, domain, co-domain, range, image, pre-image,
  - Tabular, graphical, analytical and piecewise representations,
  - Identity function, constant function, zero function, modulus function, signum function, greatest integer function, smallest integer function,
  - Injective, surjective and bijective functions,
  - Vertical test and Horizontal test,
  - Composition of functions, inverse of a function,
  - Addition and multiplication of real valued functions,
  - Polynomial function, linear function, exponential function, logarithmic function, rational function, reciprocal function,
  - Odd and Even functions.
- Graphing functions
  - Reflection, translation, dilation
  - Drawing graph of some seems to be complicated functions.

ICT CORNER-1(a)

Expected Outcome ⇒

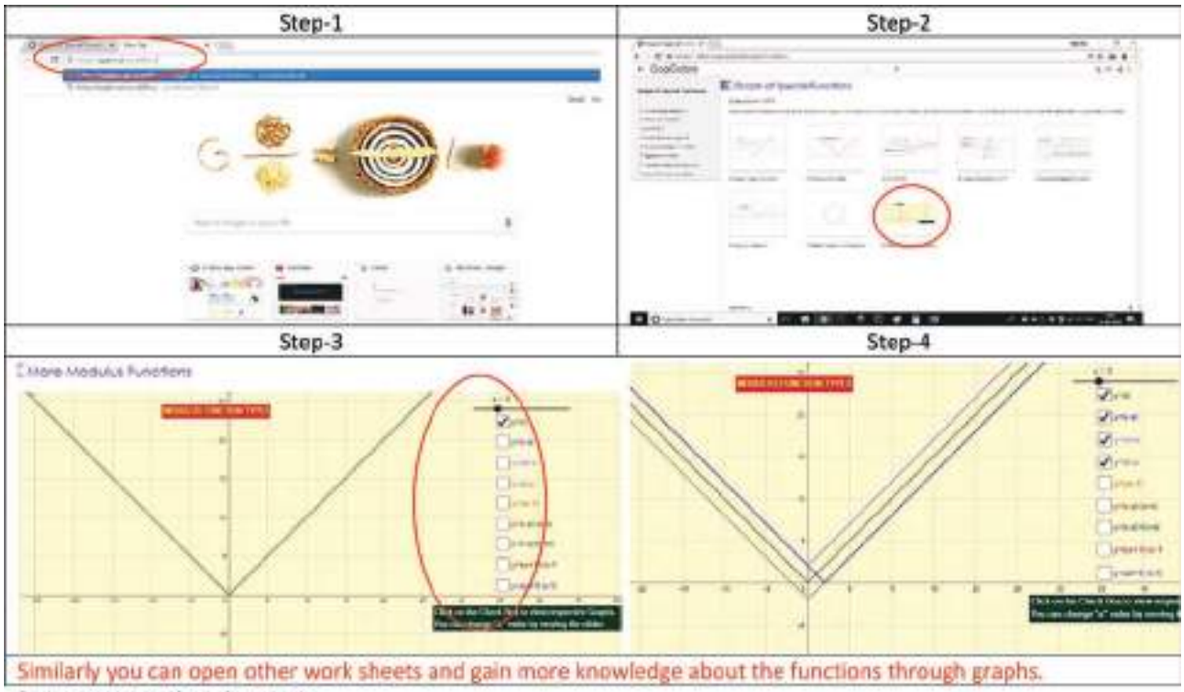


Step–1
Open the Browser and type the URL Link given below (or) Scan the QR Code.

Step–2
A workbook named “Graph of Special Functions” will open. 7 worksheets are given in this workbook related to Functions and Graph. Select a worksheet named “More Modulus Functions”

Step–3
Right side of the work sheet There are check boxes for many modulus functions

Step–4
You can click on any check box to view the Graph. Now Move the Slider a to change the function and observe.

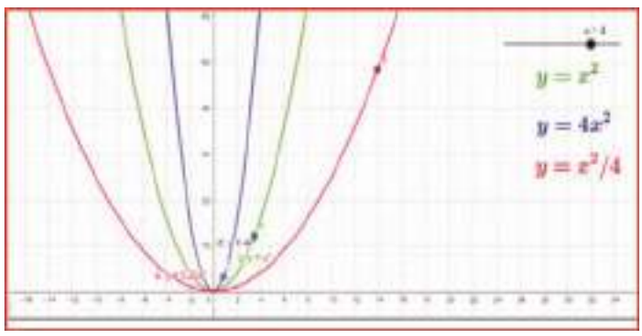


*Pictures are only indicatives.

Browse in the link Graph of Special Functions: https://ggbm.at/ucz465auor Scan the QR Code.

ICT CORNER-1(b)

Expected Outcome ⇒



Step–1

Open the Browser and type the URL Link given below (or) Scan the QR Code.

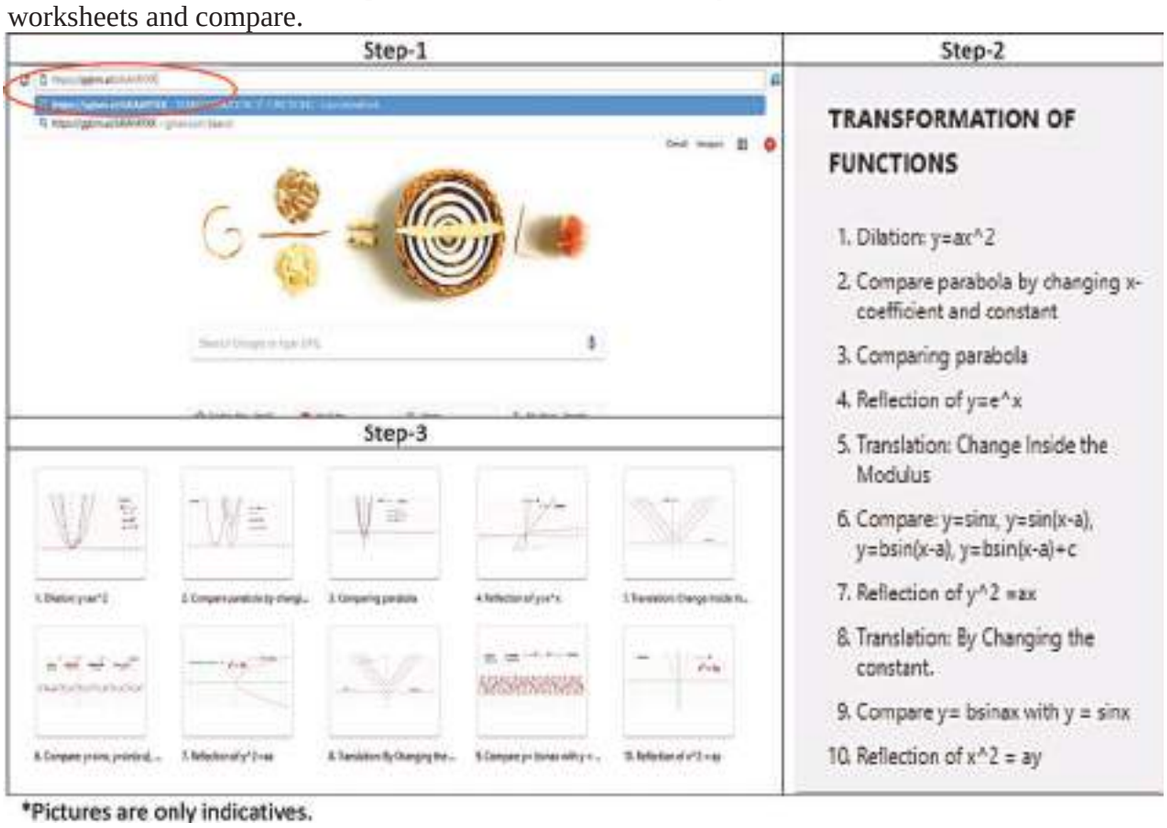
Step–2

GeoGebra Workbook called "**TRANSFORMATION OF FUNCTIONS**" will appear. In that there are 10 worksheets related to your lesson.

Step–3

Select the work sheets one-one by one and analyse the transformations 1. Translation, 2. Reflection and 3. Dilation.

Also, there is a worksheet for comparison of Sine functions. All you have to do is move the sliders in the worksheets and compare.



*Pictures are only indicatives.

Browse in the link Transformation of Functions: https://ggbm.at/bKAhRYXR

Chapter 2 Basic Algebra

22JLVA

I see it but I don't believe it.

Richard Dedekind

2.1 Introduction

Algebra is a branch of mathematics in which one expresses relations among quantities by using symbols to represent these quantities. The symbols are called the *variables*. In this class we shall allow the variables to represent real numbers only. One can carry out manipulations and computations using variables just as one does with numbers. That is, one may substitute real numbers for the variables in the expression and the resulting value will also be a real number. Once a quantity or a mathematical statement is expressed in terms of variables, it is possible to substitute specific numerical values for those variables. This makes algebra a very powerful tool. For this reason the subject of algebra has very wide application, not only within mathematics, but also in other disciplines and in real life. The notion of real numbers is fundamental to the whole of mathematics. The real number system was well understood only in the nineteenth century. The need for extending the rational numbers arose quite early in the history of mathematics. Pythagoreans knew that $\sqrt{2}$ was not a rational number. Certain constructions involving irrational numbers can be found in Shulbha Sutras, which date back to around 800 BC(BCE). Aryabhata (476-550) had found approximations to π.

Indian mathematicians like Brahmagupta (598-670) and Bhaskaracharya (1114-1185) had made contributions to the understanding of the real numbers system and algebra. In his work Brahmagupta had solved the general quadratic equation for both positive and negative roots. Bhaskaracharya solved quadratic equations with more than one unknown and found negative and irrational solutions. The most important real number **zero** was the contribution by Indians.

Rene Descartes (1596-1650) introduced the term "real" to describe roots of a polynomial distinguishing them from imaginary ones. A rigorous construction of real number system was due to Richard Dedekind (1831-1916).

Richard Dedekind
(1831-1916)

Learning Objectives

On completion of this chapter, the students are expected to know

- the concept of real numbers and their properties.
- the absolute value, polynomials, exponents, radicals, logarithms and functions of one variables involving these concepts.
- how to solve equations, inequalities involving above mentioned functions.
- how to solve linear inequalities involving two variables and representing the solutions graphically in the cartesian plane.

2.2 Real Number System

First we shall recall how the real number system was developed. We start with natural numbers $\mathbb{N}$.

2.2.1 Rational Numbers

Note that $\mathbb{N} = \{1, 2, 3, \ldots\}$ is enough for counting objects. In order to deal with loss or debts, we enlarged $\mathbb{N}$ to the set of all integers, $\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, \ldots\}$, which consists of the natural numbers, zero, and the negatives of natural numbers. We call $\{0, 1, 2, 3, \cdots\}$ as the set of whole numbers and denote it by $\mathbb{W}$. Note that it differs from $\mathbb{N}$ by just one element, namely, zero. Now imagine dividing a cake into five equal parts, which is equivalent to finding a solution of $5x = 1$. But this equation cannot be solved within $\mathbb{Z}$. Hence we have enlarged $\mathbb{Z}$ to the set $\mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \right\}$ of *ratios*; so we call each $x \in \mathbb{Q}$ as a *rational* number. Some examples of rational numbers are

$$-5, \ \frac{-7}{3}, \ 0, \ \frac{22}{7}, \ 7, \ 12.$$

We have seen in earlier classes that rational numbers are precisely the set of terminating or infinite periodic decimals. For example,

$$-5.0, \ -2.333\cdots, \ \frac{25}{99} = 0.252525\cdots, \ \frac{2}{3} = 0.66666\cdots, 7.14527836231231231\cdots$$

are rational numbers.

2.2.2 The Number Line

Let us recall "*The Number Line*". It is a horizontal line with the *origin*, to represent 0, and another point marked to the right of 0 to represent 1. The distance from 0 to 1 defines one unit of length. Now put 2 one unit to the right of 1. Similarly we put any positive rational number x to the right of 0 and x units away. Also, we put a negative rational number $-r, r > 0$, to the left of 0 by r units. Note that for any $x, y \in \mathbb{Q}$ if $x < y$, then x is to the left of y; also $x < \dfrac{x+y}{2} < y$ and hence between any two distinct rational numbers there is another rational number between them.

Question:

Have we filled the whole line with rational numbers?

The answer to the above question is "No" as the following consideration demonstrates. Consider a square whose side has length 1 unit. Then by Pythagoras theorem its diagonal has length $\sqrt{2}$ units.

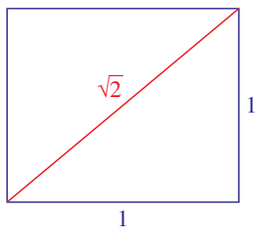


Figure 2.1

2.2.3 Irrational Numbers

Theorem 2.1: $\sqrt{2}$ is not a rational number.

Proof. Suppose that $\sqrt{2}$ is a rational number. Let $\sqrt{2} = \frac{m}{n}$, where m and n are positive integers with no common factors greater than 1. Then, we have $m^2 = 2n^2$, which implies that m^2 is even and hence m is even.

Let $m = 2k$. Then, we have $2n^2 = 4k^2$ which gives $n^2 = 2k^2$.
Thus, n is also even.
It follows, that m and n are even numbers having a common factor 2.
Thus, we arrived at a contradiction.
Hence, $\sqrt{2}$ is an irrational number. ■

Remark:

(i) Note that in the above proof we have assumed the contrary of what we wanted to prove and arrived at a contradiction. This method of proof is called '*proof by contradiction*'.
(ii) There are points on the number line that are not represented by rational numbers.
(iii) We call those numbers on the number line that do not correspond to rational numbers as *irrational numbers*. The set of all irrational numbers is denoted by $\mathbb{Q}'$ (For number line see Figure 1.2.)

Every real number is either a rational number or an irrational number, but not both. Thus, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$ and $\mathbb{Q} \cap \mathbb{Q}' = \emptyset$.

As we already knew that every terminating or infinite periodic decimal is a rational number, we see that the decimal representation of an irrational number will neither be terminating nor infinite periodic. The set $\mathbb{R}$ of real numbers can be visualized as the set of points on the number line such that if $x < y$, then x lies to left of y.

Figure 2.2 demonstrates how the square roots of 2 and 3 can be identified on a number line.

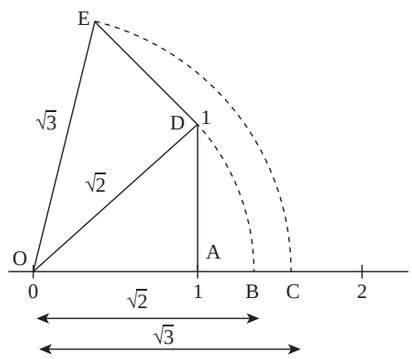


Figure 2.2

We notice that $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

As we have already observed, irrational numbers occur in real life situations. Over 2000 years ago people in the Orient and Egypt observed that the ratio of the circumference to the diameter is the same for any circle. This constant was proved to be an irrational number by Johann Heinrich Lambert in 1767. The value of π rounded off to nine decimal places is equal to 3.141592654. The values $\frac{22}{7}$ and 3.14, used in calculations (such as area of a circle or volume of a sphere) are only approximate values for π.

The number π, which is the ratio of the circumference of a circle to its diameter, is an irrational number.

Now let us recall the properties of the real number system which is the foundation for mathematics.

2.2.4 Properties of Real Numbers

(i) For any $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$ and $ab \in \mathbb{R}$.
[Sum of two real numbers is again a real number and multiplication of two real numbers is again a real number.]

(ii) For any $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$.
[While adding (or multiplying) finite number of real numbers, we can add (or multiply) by grouping them in any order.]

(iii) For all $a \in \mathbb{R}$, $a + 0 = a$ and $a(1) = a$.

(iv) For every $a \in \mathbb{R}$, $a + (-a) = 0$ and for every $b \in \mathbb{R} - \{0\}$, $b\left(\frac{1}{b}\right) = 1$.

(v) For any $a, b \in \mathbb{R}$, $a + b = b + a$ and $ab = ba$.

(vi) For $a, b, c \in \mathbb{R}$, $a(b + c) = ab + ac$.

(vii) For $a, b \in \mathbb{R}$, $a < b$ if and only if $b - a > 0$.

(viii) For any $a \in \mathbb{R}$, $a^2 \geq 0$.

(ix) For any $a, b \in \mathbb{R}$, only one of the following holds: $a = b$ or $a < b$ or $a > b$.

(x) If $a, b \in \mathbb{R}$ and $a < b$, then $a + c < b + c$ for all $c \in \mathbb{R}$.

(xi) If $a, b \in \mathbb{R}$ and $a < b$, then $ax < bx$ for all $x > 0$.

(xii) If $a, b \in \mathbb{R}$ and $a < b$, then $ay > by$ for all $y < 0$.

Exercise - 2.1

1. Classify each element of $\{\sqrt{7}, \frac{-1}{4}, 0, 3.14, 4, \frac{22}{7}\}$ as a member of $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R} - \mathbb{Q}$ or $\mathbb{Z}$.
2. Prove that $\sqrt{3}$ is an irrational number.
(Hint: Follow the method that we have used to prove $\sqrt{2} \notin \mathbb{Q}$.)
3. Are there two distinct irrational numbers such that their difference is a rational number? Justify.
4. Find two irrational numbers such that their sum is a rational number. Can you find two irrational numbers whose product is a rational number.
5. Find a positive number smaller than $\dfrac{1}{2^{1000}}$. Justify.

2.3 Absolute Value

2.3.1 Definition and Properties

As we have observed that there is an order preserving one-to-one correspondence between elements of $\mathbb{R}$ and points on the number line. Note that for each $x \in \mathbb{R}$, x and $-x$ are equal distance from the origin. The distance of the number $a \in \mathbb{R}$ from 0 on the number line is called the *absolute value* of the number a and is denoted by $|a|$. Thus, for any $x \in \mathbb{R}$, we have

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

and hence $|\cdot|$ defines a function known as absolute value function, from $\mathbb{R}$ onto $[0, \infty)$ and the graph of this function is discussed in Chapter 1.

(i) For any $x \in \mathbb{R}$, we have $|x| = |-x|$ and thus, $|x| = |y|$ if and only if $x = y$ or $x = -y$.
(ii) $|x - a| = r$ if and only if $r \geq 0$ and $x - a = r$ or $x - a = -r$.

2.3.2 Equations Involving Absolute Value

Note that *a real number a is said to be a* ***solution*** *of an equation or an inequality, if the statement obtained after replacing the variable by a is true.*

Next we shall learn solving equations involving absolute value.

Example 2.1 Solve $|2x - 17| = 3$ for x.

Solution:
$|2x - 17| = 3$. Then, we have $2x - 17 = \pm 3$ which implies $x = 10$ or $x = 7$.

Example 2.2 Solve $3|x - 2| + 7 = 19$ for x.

Solution:
$3|x - 2| + 7 = 19$. So that we have, $|x - 2| = \frac{19-7}{3} = 4$.
Thus, we have either $x - 2 = 4$ or $x - 2 = -4$.
Therefore the solutions are $x = -2$ and $x = 6$.

Example 2.3 Solve $|2x - 3| = |x - 5|$.

Solution:
We know that $|u| = |v|$ if and only if $u = v$ or $u = -v$.
Therefore, $|2x - 3| = |x - 5|$ implies $2x - 3 = x - 5$ or $2x - 3 = 5 - x$.
Solving these two equations, we get $x = -2$ and $x = \dfrac{8}{3}$.
Hence, both $x = -2$ and $x = \dfrac{8}{3}$ are solutions.

2.3.3 Some Results For Absolute Value

(i) If $x, y \in \mathbb{R}$, $|y + x| = |x - y|$, then $xy = 0$.
(ii) For any $x, y \in \mathbb{R}$, $|xy| = |x||y|$.
(iii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, for all $x, y \in \mathbb{R}$ and $y \neq 0$.
(iv) For any $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.

2.3.4 Inequalities Involving Absolute Value

Here we shall learn to solve inequalities involving absolute values. First we analyze very simple inequalities such as (i) $|x| < r$ and (ii) $|x| > r$.

(i) Let us prove that $|x| < r$ if and only if $-r < x < r$. Note that $r > 0$ as $|x| \geq 0$.
There are two possibilities to consider depending on the sign of x.

Case (1): If $x \geq 0$, then $|x| = x$, so $|x| < r$ implies $x < r$.
Case (2): If $x < 0$, then $|x| = -x$, so $|x| < r$ implies $-x < r$ that is, $x > -r$.

Therefore we have, $|x| < r$ if and only if $-r < x < r$, that is $x \in (-r, r)$.

(ii) Let us prove that $|x| > r$ if and only if $x < -r$ or $x > r$.
Consider $|x| > r$. If $r < 0$, then every $x \in \mathbb{R}$ satisfies the inequality.
For $r \geq 0$, there are two possibilities to consider.

Case (1). If $x \geq 0$, then $|x| = x > r$.
Case (2). If $x < 0$, then $|x| = -x > r$, that is, $x < -r$.

So we have $|x| > r$, if and only if $x < -r$ or $x > r$, that is, $x \in (-\infty, -r) \cup (r, \infty)$.

Remark:

(i) For any $a \in \mathbb{R}$, $|x - a| \leq r$ if and only if $-r \leq x - a \leq r$ if and only if $x \in [a - r, a + r]$.
(ii) For any $a \in \mathbb{R}$, $|x - a| \geq r$ is equivalent to $x - a \leq -r$ or $x - a \geq r$ if and only if $x \in (-\infty, a - r] \cup [a + r, \infty)$.

Example 2.4 Solve $|x - 9| < 2$ for x.

Solution:
$|x - 9| < 2$ implies $-2 < x - 9 < 2$. Thus, $7 < x < 11$.

Example 2.5 Solve $\left|\dfrac{2}{x-4}\right| > 1$, $x \neq 4$.

Solution:
From the given inequality, we have that $2 > |x - 4|$.
That is, $-2 < x - 4 < 2$ and $x \neq 4$.
Adding 4 throughout the inequality, we obtain $2 < x < 6$ and $x \neq 4$.
So the solution set is $(2, 4) \cup (4, 6)$.

Exercise - 2.2

1. Solve for x:
   (i) $|3 - x| < 7$. (ii) $|4x - 5| \geq -2$. (iii) $\left|3 - \frac{3}{4}x\right| \leq \frac{1}{4}$.
   (iv) $|x| - 10 < -3$.
2. Solve $\frac{1}{|2x-1|} < 6$ and express the solution using the interval notation.
3. Solve $-3|x| + 5 \leq -2$ and graph the solution set in a number line.
4. Solve $2|x + 1| - 6 \leq 7$ and graph the solution set in a number line.
5. Solve $\frac{1}{5}|10x - 2| < 1$.
6. Solve $|5x - 12| < -2$.

2.4 Linear Inequalities

Recall that a function of the form $f(x) = ax + b$, $a, b \in \mathbb{R}$ are constants, is called a linear function, because its graph is a straight line. Here a is the slope of the line and b is the y-intercept. If $a \neq 0$, then x-intercept $x = \frac{-b}{a}$ is obtained by solving $f(x) = ax + b = 0$.
But there are situations where we need to consider linear inequalities.
For example to describe a statement like " *A tower is not taller than fifty feet.*"
If x denotes the height of the tower in feet, then the above statement can be expressed as $x \leq 50$.

Example 2.6 Our monthly electricity bill contains a basic charge, that is independent of units consumed and a charge that depends on the units consumed. Let us say Electricity Board charges Rs.110 as basic charge and charges Rs. 4 for each unit we use. If a person wants to keep his electricity bill below Rs.250, then what should be his electricity usage?

Solution:
Let x denote the number of units used. Note that $x \geq 0$. Then, his electricity bill is Rs. $110 + 4x$. The person wants his bill to be below Rs.250. Let us solve the inequality $110 + 4x < 250$. Thus, $4x < 140$; which gives $0 \leq x < 35$.
The person should keep his usage below 35 units in order to keep his bill below Rs.250.

Example 2.7 Solve $3x - 5 \leq x + 1$ for x.

Solution:
We have $3x - 5 \leq x + 1$; which is equivalent to $2x \leq 6$. Hence we have $x \leq 3$; the solution set is $(-\infty, 3]$.

We can also solve the above inequality graphically.
Let us consider the graphs of $f(x) = 3x - 5$ and $g(x) = x + 1$ (See Figure 2.3). Now, find all the x-values for which the graph of f is below the graph of g.

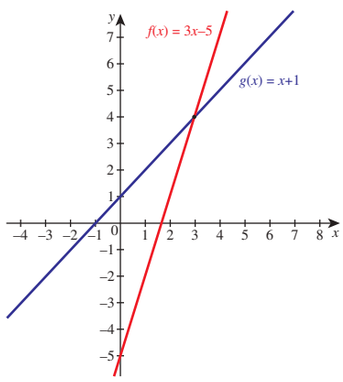


Figure 2.3

Example 2.8 Solve the following system of linear inequalities.
$3x - 9 \geq 0, \quad 4x - 10 \leq 6.$

Solution:
Note that $3x - 9 \geq 0$ implies $3x \geq 9$, by multiplying both sides by $1/3$ we get $x \geq 3$. Similarly, $4x - 10 \leq 6$ implies $4x \leq 16$ and hence $x \leq 4$.

So the solution set of $3x - 9 \geq 0, 4x - 10 \leq 6$ is the intersection of $[3, \infty)$ and $(-\infty, 4]$. Clearly, the intersection of these intervals give $[3, 4]$.

Example 2.9 A girl A is reading a book having 446 pages and she has already finished reading 271 pages. She wants to finish reading this book within a week. What is the minimum number of pages she should read per day to complete reading the book within a week?

Solution:
Let x denote the number of pages the girl should read per day. Then we need our x to satisfy $7x + 271 \geq 446$. Hence $x \geq 25$; which implies that she should read at least 25 pages per day.

In all the above examples observe that each inequality has more than one solution. Inequalities in general give rise to a range of solutions.

Exercise - 2.3

1. Represent the following inequalities in the interval notation:
   (i) $x \geq -1$ and $x < 4$ (ii) $x \leq 5$ and $x \geq -3$
   (iii) $x < -1$ or $x < 3$ (iv) $-2x > 0$ or $3x - 4 < 11$.
2. Solve $23x < 100$ when (i) x is a natural number, (ii) x is an integer.
3. Solve $-2x \geq 9$ when (i) x is a real number, (ii) x is an integer, (iii) x is a natural number.
4. Solve: (i) $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$. (ii) $\frac{5-x}{3} < \frac{x}{2} - 4$.
5. To secure A grade one must obtain an average of 90 marks or more in 5 subjects each of maximum 100 marks. If one scored $84, 87, 95, 91$ in first four subjects, what is the minimum mark one scored in the fifth subject to get A grade in the course?
6. A manufacturer has 600 litres of a 12 percent solution of acid. How many litres of a 30 percent acid solution must be added to it so that the acid content in the resulting mixture will be more than 15 percent but less than 18 percent?
7. Find all pairs of consecutive odd natural numbers both of which are larger than 10 and their sum is less than 40.
8. A model rocket is launched from the ground. The height h reached by the rocket after t seconds from lift off is given by $h(t) = -5t^2 + 100t, 0 \leq t \leq 20$. At what time the rocket is 495 feet above the ground?
9. A plumber can be paid according to the following schemes: In the first scheme he will be paid rupees 500 plus rupees 70 per hour, and in the second scheme he will be paid rupees 120 per hour. If he works x hours, then for what value of x does the first scheme give better wages?
10. A and B are working on similar jobs but their monthly salaries differ by more than Rs 6000. If B earns rupees 27000 per month, then what are the possibilities of A's salary per month?

2.5 Quadratic Functions

In earlier classes we have learnt that for any $z \in \mathbb{R}$ and $n \in \mathbb{N}$, $z^n = z \cdot z \cdot z \cdots z$ (n-times).
A function of the form $P(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ are constants and $a \neq 0$, is called a quadratic function. If $P(t) = 0$ for some $t \in \mathbb{R}$, then we say t is a ***zero*** of $P(x)$.

22THWX

2.5.1 Quadratic Formula

Is it possible to write the general quadratic function $P(x) = ax^2+bx+c$ in the form $a(x-k)^2+d$? The answer is yes. We can do this by the method called "**completing the square.**" We shall rewrite the function $P(x)$ as follows:

$$\begin{aligned}
P(x) &= ax^2+bx+c \\
&= a\left(x^2+2x\frac{b}{2a}+\frac{c}{a}\right) \\
&= a\left(x^2+2x\frac{b}{2a}+\left(\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right) \\
&= a\left(x+\frac{b}{2a}\right)^2-a\frac{b^2}{4a^2}+c \\
&= a\left(x+\frac{b}{2a}\right)^2+\left(a\left(\frac{b}{2a}\right)^2-b\frac{b}{2a}+c\right).
\end{aligned}$$

$$\text{Thus,}\quad P(x) = a\left(x+\frac{b}{2a}\right)^2+P\left(\frac{-b}{2a}\right). \qquad (1)$$

Now, to find the x- intercepts of the curve described by $P(x)$, let us solve for $P(x)=0$.

Considering $P(x)=0$ from (1) it follows that $a\left(x+\dfrac{b}{2a}\right)^2+P\left(\frac{-b}{2a}\right)=0$.

$$\begin{aligned}
a\left(x+\tfrac{b}{2a}\right)^2 &= -P\left(\tfrac{-b}{2a}\right) \\
&= -\tfrac{(-b^2+4ac)}{4a} \\
\left(x+\tfrac{b}{2a}\right)^2 &= \tfrac{b^2-4ac}{4a^2}.
\end{aligned}$$

So $x=\frac{\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}$ or $x=-\frac{\sqrt{b^2-4ac}}{2a}-\dfrac{b}{2a}$.

Hence, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$; which is called the ***quadratic formula***.

Remark:

(i) Note that $\sqrt{u}$ is defined as a real number only for $u \geq 0$.
(ii) when we write $\sqrt{u}$, we mean only the nonnegative root.

Note that $P(x)=0$ has two distinct real solutions if $b^2-4ac>0$, the roots are real and equal if $b^2-4ac=0$, and no real root if $b^2-4ac<0$.
Thus the curve intersects x-axis in two places if $b^2-4ac>0$, touches x-axis at only one point if $b^2-4ac=0$, and does not intersect x-axis at any point if $b^2-4ac<0$.
That is why $D=b^2-4ac$ is called the ***discriminant*** of the quadratic function $P(x)=ax^2+bx+c$.

(i) If α and β are roots of $ax^2+bx+c=0$, then $\alpha+\beta=\frac{-b}{a}$ and $\alpha\beta=\frac{c}{a}$.
(ii) If the discriminant b^2-4ac is negative, then the quadratic equation $ax^2+bx+c=0$, has no real roots. In this case, we have complex roots given by

$$\alpha=\frac{-b+i\sqrt{4ac-b^2}}{2a},\quad \beta=\frac{-b-i\sqrt{4ac-b^2}}{2a},\quad \text{where } i^2=-1,$$

which we will study in Higher Secondary Second year
(iii) For example, let us look at the graph of $y=x^2-4x+5$. (See Figure 2.4.)
Since the graph does not intersect the $x-$axis, $x^2-4x+5=0$ has no real roots.
(iv) We have the following table describing the nature of the roots of a quadratic equation and the sign of the discriminant $D=b^2-4ac$.

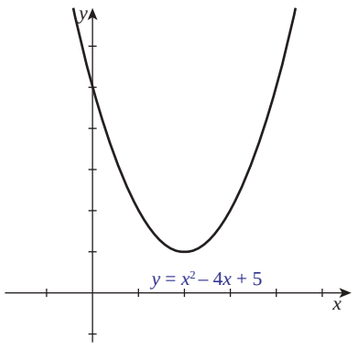


Figure 2.4

Discriminant	Nature of roots	Parabola
Positive	real and distinct	intersects x-axis at two points
Zero	real and equal	touches x-axis at one point
Negative	no real roots	does not meet x-axis

Example 2.10 If a and b are the roots of the equation $x^2 - px + q = 0$, find the value of $\frac{1}{a} + \frac{1}{b}$.

Solution:
Given that a and b are the roots of $x^2 - px + q = 0$. Then, $a + b = p$ and $ab = q$. Thus,
$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{p}{q}$.

Example 2.11 Find the complete set of values of a for which the quadratic $x^2 - ax + a + 2 = 0$ has equal roots.

Solution:
The quadratic equation $x^2 - ax + a + 2 = 0$ has equal roots.
So, its discriminant is zero. Thus, $D = b^2 - 4ac = a^2 - 4a - 8 = 0$.
So, $a = \frac{4 \pm \sqrt{48}}{2}$ which gives $a = 2 + \sqrt{12}, \quad 2 - \sqrt{12}$.

Example 2.12 Find the number of solutions of $x^2 + |x - 1| = 1$.

Solution:

Case (1). For $x \geq 1$, $|x - 1| = x - 1$.
Then the given equation reduces to $x^2 + x - 2 = 0$. Factoring we get $(x+2)(x-1) = 0$, which implies $x = -2$ or 1. As $x \geq 1$, we obtain $x = 1$.

Case (2). For $x < 1$, $|x - 1| = 1 - x$
Then the given equation becomes $x^2 + 1 - x = 1$. Thus we have $x(x - 1) = 0$ which implies $x = 0$ or $x = 1$. As $x < 1$, we have to choose $x = 0$.

Thus, the solution set is $\{0, 1\}$. Hence, the equation has two solutions.

Exercise - 2.4

1. Construct a quadratic equation with roots 7 and -3.
2. A quadratic polynomial has one of its zeros $1+\sqrt{5}$ and it satisfies $p(1)=2$. Find the quadratic polynomial.
3. If α and β are the roots of the quadratic equation $x^2+\sqrt{2}x+3=0$, form a quadratic polynomial with zeroes $\frac{1}{\alpha}, \frac{1}{\beta}$.
4. If one root of $k(x-1)^2=5x-7$ is double the other root, show that $k=2$ or -25.
5. If the difference of the roots of the equation $2x^2-(a+1)x+a-1=0$ is equal to their product, then prove that $a=2$.
6. Find the condition that one of the roots of ax^2+bx+c may be (i) negative of the other, (ii) thrice the other, (iii) reciprocal of the other.
7. If the equations $x^2-ax+b=0$ and $x^2-ex+f=0$ have one root in common and if the second equation has equal roots, then prove that $ae=2(b+f)$.
8. Discuss the nature of roots of (i) $-x^2+3x+1=0$, (ii) $4x^2-x-2=0$, (iii) $9x^2+5x=0$.
9. Without sketching the graphs, find whether the graphs of the following functions will intersect the x-axis and if so in how many points.
   (i) $y=x^2+x+2$, (ii) $y=x^2-3x-7$, (iii) $y=x^2+6x+9$.
10. Write $f(x)=x^2+5x+4$ in completed square form.

2.5.2 Quadratic Inequalities

Here we shall learn to solve the quadratic inequalities $ax^2+bx+c<0$ or $ax^2+bx+c>0$.

Steps to Solve Quadratic Inequalities:

(i) First solve $ax^2+bx+c=0$.
(ii) If there are no real solutions, then one of the above inequality holds for all $x \in \mathbb{R}$
(iii) If there are real solutions, which are called *critical points*, then label those points on the number line.
(iv) Note that these critical points divide the number line into disjoint intervals. (It is possible that there may be only one critical point.)
(v) Choose one representative number from each interval.
(vi) Substitute that these representative numbers in the inequality.
(vii) Identify the intervals where the inequality is satisfied.

Example 2.13 Solve $3x^2+5x-2 \leq 0$.

Solution:
On factorizing the quadratic polynomial we get $3(x+2)(x-\frac{1}{3}) \leq 0$. Draw the number line. Mark the critical points -2 and $\frac{1}{3}$ where the factors vanish (See Figure 2.5). On each sub-interval check the sign of $(x+2)(x-\frac{1}{3})$. To do this pick an arbitrary point anywhere in the interval. Whatever sign the resulting value has, the polynomial has the same sign throughout the whole corresponding interval. (Otherwise, there would be another critical point within the interval). This process is easily organized in the following table.

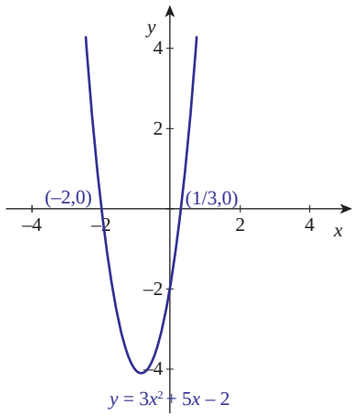


Figure 2.5

Interval	Sign of $(x+2)$	Sign of $(x-1/3)$	Sign of $3x^2+5x-2$
$(-\infty,-2)$	-	-	+
$(-2,1/3)$	+	-	-
$(1/3,\infty)$	+	+	+

You can see the inequality is satisfied in $[-2,1/3]$.

Example 2.14 Solve $\sqrt{x+14} < x+2$.

Solution:
The function $\sqrt{x+14}$ is defined for $x+14 \geq 0$. Therefore $x \geq -14$, $x+2>0$ implies $x \geq -2$.
$(x+14) < (x+2)^2$ gives $x^2+3x-10>0$.
Hence, $(x+5)(x-2)>0$. Dividing the number line with the critical points $x=-5$ and $x=2$.
Substituting a reference point in the sub-interval we get the solution set to be $x<-5$ and $x>2$.
Since $x \geq -2$, we have the solution to be $x>2$.

Example 2.15 Solve the equation $\sqrt{6-4x-x^2} = x+4$.

Solution:
The given equation is equivalent to the system
$(x+4) \geq 0$ and $6-4x-x^2=(x+4)^2$.
This implies $x \geq -4$ and $x^2+6x+5=0$. Thus, $x=-1,-5$.
But only $x=-1$ satisfies both the conditions. Hence, $x=-1$.

Exercise - 2.5

1. Solve $2x^2+x-15 \leq 0$.
2. Solve $-x^2+3x-2 \geq 0$.

2.6 Polynomial Functions

So far we have understood about linear functions and quadratic functions. Now we shall generalize these ideas. We call an expression of the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ is called a ***polynomial*** in the variable x, where $a_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n$. Here n is a non-negative integer. When $a_n \neq 0$, we say that the polynomial has **degree** n. The numbers $a_0, a_1, \ldots, a_n \in \mathbb{R}$ are called the coefficients of the polynomial. The number a_0 is called the constant term and a_n is called the leading coefficient (when it is non-zero). It is clear that:

(i) $100x^7 - \pi x^5 + 20\sqrt{2}x^2 + 7x + 1.22$ is a polynomial of degree 7.
(ii) $(17x - 3)(x + 3)(2x - \sqrt{\pi})(x + 2.3)$ is a polynomial of degree 4.
(iii) $(x^2 + x + 1)(x^3 + 2x + 2)(x^5 - 5x + \sqrt{3})$ is a polynomial of degree 10.

One may substitute specific values for x, say $x = c$ and obtain $a_nc^n + a_{n-1}c^{n-1} + \cdots + a_1c + a_0$. A function of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ is called a ***polynomial function*** which is defined from $\mathbb{R}$ to $\mathbb{R}$. We shall treat polynomial and polynomial function as one and the same.

A polynomial with degree 1 is called a ***linear polynomial***. A polynomial with degree 2 is called a ***quadratic polynomial***. A **cubic polynomial** is one that has degree three. Likewise, degree 4 and degree 5 polynomials are called ***quartic*** and ***quintic*** polynomials respectively. Note that any constant $a \neq 0$ is a polynomial of degree zero!

Two polynomials $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_n \neq 0$ and $g(x) = b_mx^m + b_{m-1}x^{m-1} + \cdots + b_0$, $b_m \neq 0$ are equal if and only if $f(x) = g(x)$ for all $x \in \mathbb{R}$. It can be proved that $f(x) = g(x)$ if and only if $n = m$ and $a_k = b_k, k = 0, 1, 2, \cdots n$. Given two polynomials, one can form their **sum** and **product**. For example if $P(x) = 2x^3 + 7x^2 - 5$ and $Q(x) = x^4 - 2x^3 + x^2 + x + 1$, then $P(x) + Q(x) = x^4 + 8x^2 + x - 4$ (by adding the corresponding coefficients of the like powers of x) and $P(x)Q(x) = 2x^7 + 3x^6 - 12x^5 + 4x^4 + 19x^3 + 2x^2 - 5x - 5$ by multiplying each term of $P(x)$ by every term of $Q(x)$. It is easy to see that the degree of $P(x)Q(x)$ is the sum of the degrees of $P(x)$ and $Q(x)$, whereas the degree of $P(x) + Q(x)$ is at most the maximum of degrees of $P(x)$ and $Q(x)$. Here is an example of the graph of a cubic polynomial function.

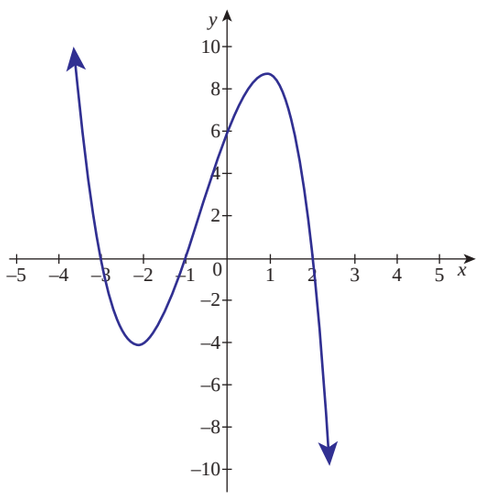


Figure 2.6

Suppose that $f(x)$ and $g(x)$ are polynomials where $g(x)$ is not zero. The quotient $\frac{f(x)}{g(x)}$ is called a **rational function,** which is defined for all $x \in \mathbb{R}$ such that $g(x) \neq 0$. In general, a rational function need not be a polynomial.

2.6.1 Division Algorithm

Given two polynomials $f(x)$ and $g(x)$, where $g(x)$ is not the zero polynomial, there exist two polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)g(x)+r(x)$ where degree of $r(x) <$ degree of $g(x)$. Here, $q(x)$ is called the quotient polynomial, and $r(x)$ is called the remainder polynomial. If $r(x)$ is the zero polynomial, then $q(x)$, $g(x)$ are factors of $f(x)$ and $f(x) = q(x)g(x)$.

These terminologies are similar to terminologies used in division done with integers.

If $g(x) = x - a$, then the remainder $r(x)$ should have degree zero and hence $r(x)$ is a constant. To determine the constant, write $f(x) = (x - a)q(x) + c$. Substituting $x = a$ we get $c = f(a)$.

Remainder Theorem

If a polynomial $f(x)$ is divided by $x-a$, then the remainder is $f(a)$. Thus the remainder $c = f(a) = 0$ if and only if $x - a$ is a factor for $f(x)$.

Definition 2.1

A real number a is said to be a ***zero of the polynomial*** $f(x)$ if $f(a) = 0$. If $x = a$ is a zero of $f(x)$, then $x - a$ is a ***factor*** for $f(x)$.

In general, if we can express $f(x)$ as $f(x) = (x - a)^k.g(x)$ where $g(a) \neq 0$, then the value of k, which depends on a, cannot exceed the degree of $f(x)$. The value k is called the ***multiplicity*** of the zero a.

(i) A polynomial function of degree n can have at most n distinct real zeros. It is also possible that a polynomial function like $P(x) = x^2 + 1$ has no real zeros at all.

(ii) Suppose that $P(x)$ is a polynomial function having rational coefficients. If $a + b\sqrt{p}$ where $a, b \in \mathbb{Q}$, p a prime, is a zero of $P(x)$, then its ***conjugate*** $a - b\sqrt{p}$ is also a zero.

Two important problems relating to polynomials are

(i) Finding zeros of a given polynomial function; and hence factoring the polynomial into linear factors and

(ii) Constructing polynomials with the given zeros and/or satisfying some additional conditions.

To address the problem of finding zeros of a polynomial function, some well known algebraic identities are useful. What is an identity?

An equation is said to be an ***identity*** if that equation remains valid for all values in its domain. An equation is called ***conditional equation*** if it is true only for some (not all) of values in its domain. Let us recall the following identities.

2.6.2 Important Identities

For all $x, a, b \in \mathbb{R}$ we have

1. $(x + a)^3 = (x + a)^2(x + a) = x^3 + 3x^2a + 3xa^2 + a^3 = x^3 + 3xa(x + a) + a^3$
2. $(x - b)^3 = x^3 - 3x^2b + 3xb^2 - b^3 = x^3 - 3xb(x - b) - b^3$ (taking $a = -b$ in (1))
3. $x^3 + a^3 = (x + a)(x^2 - xa + a^2)$
4. $x^3 - b^3 = (x - b)(x^2 + xb + b^2)$ taking $a = -b$ in (3)
5. $x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \cdots + x^{n-k-1}a^k + \cdots + a^{n-1}),\ n \in \mathbb{N}$
6. $x^n + b^n = (x + b)(x^{n-1} - x^{n-2}b + \cdots + x^{n-k-1}(-b)^k + \cdots + (-b)^{n-1}),\ n \in \mathbb{N}$

Exercise - 2.6

1. Find the zeros of the polynomial function $f(x) = 4x^2 - 25$.
2. If $x = -2$ is one root of $x^3 - x^2 - 17x = 22$, then find the other roots of equation.
3. Find the real roots of $x^4 = 16$.
4. Solve $(2x + 1)^2 - (3x + 2)^2 = 0$.

Method of Undetermined Coefficients

Now let us focus on constructing polynomials with the given information using the method of undetermined coefficients. That is, we shall determine coefficients of the required polynomial using the given conditions. The main idea here is that two polynomials are equal if and only if the coefficients of same powers of the variables in the two polynomials are equal.

Example 2.16 Find a quadratic polynomial $f(x)$ such that, $f(0) = 1, f(-2) = 0$ and $f(1) = 0$.

Solution:
Let $f(x) = ax^2 + bx + c$ be the polynomial satisfying the given conditions.
$f(0) = a(0)^2 + b(0) + c = 1$, implies that $c = 1$. Now the other two conditions $f(-2) = 0$, $f(1) = 0$ give $4a - 2b + c = 0$ and $a + b + c = 0$.
Using $c = 1$, we get $4a - 2b = -1$ and $a + b = -1$. Solving these two equations we get $a = b = -\dfrac{1}{2}$ and thus, we have $f(x) = -\dfrac{1}{2}x^2 - \dfrac{1}{2}x + 1$.

The above problem can also be solved in another way. $x = -2$, $x = 1$ are zeros of $f(x)$. Thus, $f(x) = d(x + 2)(x - 1)$ for some constant d.
Now using $f(0) = 1$ gives $-2d = 1$, hence $d = -\frac{1}{2}$. So, $f(x) = -\frac{1}{2}(x+2)(x-1) = -\frac{1}{2}x^2 - \frac{1}{2}x + 1$.

Example 2.17 Construct a cubic polynomial function with rational coefficients having zeros at $x = \frac{2}{5}, 1 + \sqrt{3}$ such that $f(0) = -8$.

Solution:
Given that $\frac{2}{5}$ and $1 + \sqrt{3}$ are zeros of $f(x)$. Thus, $1 - \sqrt{3}$ is also a zero of $f(x)$.
Let $f(x) = a(x - \frac{2}{5})[(x - (1 + \sqrt{3}))][x - (1 - \sqrt{3})] = a(x - \frac{2}{5})[(x - 1)^2 - 3]$.
Using $f(0) = -8$, we have, $\left(-\frac{2}{5}a\right)(-2) = -8$ which give $a = -10$.
Thus the required polynomial is $f(x) = (-10)(x - \frac{2}{5})[x^2 - 2x - 2] = -10x^3 + 24x^2 + 12x - 8$.

Example 2.18 Prove that $ap + q = 0$ if $f(x) = x^3 - 3px + 2q$ is divisible by $g(x) = x^2 + 2ax + a^2$.

Solution:
Note that the degree of $f(x)$ is 3 and the leading coefficient is 1. Since $g(x)$ divides $f(x)$, we have $f(x) = (x + b)g(x)$, for some $b \in \mathbb{R}$. Thus, $x^3 - 3px + 2q = (x + b)(x^2 + 2ax + a^2)$.
Equating like coefficients on both sides, we have $2a + b = 0$, $a^2 + 2ab = -3p$ and $2q = ba^2$. Thus, $b = -2a$, $p = a^2$, and $q = -a^3$.
Now, $q = -a^3 = -a(a^2) = -ap$, which gives $ap + q = 0$.

Example 2.19 Use the method of undetermined coefficients to find the sum of $1+2+3+\cdots+(n-1)+n, n \in \mathbb{N}$

Solution:

$$\begin{aligned}
\text{Let } S(n) &= n+(n-1)+(n-2)+\cdots+2+1 \\
&= n+(n-1)+(n-2)+\cdots+[n-(n-2)]+[n-(n-1)] \\
&= n\left[1+\frac{n-1}{n}+\frac{n-2}{n}+\cdots+\frac{n-(n-2)}{n}+\frac{n-(n-1)}{n}\right] \\
&\leq n[1+1+\cdots+1] \quad \text{since } \frac{n-1}{n}<1,\ \frac{n-2}{n}<1,\cdots \\
\text{Thus, } S(n) &\leq n^2.
\end{aligned}$$

$$\begin{aligned}
\text{Let } S(n) &= a+bn+cn^2, \text{ where } a,b,c, \in \mathbb{R}. \\
\text{Now, } S(n+1)-S(n) &= n+1 \\
a+b(n+1)+c(n+1)^2-[a+bn+cn^2] &= n+1 \\
b+2cn+c &= n+1
\end{aligned}$$

Thus, $b+c=1$ and $2c=1$ (Equating like coefficients) which give $b=\frac{1}{2};\ c=\frac{1}{2}$

Now, $S(1)=1,\quad a+b+c=1$ which gives $a=0$

Hence, $S(n)=\frac{1}{2}n+\frac{1}{2}n^2=\frac{n(n+1)}{2}, n \in \mathbb{N}$.

Example 2.20 Find the roots of the polynomial equation $(x-1)^3(x+1)^2(x+5)=0$ and state their multiplicity.

Solution:
Let $f(x)=(x-1)^3(x+1)^2(x+5)=0$. Clearly, we have $x=1,-1,-5$.

Hence, the roots are 1 with multiplicity 3, -1 with multiplicity 2 and -5 with multiplicity 1.

When the root has multiplicity 1, it is called a simple root.

Example 2.21 Solve $x=\sqrt{x+20}$ for $x \in \mathbb{R}$.

Solution:
Observe that $\sqrt{x+20}$ is defined only if $x+20 \geq 0$.
By definition, $(x+20) \geq 0$. So x is positive.
Now squaring we get $x^2=x+20$. $x^2-x-20=0$
$(x-5)(x+4)=0$, which gives $x=5, x=-4$
Since, x is positive, the required solution is $x=5$.

Example 2.22 The equations $x^2 - 6x + a = 0$ and $x^2 - bx + 6 = 0$ have one root in common. The other root of the first and the second equations are integers in the ratio $4 : 3$. Find the common root.

Solution:
Let α be the common root.
Let $\alpha, 4\beta$ be the roots of $x^2 - 6x + a = 0$.
Let $\alpha, 3\beta$ be the roots of $x^2 - bx + 6 = 0$.
Then, $4\alpha\beta = a$ and $3\alpha\beta = 6$ which give $\alpha\beta = 2$ and $a = 8$.
The roots of $x^2 - 6x + 8 = 0$ are $2, 4$.
If $\alpha = 2$, then $\beta = 1$
If $\alpha = 4$, then $\beta = \frac{1}{2}$ which is not an integer.
Hence, the common root is 2.

Example 2.23 Find the values of p for which the difference between the roots of the equation $x^2 + px + 8 = 0$ is 2.

Solution:
Let α and β be the roots of the equation $x^2 + px + 8 = 0$.
Then, $\alpha + \beta = -p$, $\alpha\beta = 8$ and $|\alpha - \beta| = 2$.
Now, $(\alpha + \beta)^2 - 4\alpha\beta = (\alpha - \beta)^2$, which gives $p^2 - 32 = 4$. Thus, $p = \pm 6$.

Exercise - 2.7

1. Factorize: $x^4 + 1$. (*Hint:* Try completing the square.)
2. If $x^2 + x + 1$ is a factor of the polynomial $3x^3 + 8x^2 + 8x + a$, then find the value of a.

2.7 Rational Functions

A rational expression of x is defined as the ratio of two polynomials in x, say $P(x)$ and $Q(x)$ where $Q(x) \neq 0$. Examples of rational expressions are $\dfrac{2x+1}{x^2+x+1}$, $\dfrac{x^4+1}{x^2+1}$ and $\dfrac{x^2+x}{x^2-5x+6}$.

If the degree of the numerator $P(x)$ is equal to or larger than that of the denominator $Q(x)$, then we can write $P(x) = f(x)Q(x) + r(x)$ where $r(x) = 0$ or the degree of $r(x)$ is less than that of $Q(x)$. So $\dfrac{P(x)}{Q(x)} = f(x) + \dfrac{r(x)}{Q(x)}$.

2.7.1 Rational Inequalities

Example 2.24 Solve $\dfrac{x+1}{x+3} < 3$.

Solution:
Subtracting 3 from both sides we get $\dfrac{x+1}{x+3} - 3 < 0$.

$$\begin{aligned} \frac{x+1-3(x+3)}{x+3} &< 0 \\ \frac{-2x-8}{x+3} &< 0 \\ \frac{x+4}{x+3} &> 0 \end{aligned}$$

Thus, $x+4$ and $x+3$ are both positive or both negative.

So let us find out the signs of $x+3$ and $x+4$ as follows

x	$x+3$	$x+4$	$\frac{x+4}{x+3}$
$x<-4$	$-$	$-$	$+$
$-4<x<-3$	$-$	$+$	$-$
$x>-3$	$+$	$+$	$+$
$x=-4$	$-$	0	0

So the solution set is given by $(-\infty, -4) \cup (-3, \infty)$.

The above type of rational inequality problem can also be solved by plotting the signs of various factors on the intervals of the number line.

Exercise - 2.8

1. Find all values of x for which $\dfrac{x^3(x-1)}{(x-2)} > 0$.
2. Find all values of x that satisfies the inequality $\dfrac{2x-3}{(x-2)(x-4)} < 0$.
3. Solve $\dfrac{x^2-4}{x^2-2x-15} \leq 0$.

2.7.2 Partial Fractions

A rational expression $\dfrac{f(x)}{g(x)}$ is called a proper fraction if the degree of $f(x)$ is less than degree of $g(x)$, where $g(x)$ can be factored into linear factors and quadratic factors without real zeros. Now $\frac{f(x)}{g(x)}$ can be expressed in simpler terms, namely, as a sum of expressions of the form

(i) $\dfrac{A_1}{(x-a)} + \dfrac{A_2}{(x-a)^2} + \cdots + \dfrac{A_k}{(x-a)^k}$ if $x-a$ divides $g(x)$ and

(ii) $\dfrac{(B_1x+C_1)}{(x^2+ax+b)} + \dfrac{(B_2x+C_2)}{(x^2+ax+b)^2} + \cdots + \dfrac{(B_kx+C_k)}{(x^2+ax+b)^k}$ if x^2+ax+b has no real zeros and (x^2+ax+b) divides $g(x)$.

The resulting expression of $\dfrac{f(x)}{g(x)}$ is called the *partial fraction decomposition*. Such a decomposition is *unique* for a given rational function.

This method is useful in doing Integral calculus. So let us discuss some examples.

Example 2.25 Resolve into partial fractions: $\dfrac{x}{(x+3)(x-4)}$.

Solution:

Let $\dfrac{x}{(x+3)(x-4)} = \dfrac{A}{x+3} + \dfrac{B}{x-4}$ where A and B are constants.

Then, $\dfrac{x}{(x+3)(x-4)} = \dfrac{A(x-4)+B(x+3)}{(x+3)(x-4)}$, which gives $x = A(x-4)+B(x+3)$.

When $x = 4$, we have $B = \dfrac{4}{7}$.

When $x = -3$, we have $A = \dfrac{3}{7}$

Hence, $\dfrac{x}{(x+3)(x-4)} = \dfrac{3}{7(x+3)} + \dfrac{4}{7(x-4)}$.

The above procedure can be carried out if the denominator has all its zeros in $\mathbb{R}$ which are all distinct.

Example 2.26 Resolve into partial fractions: $\dfrac{2x}{(x^2+1)(x-1)}$.

Solution:

In this case, note that the denominator has a factor x^2+1 which does not have real zeros.

Let $\dfrac{2x}{(x^2+1)(x-1)} = \dfrac{A}{(x-1)} + \dfrac{Bx+C}{x^2+1}$

where A, B, C are constants.

We have, $2x = A(x^2+1) + (Bx+C)(x-1)$.

When $x = 1$, we get $A = 1$.

When $x = 0$, we have $A - C = 0$ and hence $A = C = 1$.

When $x = -1$, we have $2A - 2(C-B) = -2$, which gives $B = -1$.

Thus, $\dfrac{2x}{(x^2+1)(x-1)} = \dfrac{1}{(x-1)} + \dfrac{1-x}{x^2+1}$

We now illustrate the situation when denominator has a real zeros with multiplicity more than one.

Example 2.27 Resolve into partial fractions: $\dfrac{x+1}{x^2(x-1)}$.

Solution:

Let $\dfrac{x+1}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1}$.

Then, $x+1 = Ax(x-1) + B(x-1) + Cx^2$.

When $x = 0$, we have $B = -1$ and when $x = 1$, we get $C = 2$.

When $x = -1$, we have $2A - 2B + C = 0$ which gives $A = -2$.

Thus, $\dfrac{x+1}{x^2(x-1)} = \dfrac{-2}{x} - \dfrac{1}{x^2} + \dfrac{2}{x-1}$.

Exercise - 2.9

Resolve the following rational expressions into partial fractions.

1. $\dfrac{1}{x^2-a^2}$
2. $\dfrac{3x+1}{(x-2)(x+1)}$
3. $\dfrac{x}{(x^2+1)(x-1)(x+2)}$
4. $\dfrac{x}{(x-1)^3}$
5. $\dfrac{1}{x^4-1}$
6. $\dfrac{(x-1)^2}{x^3+x}$
7. $\dfrac{x^2+x+1}{x^2-5x+6}$
8. $\dfrac{x^3+2x+1}{x^2+5x+6}$
9. $\dfrac{x+12}{(x+1)^2(x-2)}$
10. $\dfrac{6x^2-x+1}{x^3+x^2+x+1}$
11. $\dfrac{2x^2+5x-11}{x^2+2x-3}$
12. $\dfrac{7+x}{(1+x)(1+x^2)}$

2.7.3 Graphical Representation of Linear Inequalities

A straight line $ax+by=c$ divides the Cartesian plane into two parts. Each part is an half plane. A vertical line $x=c$ will divide the plane in left and right half planes and a horizontal line $y=k$ will divide the plane into upper and lower half planes.

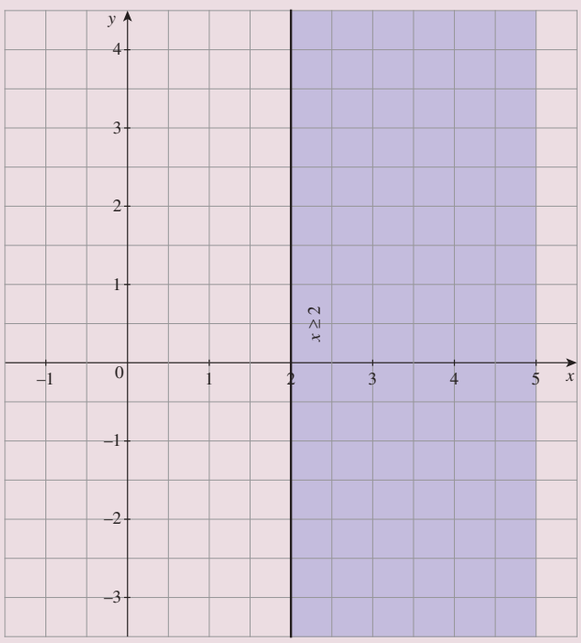
A point in the cartesian plane which is not on the line $ax+by=c$ will lie in exactly one of the two half planes determined by the line and satisfies one of the inequalities $ax+by<c$ or $ax+by>c$.

To identify the half plane represented by $ax+by<c$, choose a point P in any one of the half planes and substitute the coordinates of P in the inequality.

If the inequality is satisfied, then the required half plane is the one that contains P; otherwise the required half plane is the one that does not contain P. When $c \neq 0$, it is most convenient to take P to be the origin.

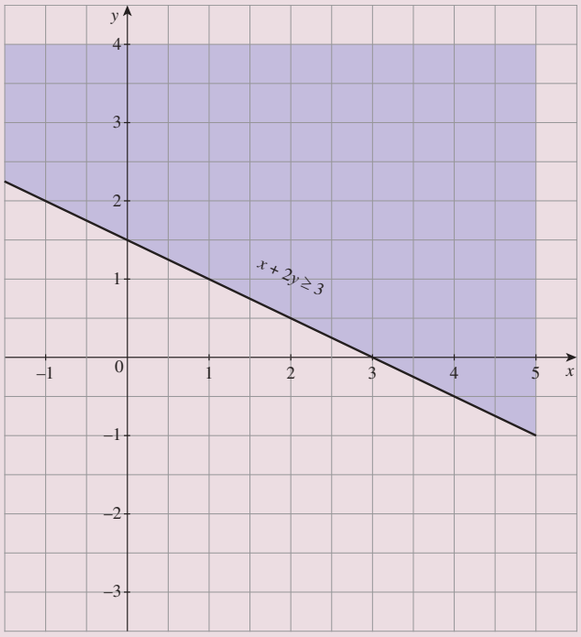
Example 2.28 Shade the region given by the inequality $x \geq 2$.

Solution:
First we consider equation $x=2$. It is a line parallel to y axis at a distance of 2 units from it. This line divides the cartesian plane into two parts. Substituting $(0,0)$ in the inequality we get $0 \geq 2$ which is false. Hence the region which does not contain the origin is represented by the inequality $x \geq 2$. The shaded region is the required solution set of the given inequality. Since $x \geq 2$, the points on the line $x=2$ are also solutions.



Example 2.29 Shade the region given by the linear inequality $x + 2y > 3$.

Proof. The line $x + 2y = 3$ divides the cartesian plane into two half planes. To find the half plane represented by $x + 2y > 3$ substitute a point in one of the half planes in the inequality and check whether it is satisfied. Let us substitute $(0, 0)$ in the inequality. We get $0 > 3$ which is false. Hence, the region given by $x + 2y > 3$ is the half plane which does not contain the origin.



Example 2.30 Solve the linear inequalities and exhibit the solution set graphically:

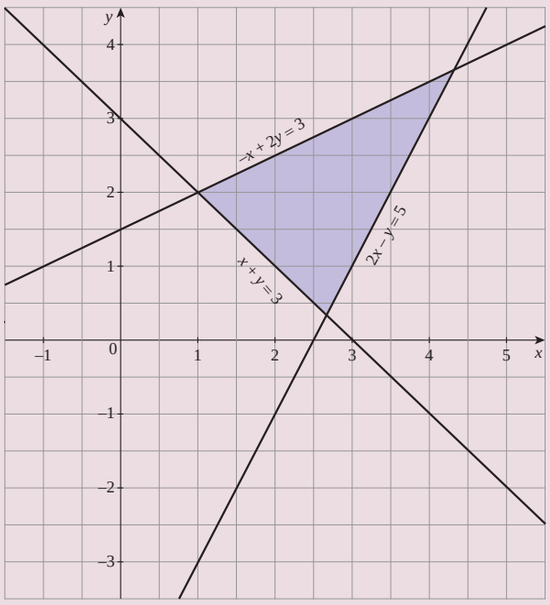
$$x + y \geq 3,\ 2x - y \leq 5,\ -x + 2y \leq 3.$$

Solution:
Observe that a straight line can be drawn if we identify any two points on it. For example, $(3, 0)$ and $(0, 3)$ can be easily identified as two points on the straight line $x + y = 3$.
Draw the three straight lines $x + y = 3, 2x - y = 5$ and $-x + 2y = 3$.
Now $(0, 0)$ does not satisfy $x + y \geq 3$. Thus, the half plane bounded by $x + y = 3$, not containing the origin, is the solution set of $x + y \geq 3$.

Similarly, the half-plane bounded by $2x - y \leq 5$ containing the origin represents the solution set of the $2x - y \leq 5$.

The region represented by $-x + 2y \leq 3$ is the half space bounded by the straight line $-x + 2y = 3$ that contains the origin.

The region common to the above three half planes represents the solution set of the given linear inequalities.



Exercise - 2.10

Determine the region in the plane determined by the inequalities:

(1) $x \leq 3y, \quad x \geq y.$
(2) $y \geq 2x, \quad -2x + 3y \leq 6.$
(3) $3x + 5y \geq 45, x \geq 0, y \geq 0.$
(4) $2x + 3y \leq 35, y \geq 2, x \geq 5.$
(5) $2x + 3y \leq 6,\ x + 4y \leq 4,\ x \geq 0, y \geq 0.$
(6) $x - 2y \geq 0,\ 2x - y \leq -2,\ x \geq 0, y \geq 0.$
(7) $2x + y \geq 8, x + 2y \geq 8, x + y \leq 6.$

2.8 Exponents and Radicals

First we shall consider exponents.

2.8.1 Exponents

Let $n \in \mathbb{N}$, $a \in \mathbb{R}$. Then $a^n = a \cdot a \cdots a$ (n times). If m is a negative integer and the real number $a \neq 0$, then $a^m = \dfrac{1}{a^{-m}}$.

Note that for any $a \neq 0$, we have $\dfrac{a}{a} = a^{1-1} = a^0 = 1$. It is also easy to see the following properties.

Properties of Exponents

(i) For $m, n \in \mathbb{Z}$ and $a \neq 0$, we have $a^m a^n = a^{m+n}$.
(ii) For $m, n \in \mathbb{Z}$ and $a \neq 0$, we have $\dfrac{a^m}{a^n} = a^{m-n}$.

2.8.2 Radicals

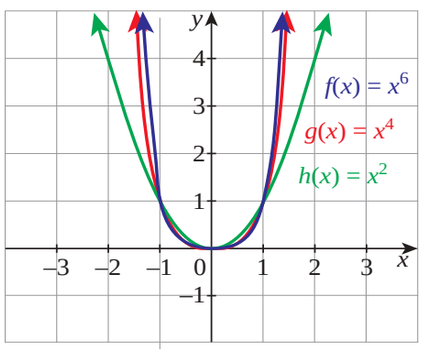
Question:

For $a \neq 0$ and $r \in \mathbb{Q}$, is it possible to define a^r?

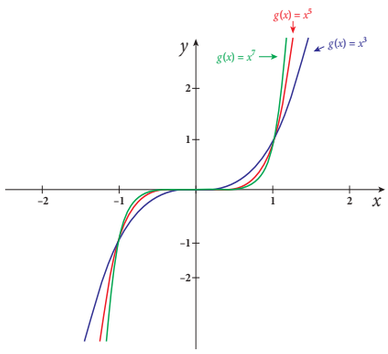
First let us consider the case when $r = \dfrac{1}{n}$, $n \in \mathbb{N}$. Suppose there is a real number $y \in \mathbb{R}$ such that $y = a^{\frac{1}{n}}$. Then we must have $y^n = a$.

This problem is basically to finding inverse function of $y = x^n$. In order to understand better let us consider the graphs of the following functions:

(i) $f(x) = x^{2n}$, $n \in \mathbb{N}$



(ii) $g(x) = x^{2n+1}$, $n \in \mathbb{N}$



From these two figures it is clear that the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^{2n+1}, n \in \mathbb{N}$ is one-to-one and onto and hence its inverse function from $\mathbb{R}$ onto $\mathbb{R}$ exists. But $f : \mathbb{R} \to [0, \infty)$ given by $f(x) = x^{2n}, n \in \mathbb{N}$ is onto but not one-to-one. However, f is one-to-one and onto if we restrict its domain to $[0, \infty)$. This is helpful in understanding nth root of a real number. So we have two cases;

Case 1 When n is even.

In this case $y^n = a$ is not meaningful when $a < 0$. So no such y exists when $a < 0$.

Assume that $a > 0$. If y is a solution to $x^n = a$, then $-y$ is also solution to $x^n = a$.

Case 2 When n is odd.

In this case no such problem arises as in Case 1. For $y \in \mathbb{R}$, there is a unique $x \in \mathbb{R}$ such that $y = x^n$.

Based on the above observation we define radicals as follow.

Definition 2.2

(i) For $n \in \mathbb{N}$, n even, and $b > 0$, there is a unique $a > 0$ such that $a^n = b$.

(ii) For $n \in \mathbb{N}$, n odd, $b \in \mathbb{R}$, there is a unique $a \in \mathbb{R}$ such that $a^n = b$. In both cases a is called the nth root of b or radical and is denoted by $b^{1/n}$ or $\sqrt[n]{b}$

(i) If $n = 2$, then nth root is called the square root; if $n = 3$, then it is called cube root.

(ii) Observe that the equation $x^2 = a^2$, has two solutions $x = a$, $x = -a$; but $\sqrt{a^2} = |a|$.

(iii) Properties of exponents given above are still valid for radicals provided each of the individual terms are defined.

(iv) Note that for $n \in \mathbb{N}$ and $a \neq 0$ we have

$$(a^n)^{1/n} = \begin{cases} |a| & \text{if } n \text{ is even ,} \\ a & \text{if } n \text{ is odd .} \end{cases}$$

For example, $\sqrt[4]{(-2)^4} = 16^{1/4} = 2$, $343^{1/3} = 7$ and $(-1000)^{\frac{1}{3}} = -10$.

For any rational $r = \dfrac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$, with gcd$(m, n) = 1$ and for $a > 0$ we define $a^r = a^{\frac{m}{n}} = (a^{1/n})^m$.

For example, $49^{3/2} = (49^{1/2})^3 = 7^3 = 343$. But $(-49)^{3/2}$ has no meaning in real number system because there is no real number x such that $x^2 = (-49)^3$.

2.8.3 Exponential Function

Observe that for any $a > 0$ and $x \in \mathbb{R}$, a^x can be defined. If $a = 1$, we define $1^x = 1$. So we shall consider a^x, $x \in \mathbb{R}$ for $0 < a \neq 1$. Here a^x is called **exponential function with base** a. Note that a^x may not be defined if $a < 0$ and $x = \dfrac{1}{m}$ for even $m \in \mathbb{N}$. This is why we restrict to $a > 0$. Also, $a^x > 0$ for all $x \in \mathbb{R}$. It does also satisfy the following:

Properties of Exponential Function

For $a, b > 0$ and $a \neq 1 \neq b$

(i) $a^{x+y} = a^x\, a^y$ for all $x,\ y \in \mathbb{R}$,

(ii) $\dfrac{a^x}{a^y} = a^{x-y}$ for all $x,\ y \in \mathbb{R}$,

(iii) $(a^x)^y = a^{xy}$ for all $x,\ y \in \mathbb{R}$,
(iv) $(ab)^x = a^x b^x$ for all $x \in \mathbb{R}$,
(v) $a^x = 1$ if and only if $x = 0$.

1. Let us consider $f(x) = a^x, x \in \mathbb{R}$ where $a = 2$.
   Now $f(x) = 2^x,\ x \in \mathbb{R}$. Let us show that f is one-to-one and onto.
   Suppose $f(u) = f(v)$ for some $u, v \in \mathbb{R}$. Then, we have $2^u = 2^v$, which implies that $\dfrac{2^u}{2^v} = 1$, $\Rightarrow 2^{u-v} = 1$.
   So, $u - v = 0$ and hence $u = v$. Thus f is a one-to-one function.

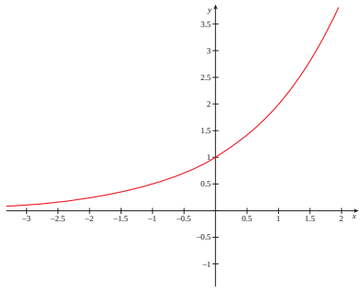


Figure 2.7: $f(x) = 2^x$

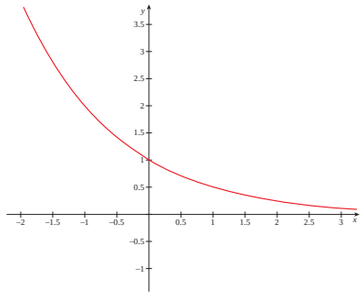


Figure 2.8: $f(x) = \dfrac{1}{2^x}$

From the graph it is clear that values of $f(x) = 2^x$ increase as x values increase and the range of f is $(0,\ \infty)$. So as $2^0 = 1$, we have $2^x > 1$ for all $x > 0$ and $2^x < 1$ for all $x < 0$. Observe that $f : \mathbb{R} \to (0, \infty)$ is onto.

2. Let us consider $a = \dfrac{1}{2}$. Let $g(x) = \left(\dfrac{1}{2}\right)^x = \dfrac{1}{2^x},\ x \in \mathbb{R}$.
   From the graph it is clear that the values of $g(x) = \left(\dfrac{1}{2}\right)^x$ decrease as x values increase and $g(\mathbb{R}) = (0,\ \infty)$. Also, $g(0) = 1$ we have $g(x) > 1$ for all $x < 0$ and $g(x) < 1$ for all $x > 0$.

Remark: Exactly same arguments as above would show that an exponential function $f(x) = a^x$, for any base $0 < a \neq 1$, is one-to-one and onto with domain $\mathbb{R}$ and codomain $(0,\ \infty)$.

A Special Exponential Function

Among all exponential functions, $f(x) = e^x,\ x \in \mathbb{R}$ is the most important one as it has applications in many areas like mathematics, science and economics. Then what is this e? The following illustration from compounding interest problem leads to the constant e.

Illustration

2.8.3.1 Compound Interest

Recall that if P is the principal, $r = \dfrac{\text{interest rate}}{100}$, n is the number of compounding periods in a year and t is the number of years, then $A = P\left(1 + \frac{r}{n}\right)^{nt}$ gives the total amount after t years. If $n = 4$, then it is compounded quarterly (the interest is added to the existing principal for three months in a year). If $n = 12$, then compounded monthly, $n = 365$ means compounded daily. We can compound every hour, every minute etc. We know that if P and r are fixed and the number of compounding periods in a year increases, then the total amount also increases. Let us consider the case with $P = 1$, $r = 1$ and $t = 1$. Then, we have $A_n = \left(1 + \frac{1}{n}\right)^n$. We want to understand how big it gets as n gets really large. Let us make a table with different values of $n = 10,\ 100,\ 10000,\ 100000,\ 100000000$.

n	10	100	10000	100000	100000000
A_n	2.593742460	2.704813829	2.718145927	2.718268237	2.718281815

We notice that as n gets really large, A_n values seem to be getting closer to 2.718281815..... Actually A_n values approach a real number e, an irrational number. 2.718281815 is an approximation to e. So the compound interest formula becomes $A = Pe^{rt}$, where r is the interest rate and P is the principal and t is the number of years. This is called Continuous Compounding.

Example 2.31 (i) Simplify: $(x^{1/2}y^{-3})^{1/2}$; where $x, y \geq 0$.
(ii) Simplify: $\sqrt{x^2 - 10x + 25}$.

Solution:
(i) Since $x, y \geq 0$, we have $(x^{1/2}y^{-3})^{1/2} = x^{1/4}/y^{3/2}$.
(ii) Observe that $\sqrt{x^2 - 10x + 25} = \sqrt{(x-5)^2} = |x-5|$.

(i) $(x^{1/4})^4 = x$ but $(y^4)^{1/4} = |y|$.
Observe that $x^{1/4}$ is defined only when x is positive. But y^4 is defined even when $y < 0$.
Now $(y^4)^{1/4}$ is a positive number whose fourth power equals y^4. So it has to be $|y|$.

(ii) $(x^8.y^4)^{1/4} = x^2|y|$.

(iii) Let u, v, b be rational numbers where b is positive.
Let us suppose they are not squares of rational numbers.
Then $u + v\sqrt{b}, u - v\sqrt{b}$ are called **conjugates**.
Observe that $(u + v\sqrt{b})(u - v\sqrt{b}) = u^2 - bv^2$ is now rational.
Thus, if an expression such as $u + v\sqrt{b}$ appears in the denominator we can multiply both the numerator and denominator by its conjugate, namely, $u - v\sqrt{b}$, to get a rational number in the denominator.

(iv) Using $(u\sqrt{a} - v\sqrt{b})(u\sqrt{a} + v\sqrt{b}) = u^2a - v^2b$, it is possible to simplify expressions when $u\sqrt{a} + v\sqrt{b}$ occurs in the denominator.

Example 2.32 Rationalize the denominator of $\dfrac{\sqrt{5}}{(\sqrt{6}+\sqrt{2})}$.

Solution:
Multiplying both numerator and denominator by $(\sqrt{6} - \sqrt{2})$,we get

$$\frac{\sqrt{5}}{(\sqrt{6}+\sqrt{2})} = \frac{\sqrt{5}(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})} = \frac{(\sqrt{30}-\sqrt{10})}{4}.$$

Example 2.33 Find the square root of $7 - 4\sqrt{3}$.

Solution:

Let $\sqrt{7 - 4\sqrt{3}} = a + b\sqrt{3}$ where a, b are rationals.
Squaring on both sides, we get $7 - 4\sqrt{3} = a^2 + 3b^2 + 2ab\sqrt{3}$. So, $a^2 + 3b^2 = 7$ and $2ab = -4$.

Therefore $a = -2/b$.
From $a^2 + 3b^2 = 7$, we get $(-2/b)^2 + 3b^2 = 7$, which gives $4/b^2 + 3b^2 = 7$ or $3b^4 - 7b^2 + 4 = 0$.
Solving for b^2 we get $b^2 = \frac{(7 \pm \sqrt{49-48})}{6}$, which gives $b^2 = 1$ or $b^2 = \frac{4}{3}$.
Thus, $b = \pm 1$ or $b = \pm \frac{2}{\sqrt{3}}$.
Since b is rational, we have $b = \pm 1$ and hence the corresponding values of a are ∓ 2.
Since $\sqrt{7 - 4\sqrt{3}} > 0$, we have $\sqrt{7 - 4\sqrt{3}} = 2 - \sqrt{3}$.

It is not always possible to express square roots of $u + v\sqrt{b}$ where u, v are rationals, in the form $x + y\sqrt{b}$ with x, y rationals. For example, the square root of $1 + \sqrt{2}$ is not of the form $a + b\sqrt{2}$ with a, b rationals.

Exercise - 2.11

1. Simplify:

   (i) $(125)^{\frac{2}{3}}$, (ii) $16^{\frac{-3}{4}}$, (iii) $(-1000)^{\frac{-2}{3}}$, (iv) $(3^{-6})^{\frac{1}{3}}$, (v) $\dfrac{27^{\frac{-2}{3}}}{27^{\frac{-1}{3}}}$.
2. Evaluate $\left(\left((256)^{-1/2}\right)^{\frac{-1}{4}}\right)^3$.
3. If $(x^{1/2} + x^{-1/2})^2$ = 9/2, then find the value of $(x^{1/2} - x^{-1/2})$ for $x > 1$.
4. Simplify and hence find the value of n: $3^{2n}9^2 3^{-n}/3^{3n} = 27$.
5. Find the radius of the spherical tank whose volume is $32\pi/3$ units.
6. Simplify by rationalising the denominator. $\dfrac{7 + \sqrt{6}}{3 - \sqrt{2}}$.
7. Simplify $\dfrac{1}{3 - \sqrt{8}} - \dfrac{1}{\sqrt{8} - \sqrt{7}} + \dfrac{1}{\sqrt{7} - \sqrt{6}} - \dfrac{1}{\sqrt{6} - \sqrt{5}} + \dfrac{1}{\sqrt{5} - 2}$.
8. If $x = \sqrt{2} + \sqrt{3}$ find $\dfrac{x^2 + 1}{x^2 - 2}$.

2.9 Logarithm

We have seen that, with a base $0 < a \neq 1$, the exponential function $f(x) = a^x$ is defined on $\mathbb{R}$ having range $(0, \infty)$. We also observed that $f(x)$ is a bijection, hence it has an inverse. We call this inverse function as ***logarithmic function*** and is denoted by $\log_a(.)$. Let us discuss this function further. Note that if $f(x)$ takes x to $y = a^x$, then $\log_a(.)$ takes y to x. That is, for $0 < a \neq 1$, we have

$$y = a^x \qquad \text{is equivalent to} \qquad \log_a y = x.$$

For example, since $3^4 = 81$ we have $\log_3(81) = 4$. In other words, with fixed a, given a real number y, logarithm finds the exponent x satisfying $a^x = y$. This is useful in addressing practical problems like, "how long will it take for certain investment to reach a fixed amount?" Logarithm is also very useful in multiplying very small or big numbers.

(i) Note that exponential function a^x is defined for all $x \in \mathbb{R}$ and $a^x > 0$ and so $\log_a(\cdot)$ defined **only for positive real numbers**.
(ii) Also, $a^0 = 1$ for any base a and hence $\log_a(1) = 0$ for any base a.

2.9.1 Properties of Logarithm

(i) $a^{\log_a x} = x$ for all $x \in (0, \infty)$ and $\log_a (a^y) = y$ for all $y \in \mathbb{R}$.

(ii) For any $x, y > 0$, $\log_a(xy) = \log_a x + \log_a y$. (Product Rule)

(iii) For any $x, y > 0$, $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$. (Quotient Rule)

(iv) For any $x > 0$ and $r \in \mathbb{R}$, $\log_a x^r = r \log_a x$. (Power Rule)

(v) For any $x > 0$, with a and b as bases, $\log_b x = \dfrac{\log_a x}{\log_a b}$. (Change of base formula.)

Proof. Since exponential function with base a and logarithm function with base a are inverse of each other,

(i) follows by using the definitions.

(ii) For $x, y > 0$ let $\log_a x = u$, $\log_a y = v$, and $\log_a(xy) = w$. Rewriting these in the exponential form we obtain $a^u = x$, $a^v = y$, and, $a^w = xy$. So, $a^w = xy = a^u a^v = a^{u+v}$; thus $w = u + v$. Thus, we obtain $\log_a(xy) = \log_a x + \log_a y$.

(iii) Let $\log_a x = u$, $\log_a y = v$, and $\log_a \dfrac{x}{y} = w$. Then $a^u = x$, $a^v = y$ and $a^w = \dfrac{x}{y}$. Hence, $a^w = \dfrac{x}{y} = \dfrac{a^u}{a^v} = a^{u-v}$; which implies $w = u - v$.

Thus, we obtain $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$.

(iv) Let $\log_a x = u$. Then $a^u = x$ and therefore, $x^r = (a^u)^r = a^{ru}$. Thus, $\log_a x^r = ru = r \log_a x$.

(v) Let $\log_b x = v$. We have $b^v = x$. Taking logarithm with base a on both sides we get $\log_a b^v = \log_a x$.

On the other hand $\log_a b^v = v \log_a b$ by the Power rule. Therefore, $v \log_a b = \log_a x$.
Hence $\log_b x = \dfrac{\log_a x}{\log_a b}$, $b > 0$. This completes the proof.

■

Remark:

(i) If $a = 10$, then the corresponding logarithmic function $\log_{10} x$ is called the **common logarithm**.

(ii) If $a = e$,(an irrational number, approximately equal to 2.718), then the corresponding logarithmic function $\log_e x$ is called the **natural logarithm**. It is denoted by $\ln x$. These above particular cases of logarithmic functions are used very much in other sciences and engineering. Particularly, the natural logarithm occurs very naturally. When we write $\log x$ we mean $\log_e x$.

(iii) If $a = 2$, then the corresponding logarithmic function $\log_2 x$ called the binary logarithm, which is used in computer science.

(iv) Observe that $\log_a 35 = \log_a(7 * 5) = \log_a 7 + \log_a 5$; $\log_a \dfrac{50}{3} = \log_a 50 - \log_a 3$.

$\log_a 22^x = x \log_a 22$; $\log_5 50 = \dfrac{\log_{10} 50}{\log_{10} 5}$.

(v) Observe the graph of the logarithmic and exponential functions.

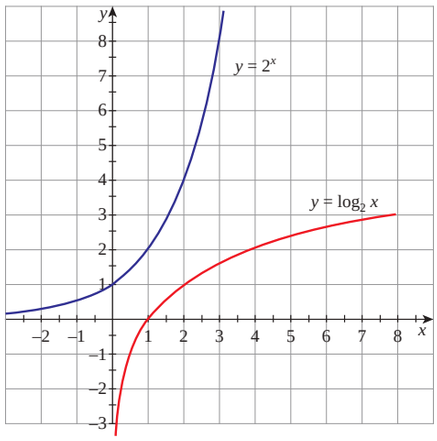


Figure 2.9

Example 2.34 Find the logarithm of 1728 to the base $2\sqrt{3}$.

Solution:
Let $\log_{2\sqrt{3}} 1728 = x$.
Then we have $(2\sqrt{3})^x = 1728 = 2^6\ 3^3 = 2^6\ (\sqrt{3})^6$.
Hence, $(2\sqrt{3})^x = (2\sqrt{3})^6$.
Therefore $x = 6$. That is, $\log_{2\sqrt{3}} 1728 = 6$.

Example 2.35 If the logarithm of 324 to base a is 4, then find a.

Solution:
We are given $\log_a 324 = 4$, which gives
$a^4 = 324 = 3^4\ (\sqrt{2})^4$. Therefore $a = 3\sqrt{2}$.

Example 2.36 Prove $\log \dfrac{75}{16} - 2\log \dfrac{5}{9} + \log \dfrac{32}{243} = \log 2$.

Solution:
Using the properties of logarithm, we have
$\log \dfrac{75}{16} - 2\log \dfrac{5}{9} + \log \dfrac{32}{243} = \log 75 - \log 16 - 2\log 5 + 2\log 9 + \log 32 - \log 243$. (By Quotient rule)
$= \log 3 + \log 25 - \log 16 - \log 25 + \log 81 + \log 16 + \log 2 - \log 81 - \log 3$
$= \log 2$.

Example 2.37 If $\log_2 x + \log_4 x + \log_{16} x = \dfrac{7}{2}$, find the value of x.

Solution:
Note that $x > 0$.
$\log_2 x + \log_4 x + \log_{16} x = \dfrac{7}{2}$ becomes $\dfrac{1}{\log_x 2} + \dfrac{1}{\log_x 4} + \dfrac{1}{\log_x 16} = \dfrac{7}{2}$. (change of base rule)
Thus $\dfrac{1}{a} + \dfrac{1}{2a} + \dfrac{1}{4a} = \dfrac{7}{2}$ where $a = \log_x 2$. That is $\dfrac{7}{4a} = \dfrac{7}{2}$.

Thus, $a = \dfrac{1}{2}$ and so, $\log_x 2 = \dfrac{1}{2}$ which gives $x^{\frac{1}{2}} = 2$.
Thus, $x = 2^2 = 4$.

Example 2.38 Solve $x^{\log_3 x} = 9$.

Solution:
Let $\log_3 x = y$.
Then $x = 3^y$ and so, $3^{y^2} = 9$.
Thus, $y^2 = 2$, which implies $y = \sqrt{2}, \quad -\sqrt{2}$. Hence, $x = 3^{\sqrt{2}}, \ \ 3^{-\sqrt{2}.}$

Example 2.39 Compute $\log_3 5 \log_{25} 27$.

Solution:
$\log_3 5 \log_{25} 27 = \log_3 5 \log_{25} 3^3$.
$= \log_3 5 \times 3 \log_{25} 3$ (by exponent rule)
$= 3 \log_{25} 5 = \frac{3}{\log_5 25} = \frac{3}{2 \log_5 5} = \frac{3}{2}$.

Example 2.40 Given that $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$ (approximately), find the number of digits in $2^8.3^{12}$.

Solution:
Suppose that $N = 2^8 3^{12}$ has $n+1$ digits. Then N can be written as $10^n \times b$ where $1 \leq b < 10$. Taking logarithm to the base 10, we get
$\log N = \log(10^n\ b) = n \log 10 + \log b = n + \log b$.
On the other hand,
$\log N = \log 2^8 3^{12} = 8 \log 2 + 12 \log 3 = 8 \times 0.30103 + 12 \times 0.47712 = 8.13368$.
Thus, we get $n + \log b = 8.13368$. Since $1 \leq b < 10$ the number of digits is 9.

Exercise - 2.12

1. Let $b > 0$ and $b \neq 1$. Express $y = b^x$ in logarithmic form. Also state the domain and range of the logarithmic function.
2. Compute $\log_9 27 - \log_{27} 9$.
3. Solve $\log_8 x + \log_4 x + \log_2 x = 11$.
4. Solve $\log_4 2^{8x} = 2^{\log_2 8}$.
5. If $a^2 + b^2 = 7ab$, show that $\log \dfrac{a+b}{3} = \dfrac{1}{2}(\log a + \log b)$.
6. Prove $\log \dfrac{a^2}{bc} + \log \dfrac{b^2}{ca} + \log \dfrac{c^2}{ab} = 0$.
7. Prove that $\log 2 + 16 \log \dfrac{16}{15} + 12 \log \dfrac{25}{24} + 7 \log \dfrac{81}{80} = 1$.
8. Prove $\log_{a^2} a \ \log_{b^2} b \ \log_{c^2} c = \frac{1}{8}$.
9. Prove $\log a + \log a^2 + \log a^3 + \cdots + \log a^n = \dfrac{n(n+1)}{2} \log a$.

10. If $\dfrac{\log x}{y-z} = \dfrac{\log y}{z-x} = \dfrac{\log z}{x-y}$, then prove that $xyz = 1$.
11. Solve $\log_2 x - 3\log_{\frac{1}{2}} x = 6$.
12. Solve $\log_{5-x}(x^2 - 6x + 65) = 2$.

2.10 Application of Algebra in Real Life

Algebra is used in many aspects of life. Financial planning is an area in daily life where algebra is used. Algebra concepts are used to calculate interest rates by bankers and as well as for calculating loan repayments. They are used to predict growth of money. Physical fitness is another area where calculations are made to determine the right amount of food intake for an individual taking into consideration such as the height, body mass of the person etc. Doctors use algebra in measuring drug dosage depending on age and weight of an individual. Architects depend on algebra to design buildings while civil engineers use it to design roads, bridges and tunnels. Algebra is needed to convert items to scale so that the structures designed have the correct proportions. It is used to programme computers and phones . Let us see some examples. Because of the extra-ordinary range of sensitivity of the human ear (a range of over 1000 million millions to one), it is useful to use logarithmic scale to measure sound intensity over this range. The unit of measure decibel is named after the inventor of the telephone Alexander Graham Bell.

If we know the population in the world today, the growth, which is rapid, can be measured by approximating to an exponential function. The radioactive carbon-14 is an organism which decays according to an exponential formula.

Exercise - 2.13

23CA18

Choose the correct or the most suitable answer.

1. If $|x+2| \leq 9$, then x belongs to
   (1) $(-\infty, -7)$ (2) $[-11, 7]$ (3) $(-\infty, -7) \cup [11, \infty)$ (4) $(-11, 7)$

2. Given that x, y and b are real numbers $x < y, b > 0$, then
   (1) $xb < yb$ (2) $xb > yb$ (3) $xb \leq yb$ (4) $\frac{x}{b} \geq \frac{y}{b}$

3. If $\dfrac{|x-2|}{x-2} \geq 0$, then x belongs to
   (1) $[2, \infty)$ (2) $(2, \infty)$ (3) $(-\infty, 2)$ (4) $(-2, \infty)$

4. The solution of $5x - 1 < 24$ and $5x + 1 > -24$ is
   (1) $(4, 5)$ (2) $(-5, -4)$ (3) $(-5, 5)$ (4) $(-5, 4)$

5. The solution set of the following inequality $|x-1| \geq |x-3|$ is
   (1) $[0, 2]$ (2) $[2, \infty)$ (3) $(0, 2)$ (4) $(-\infty, 2)$

6. The value of $\log_{\sqrt{2}} 512$ is

(1) 16 (2) 18 (3) 9 (4) 12

7. The value of $\log_3 \frac{1}{81}$ is

(1) -2 (2) -8 (3) -4 (4) -9

8. If $\log_{\sqrt{x}} 0.25 = 4$, then the value of x is

(1) 0.5 (2) 2.5 (3) 1.5 (4) 1.25

9. The value of $\log_a b \ \log_b c \ \log_c a$ is

(1) 2 (2) 1 (3) 3 (4) 4

10. If 3 is the logarithm of 343, then the base is

(1) 5 (2) 7 (3) 6 (4) 9

11. Find a so that the sum and product of the roots of the equation $2x^2 + (a-3)x + 3a - 5 = 0$ are equal is

(1) 1 (2) 2 (3) 0 (4) 4

12. If a and b are the roots of the equation $x^2 - kx + 16 = 0$ and satisfy $a^2 + b^2 = 32$, then the value of k is

(1) 10 (2) -8 (3) $-8, 8$ (4) 6

13. The number of solutions of $x^2 + |x-1| = 1$ is

(1) 1 (2) 0 (3) 2 (4) 3

14. The equation whose roots are numerically equal but opposite in sign to the roots of $3x^2 - 5x - 7 = 0$ is

(1) $3x^2 - 5x - 7 = 0$ (2) $3x^2 + 5x - 7 = 0$ (3) $3x^2 - 5x + 7 = 0$ (4) $3x^2 + x - 7$

15. If 8 and 2 are the roots of $x^2 + ax + c = 0$ and $3, 3$ are the roots of $x^2 + dx + b = 0$, then the roots of the equation $x^2 + ax + b = 0$ are

(1) $1, 2$ (2) $-1, 1$ (3) $9, 1$ (4) $-1, 2$

16. If a and b are the real roots of the equation $x^2 - kx + c = 0$, then the distance between the points $(a, 0)$ and $(b, 0)$ is

(1) $\sqrt{k^2 - 4c}$ (2) $\sqrt{4k^2 - c}$ (3) $\sqrt{4c - k^2}$ (4) $\sqrt{k - 8c}$

17. If $\dfrac{kx}{(x+2)(x-1)} = \dfrac{2}{x+2} + \dfrac{1}{x-1}$, then the value of k is

(1) 1 (2) 2 (3) 3 (4) 4

18. If $\frac{1-2x}{3+2x-x^2} = \frac{A}{3-x} + \frac{B}{x+1}$, then the value of $A+B$ is

(1) $\frac{-1}{2}$ (2) $\frac{-2}{3}$ (3) $\frac{1}{2}$ (4) $\frac{2}{3}$

19. The number of roots of $(x+3)^4 + (x+5)^4 = 16$ is

(1) 4 (2) 2 (3) 3 (4) 0

20. The value of $\log_3 11 \cdot \log_{11} 13 \cdot \log_{13} 15 \cdot \log_{15} 27 \cdot \log_{27} 81$ is

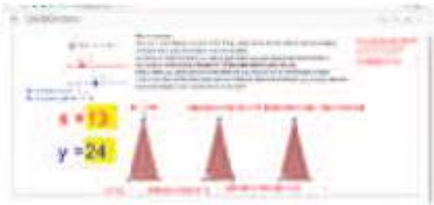
(1) 1 (2) 2 (3) 3 (4) 4

Summary

- π and $\sqrt{p}$, where p is a prime number, are some irrational numbers.
- $|x-a| = r$ if and only if $r \geq 0$ and $x - a = \pm r$.
- $|x-a| \leq r$ if and only if $-r \leq x - a \leq r$ or $a - r \leq x \leq a + r$.
- $|x-a| > r$ implies $x < a - r$ and $x > a + r$ (or) $x \in (-\infty, a-r) \cup (a+r, \infty)$
- inequalities, in general, have more than one solution.
- The nature of roots of $ax^2 + bx + c = 0$ is determined by the discriminant $D = b^2 - 4ac$.
- A real number a is a zero of a polynomial function $f(x)$ if and only if $(x-a)$ is a factor of $f(x)$.
- If degree of $f(x)$ is less than the degree of $g(x)$, then $\frac{f(x)}{g(x)}$ can be written as sum of its partial fractions.
- In general exponential functions and logarithmic functions are inverse functions to each other.

ICT CORNER-2(a)

Expected Outcome ⇒



Step–1

Open the Browser and type the URL Link given below (or) Scan the QR Code.

Step–2

GeoGebra Work Sheet called “Hill and Flower Puzzle” will appear. Puzzle Detail

(a)You have some flowers in your hand. If you climb up the hill the flowers will be doubled and also, when you climb down the hill it will be doubled. (b)At the top of each hill there is a idol of god where you have to put some flowers. (c)you have to climb and put flowers in all the three idols in each hill top.

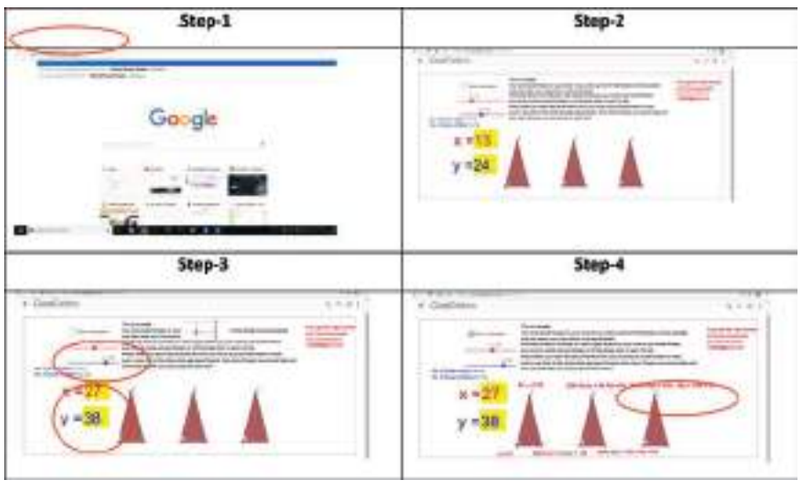
finally, when you reach the top of the third hill you have to put all the flowers in hand such a way that all the three Idols get equal flowers. How many flowers you should take and how many flowers you should put on each Idol?

Step–3

You can think of the no. of flowers taken by you as X value and no. of flowers offered to the god as Y value. And adjust the sliders in the page. Simply by thinking you cannot solve the puzzle.

Step–4

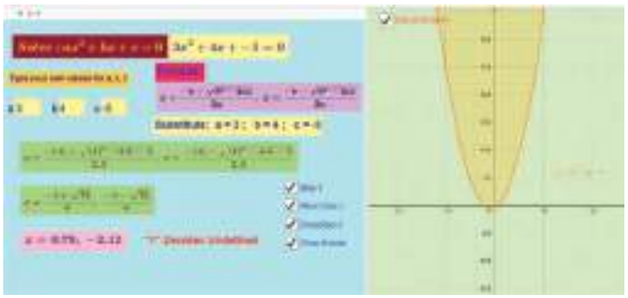
Now is the time for you to recognise the need of algebra. Think of the way to use Algebra. Otherwise Click on the box Show Calculation. Algebra calculation at each level is seen. Now you have to Identify the equation to solve the puzzle. Note: The result will be a ratio. .Step-1

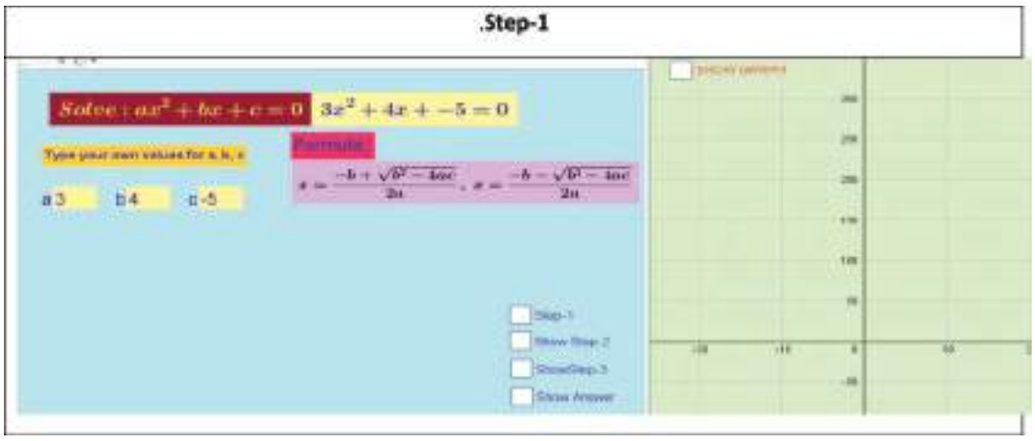


Browse in the link Hill and Flower Puzzle: https://ggbm.at/KmrE5vHsor Scan the QR Code.

ICT CORNER-2(b)

Expected Outcome ⇒





Step–1

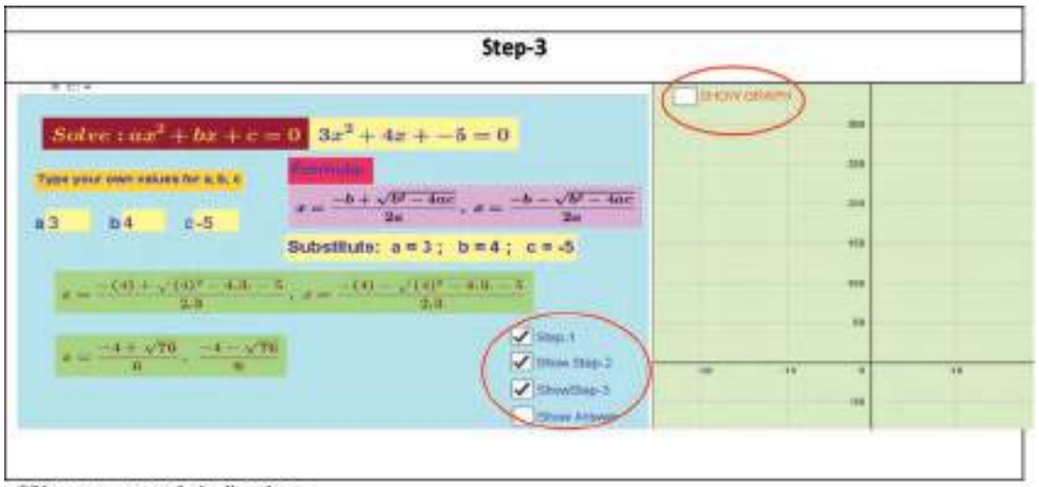
Open the Browser and type the URL Link given below (or) Scan the QR Code. GeoGebra work book named High School Algebra will open. In that several work sheets are given, choose any worksheet you want, for example open the work sheet “Quadratic Equation” solving by formula.

In the work sheet you can enter any value between −20 and 20 for a, b and c. You yourself work out the answer using the formula given.(? mark indicates Undefined answer

Step–2

Now click on the answer to check. You can click the check box one by one to see the steps. Finally, on right hand side click Show Graph to view the graph. Compare the graph with your answer.)

(The curve where it cuts the x-axis is the answer).



*Pictures are only indicatives.

Browse in the link Hill and Flower Puzzle: https://ggbm.at/N4kX9QJqor Scan the QR Code.

Chapter 3 Trigonometry

23L62V

"When I trace at my pleasure the windings to and fro of the heavenly bodies,
I no longer touch the earth with my feet"

Ptolemy

3.1 Introduction

Ptolemy of Alexandaria (AD 90-168)

Trigonometry is primarily a branch of Mathematics that studies relationship involving sides and angles of triangles. The word trigonometry stems from the Greek word trigonon which means triangle and metron which means to measure. So, literally trigonometry is the study of measuring triangles. Greek mathematicians used trigonometric ratios to determine unknown distances. The Egyptians on the other hand used a primitive form of trigonometry for building Pyramids in second millinium BC(BCE). Aristarchus (310-250 BC(BCE)) used trigonometry to determine the distances of Moon and Sun.

Eratosthenes (276-195 BC(BCE)) was t he f irst p erson to c alculate th e ea rth's circumference, whichhedidbyapplyingameasuringsystemusing stadia, a Ptolemy of Alexandaria (AD(CE) 90-168) standard unit of measurement during that period. The general principles of Trigonometry were formulated by the Greek astronomer Hipparchus (190-120 BC(BCE)) and he is credited as the founder of trigonometry. His ideas were used by Ptolemy of Alexandria (AD 90-168) leading to the development of Ptolemy theory of Astronomy. The most significant development of Trigonometry in ancient times was in India. Indian Mathematician and Astronomer Aryabhata (AD(CE) 476-550) defined sine, cosine, inverse cosine, inverse sine and he gave mathematical results in the form of 108 verses which included a formula for the area of a triangle.Mathematicians Brahmagupta (598 AD(CE)), Bhaskara I (600 AD(CE)) and Bhaskara II (1114 AD(CE)) are other Ancient Indians who contributed significantly to develop Trigonometry. Trigonometry was developed as a separate branch of Mathematics through the works of Johann Bernoulli (1667-1748) and Leonhard Euler (1707-1783). Euler established the fundamental results connecting trigonometric functions and complex exponential. Joseph Fourier (1768-1830) made important contribution to the study of trigonometric series. His invention of Fourier series has a wide range of applications especially in vibration analysis, electrical engineering, acoustics, optics, signal processing, image processing and quantum mechanics. In modern times, trigonometric functions are developed as mathematical functions of angular magnitudes, through the medium of which many kinds of geometrical and algebraic investigations are carried out in every branch of Mathematics and applications. Our GPS system in cars and mobile phones is based on trigonometric calculations. Advanced medical scanning procedures such as CT and MRI, used in detecting tumors, involve sine and cosine functions.

Learning Objectives

At the end of this chapter, students are expected to know

- the limitations of right triangle trigonometric ratios as they involve acute angles.
- the necessity for the study of radian measure of an angle and its advantage over degree measure.
- how unit circle is used to define trigonometric functions of real numbers.
- various trigonometric identities, their relationships and applications.
- the principal solution and general solution of a trigonometric equation.
- how to solve trigonometric equations.
- law of sines, law of cosines in triangles and their applications in real life situations.
- how to solve an oblique triangle using law of sines and law of cosines.
- application of Heron's formula and how to compute area of a triangle without finding its altitude.
- the existence of inverse trigonometric functions and their domains and ranges.

Let us recall the basics of trigonometric ratios using acute angles and their properties, which were discussed in earlier classes.

3.2 A recall of basic results

In earlier classes, we have learnt trigonometric ratios using a right triangle and proved trigonometric identities for an acute angle. One wonders, how the distance between planets, heights of Mountains, distance between far off objects like Earth and Sun, heights of tall buildings, the speed of supersonic jets are measured or calculated. Interestingly, such distances or heights are calculated applying the trigonometric ratios which were derived for acute angles. Our aim is to develop trigonometric functions defined for any real number and use them in all branches of mathematics, in particular, in calculus. First, let us recall the definition of angle and degree measure of an angle.

3.2.1 Angles

The angle AOB is a measure formed by two rays OA and OB sharing the common point O as shown in the Figure 3.1. The common point O is called the **vertex** of the angle. If we rotate the ray OA about its vertex O and takes the position OB, then OA and OB respectively are called the initial side and the terminal side of the angle produced. An anticlockwise rotation generates a positive angle (angle with positive sign), while a clockwise rotation generates a negative angle (angle with negative sign).

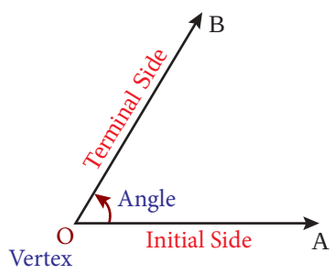


Figure 3.1

One full anticlockwise (or clockwise) rotation of OA back to itself is called one **complete rotation or revolution**.

3.2.2 Different Systems of measurement of angle

There are three different systems for measuring angles.

(i) **Sexagesimal system**

The **Sexagesimal system** is the most prevalent system of measurement where a right angle is divided into 90 equal parts called Degrees. Each degree is divided into 60 equal parts called **Minutes**, and each minute into 60 equal parts called **Seconds.**

The symbols 1°, $1'$ and $1''$ are used to denote a degree, a minute and a second respectively.

(ii) **Centesimal system**

In the **Centesimal system** , the right angle is divided into 100 equal parts, called **Grades**; each grade is subdivided into 100 Minutes, and each minute is subdivided into 100 Seconds. The symbol 1^g is used to denote a grade.

(iii) **Circular system**

In the **circular system** , the radian measure of an angle is introduced using arc lengths in a circle of radius r. Circular system is used in all branches of Mathematics and in other applications in Science. The symbol 1^c is used to denote 1 radian measure.

3.2.3 Degree Measure

The degree is a unit of measurement of angles and is represented by the symbol $^\circ$. In degrees, we split up one complete rotation into 360 equal parts and each part is one degree, denoted by 1°. Thus, 1° is 1/360 of one complete rotation. To measure a fraction of an angle and also for accuracy of measurement of angles, **minutes** and **seconds** are introduced. One minute $(1')$ corresponds to 1/60 of a degree and in turn a second $(1'')$ corresponds to 1/60 of a minute (or) 1/3600 of a degree.

We shall classify a pair of angles in the following way for better understanding and usages.

(i) Two angles that have the exact same measure are called **congruent angles**.
(ii) Two angles that have their measures adding to 90° are called **complementary angles**.
(iii) Two angles that have their measures adding to 180° are called **supplementary angles**.
(iv) Two angles between 0° and 360° are **conjugate** if their sum equals 360°.

(i) The concept of degrees, minutes and seconds, is analogous to the system of time measurement where we think of a degree representing one hour.

(ii) Observe that

$$59.0854^\circ = 59^\circ + 0.0854^\circ$$
$$0.0854^\circ = .0854^\circ \times \frac{60'}{1^\circ} = 5.124'$$
$$5.124' = 5' + 0.124'$$
$$0.124' = 0.124' \times \frac{60''}{1'} = 7.44''$$
$$\text{Thus, } 59.0854^\circ = 59^\circ 5' 7.44''$$

(iii) Also notice that

$$34^\circ 51' 35'' = 34.8597^\circ$$
$$\text{and } 90^\circ - 36^\circ 18' 47'' = 53^\circ 41' 13''$$

3.2.4 Angles in Standard Position

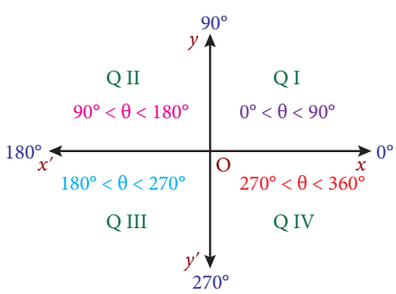


Figure 3.2

An angle is said to be in **standard position** if its vertex is at the origin and its initial side is along the positive x-axis. An angle is said to be in the first quadrant, if in the standard position, its terminal side falls in the first quadrant. Similarly, we can define for the other three quadrants. Angles in standard position having their terminal sides along the x-axis or y-axis are called **quadrantal angles**. Thus, $0^\circ, 90^\circ, 180^\circ, 270^\circ$ and 360° are quadrantal angles.

The degree measurement of a quadrantal angle is a multiple of 90°.

3.2.5 Coterminal angles

One complete rotation of a ray in the anticlockwise direction results in an angle measuring of $360°$. By continuing the anticlockwise rotation, angles larger than $360°$ can be produced. If we rotate in clockwise direction, negative angles are produced. Angles $57°$, $417°$ and $-303°$ have the same initial side and terminal side but with different amount of rotations, such angles are called **coterminal angles**. Thus, angles in standard position that have the same terminal sides are coterminal angles . Hence, if α and β are coterminal angles, then $\beta = \alpha + k(360°)$, k is an integer. The measurements of coterminal angles differ by an integral multiple of $360°$.

For example, $417°$ and $-303°$ are coterminal because $417° - (-303°) = 720° = 2\,(360°)$.

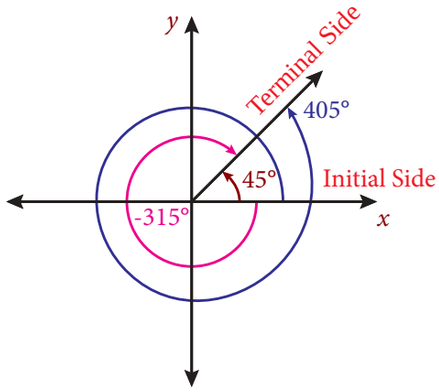


Figure 3.3

(i) Observe that $45°, -315°$ and $405°$ lie in the first quadrant.
(ii) The following pairs of angles are coterminal angles $(30°, 390°); (280°, 1000°)$ and $(-85°, 275°)$.

3.2.6 Basic Trigonometric ratios using a right triangle

We know that six ratios can be formed using the three lengths a, b, c of sides of a right triangle ABC. Interestingly, these ratios lead to the definitions of six basic trigonometric functions.

First, let us recall the trigonometric ratios which are defined with reference to a right triangle.

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \quad ; \quad \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

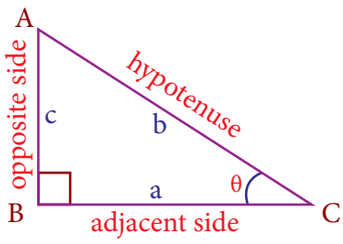


Figure 3.4

With the help of $\sin\theta$ and $\cos\theta$, the remaining trigonometric ratios $\tan\theta$, $\cot\theta$, cosec θ and $\sec\theta$ are determined by using the relations

$$\tan\theta = \frac{\sin\theta}{\cos\theta}\,, \quad \text{cosec}\,\theta = \frac{1}{\sin\theta}\,, \quad \sec\theta = \frac{1}{\cos\theta}\,, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}\,.$$

3.2.7 Exact values of trigonometric functions of widely used angles

Let us list out the values of trigonometric functions at known angles.

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

(i) The values given above are all exact.
(ii) We observe that $\sin 30°$ and $\cos 60°$ are equal. Also $\sin 60°$ and $\cos 30°$ are equal.

(iii) The value of reciprocal ratios namely cosecant, secant and cotangent can be obtained using the above table.
(iv) The result $\cos 90^\circ = 0$ does not allow us to define $\tan 90^\circ$ and $\sec 90^\circ$.
(v) Similarly $\sin 0^\circ = 0$ does not permit us to define cosec 0° and $\cot 0^\circ$.

3.2.8 Basic Trigonometric Identities

A trigonometric identity represents a relationship that is always true for all admissible values in the domain. For example $\sec\theta = \dfrac{1}{\cos\theta}$ is true for all admissible values of θ. Hence, this is an identity. However, $\sin\theta = \frac{1}{2}$ is not an identity, since the relation fails when $\theta = 60^\circ$. Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry which are being used in solving trigonometric equations. The most important part of working with identities, is to manipulate them with the help of a variety of techniques from algebra.

Let us recall the fundamental identities (Pythagorean identities) of trigonometry, namely,

$$\cos^2\theta + \sin^2\theta = 1$$
$$\sec^2\theta - \tan^2\theta = 1$$
$$\text{cosec}^2\theta - \cot^2\theta = 1$$

(i) $\sin^2\theta$ is the commonly used notation for $(\sin\theta)^2$, likewise for other trigonometric ratios.
(ii) $\sec^2\theta - \tan^2\theta = 1$ is meaningless when $\theta = 90^\circ$. But still it is an identity and true for all values of θ for which $\sec\theta$ and $\tan\theta$ are defined. Thus, an identity is an equation that is true for all values of its domain values.
(iii) When we write $\dfrac{\sin\theta}{1+\cos\theta}$, we understand that the expression is valid for all values of θ for which $1+\cos\theta \neq 0$.

Example 3.1 Prove that $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1+\sin\theta}{\cos\theta}$.

Solution:

$$\begin{aligned}\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} &= \frac{\tan\theta + \sec\theta - \left(\sec^2\theta - \tan^2\theta\right)}{\tan\theta - \sec\theta + 1}\\ &= \frac{(\tan\theta + \sec\theta)\left[1 - (\sec\theta - \tan\theta)\right]}{\tan\theta - \sec\theta + 1}\\ &= \tan\theta + \sec\theta = \frac{1+\sin\theta}{\cos\theta}.\end{aligned}$$

Example 3.2 Prove that $(\sec A - \text{cosec}A)(1 + \tan A + \cot A) = \tan A \sec A - \cot A\ \text{cosec}A$.

Solution:

$$\begin{aligned}\text{L.H.S.} &= \left(\frac{1}{\cos A} - \frac{1}{\sin A}\right)\left[1 + \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right]\\ &= \frac{\sin^3 A - \cos^3 A}{\sin^2 A \cos^2 A} \qquad \text{...(i)}\\ \text{R.H.S.} &= \frac{\sin A}{\cos^2 A} - \frac{\cos A}{\sin^2 A} = \frac{\sin^3 A - \cos^3 A}{\sin^2 A \cos^2 A} \qquad \text{...(ii)}\end{aligned}$$

From (i) and (ii), we get the required result.

Example 3.3 Eliminate θ from $a\cos\theta = b$ and $c\sin\theta = d$, where a, b, c, d are constants.

Solution:
Squaring and adding $ac\cos\theta = bc$ and $ac\sin\theta = ad$, we get

$$a^2c^2 = b^2c^2 + a^2d^2.$$

Exercise - 3.1

1. Identify the quadrant in which an angle of each given measure lies

   (i) 25° (ii) 825° (iii) -55° (iv) 328° (v) -230°

2. For each given angle, find a coterminal angle with measure of θ such that $0^\circ \leq \theta < 360^\circ$

   (i) 395° (ii) 525° (iii) 1150° (iv) -270° (v) -450°

3. If $a\cos\theta - b\sin\theta = c$, show that $a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$.

4. If $\sin\theta + \cos\theta = m$, show that $\cos^6\theta + \sin^6\theta = \dfrac{4-3\left(m^2-1\right)^2}{4}$, where $m^2 \leq 2$.

5. If $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$, prove that

   (i) $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$ (ii) $\dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha} = 1.$

6. If $y = \dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}$, then prove that $\dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha} = y$.

7. If $x = \sum_{n=0}^{\infty}\cos^{2n}\theta$, $y = \sum_{n=0}^{\infty}\sin^{2n}\theta$ and $z = \sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{2n}\theta$, $0 < \theta < \dfrac{\pi}{2}$, then show that $xyz = x+y+z$.

   [Hint: Use the formula $1 + x + x^2 + x^3 + \ldots = \dfrac{1}{1-x}$, where $|x| < 1$].

8. If $\tan^2\theta = 1-k^2$, show that $\sec\theta + \tan^3\theta\,\text{cosec}\,\theta = \left(2-k^2\right)^{3/2}$. Also, find the values of k for which this result holds.

9. If $\sec\theta + \tan\theta = p$, obtain the values of $\sec\theta, \tan\theta$ and $\sin\theta$ in terms of p.

10. If $\cot\theta\,(1+\sin\theta) = 4m$ and $\cot\theta\,(1-\sin\theta) = 4n$, then prove that $\left(m^2-n^2\right)^2 = mn$.

11. If $\text{cosec}\,\theta - \sin\theta = a^3$ and $\sec\theta - \cos\theta = b^3$, then prove that $a^2b^2\left(a^2+b^2\right) = 1$.

12. Eliminate θ from the equations $a\sec\theta - c\tan\theta = b$ and $b\sec\theta + d\tan\theta = c$.

3.3 Radian Measure

Initially right triangles were used to define trigonometric ratios and angles were measured in degrees. But right triangles have limitations as they involve only acute angles. In degrees a full rotation corresponds to 360° where the choice of 360 was made thousands of years prior to the Babylonians. They might have chosen 360 based on the number of days in a year. But it does have the nice property of breaking into smaller angles like $30^\circ, 45^\circ, 60^\circ$, 90° and 180°. In 17th century,

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trigonometry was extended to Physics and Chemistry where it required trigonometric functions whose domains were sets of real numbers rather than angles. This was accomplished by using correspondence between an angle and length of the arc on a unit circle. Such a measure of angle is termed as **radian measure** . For theoretical applications, the radian is the most common system of angle measurement. Radians are common unit of measurement in many technical fields, including calculus. The most important irrational number π plays a vital role in radian measures of angles. Let us introduce the radian measure of an angle.

Definition 3.1

The radian measure of an angle is the ratio of the arc length it subtends, to the radius of the circle in which it is the central angle.

Consider a circle of radius r. Let s be the arc length subtending an angle θ at the centre.

Then, $$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r} \text{ radians. Hence, } s = r\theta.$$

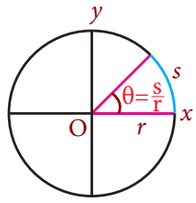


Figure 3.5

(i) All circles are similar. Thus, for a given central angle in any circle, the ratio of the intercepted arc length to the radius is always constant.

(ii) When $s = r$, we have an angle of 1 radian. Thus, one radian is the angle made at the centre of a circle by an arc with length equal to the radius of the circle.

(iii) Since the lengths s and r have same unit, θ is unitless and thus, we do not use any notation to denote radians.

(iv) $\theta = 1$ radian measure, if $s = r$
$\theta = 2$ radian measure, if $s = 2r$
Thus, in general $\theta = k$ radian measure, if $s = kr$.
Hence, radian measure of an angle tells us how many radius lengths, we need to sweep out along the circle to subtend the angle θ.

(v) Radian angle measurement can be related to the edge of the unit circle. In radian system, we measure an angle by measuring the distance travelled along the edge of the unit circle to where the terminal side of the angle intersect the unit circle .

3.3.1 Relationship between Degree and Radian Measures

We have degree and radian units to measure angles. One measuring unit is better than another if it can be defined in a simpler and more intuitive way. For example, in measuring temperature, Celsius unit is better than Fahrenheit as Celsius was defined using 0° and 100° for freezing and boiling points of water. Radian measure is better for conversion and calculations. Radian measure is more convenient for analysis whereas degree measure of an angle is more convenient to communicate the concept between people. Greek Mathematicians observed the relation of π which arises from circumference of a circle and thus, π plays a crucial role in radian measure.

In unit circle, a full rotation corresponds to 360° whereas, a full rotation is related to 2π radians, the circumference of the unit circle. Thus, we have the following relations:

$$2\pi \text{ radians } = 360^\circ, \text{ which reduces to } \pi \text{ radians } = 180^\circ.$$

$$\text{Thus, } 1 \text{ radian } = \left(\frac{180}{\pi}\right)^\circ \quad \text{or} \quad 1^\circ = \frac{\pi}{180} \text{ radians.}$$

$$x \text{ radian } = \left(\frac{180x}{\pi}\right)^\circ \quad \text{or} \quad x^\circ = \frac{\pi x}{180} \text{ radians.}$$

Observe that the scale used in radians is much smaller than the scale in degrees. The smaller scale makes the graphs of trigonometric functions more visible and usable. The above relation gives a way to convert radians into degrees or degrees into radians.

(i) The ratio of the circumference of any circle to its diameter is always a constant. This constant is denoted by the irrational number π.

(ii) Mark a point P on a unit circle and put the unit circle on the number line so that P touches the number 0. Allow the circle to roll along the number line. The point P will

touch the number 2π on the number line when the circle rolls to one complete revolution to the right.

(iii) If the unit of angle measure is not specified, then the angle is understood to be in radians.

(iv) Consider a sector of a circle with radius r. If θ is the central angle of the sector, then

$$\text{Area of the sector} = \begin{cases} \left(\dfrac{\pi r^2}{360^\circ}\right)\theta & \text{in degree measure} \\ \left(\dfrac{\pi r^2}{2\pi}\right)\theta = \dfrac{r^2\theta}{2} & \text{in radian measure} \end{cases}$$

Clearly, the calculation in radian measure is much easier to work with.

(v) The values of π and $\dfrac{22}{7}$ correct to four decimal places are 3.1416 and 3.1429 respectively. Thus, π and $\dfrac{22}{7}$ are approximately equal correct upto two decimal places. Hence, $\pi \approx \dfrac{22}{7}$.

(vi)

$$\begin{aligned} 1 \text{ radian} &\approx 57^\circ 17' 45'' \text{ and } 1^\circ \approx 0.017453 \text{ radian} \\ 1' &= \left(\frac{\pi}{180 \times 60}\right) \text{radian} \approx 0.000291 \text{ radian.} \\ 1'' &= \left(\frac{\pi}{180 \times 60 \times 60}\right) \text{radian} \approx 0.000005 \text{ radian.} \end{aligned}$$

(vii) The radian measures and the corresponding degree measures for some known angles are given in the following table

Radians	0	1	0.017453	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Degrees	0°	$57^\circ 17' 45''$	1°	30°	45°	60°	90°	180°	270°	360°

(viii) $\sin 90^\circ = 1$ but $\sin 90 \neq 1$ (in radian measure).

Example 3.4 Convert (i) 18° to radians (ii)-108° to radians.

Solution:

Now, $180^\circ = \pi$ radians gives $1^\circ = \dfrac{\pi}{180}$ radians

(i) $18^\circ = \dfrac{\pi}{180} \times 18$ radians $= \dfrac{\pi}{10}$ radians

(ii) $-108^\circ = \dfrac{\pi}{180} \times (-108)$ radians $= -\dfrac{3\pi}{5}$ radians.

Example 3.5 Convert (i) $\dfrac{\pi}{5}$ radians to degrees (ii) 6 radians to degrees.

Solution:

We know that π radians $= 180^\circ$ and thus,

(i) $\dfrac{\pi}{5}$ radians $= \dfrac{180^\circ}{5} = 36^\circ$

(ii) 6 radians $= \left(\dfrac{180}{\pi} \times 6\right)^\circ \approx \left(\dfrac{7 \times 180}{22} \times 6\right)^\circ = \left(343\dfrac{7}{11}\right)^\circ$.

Example 3.6 Find the length of an arc of a circle of radius $5\ cm$ subtending a central angle measuring $15°$.

Solution:
Let s be the length of the arc of a circle of radius r subtending a central angle θ.
Then $s = r\theta$.

We have, $\theta = 15° = 15 \times \frac{\pi}{180} = \frac{\pi}{12}$ radians

So that, $s = r\theta$ gives $s = 5 \times \frac{\pi}{12} = \frac{5\pi}{12}\ cm$

In the product $r\theta$, θ must always be in radians.

Example 3.7 If the arcs of same lengths in two circles subtend central angles $30°$ and $80°$, find the ratio of their radii.

Solution:
Let r_1 and r_2 be the radii of the two given circles and l be the length of the arc.

$\theta_1 = 30° = \frac{\pi}{6}$ radians

$\theta_2 = 80° = \frac{4\pi}{9}$ radians

Given that $l = r_1\theta = r_2\theta$

Thus, $\frac{\pi}{6} r_1 = \frac{4\pi}{9} r_2$

$\frac{r_1}{r_2} = \frac{8}{3}$ which implies $r_1 : r_2 = 8 : 3$.

Exercise - 3.2

1. Express each of the following angles in radian measure:

   (i) $30°$ (ii) $135°$ (iii) $-205°$ (iv) $150°$ (v) $330°$.

2. Find the degree measure corresponding to the following radian measures

   (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{9}$ (iii) $\frac{2\pi}{5}$ (iv) $\frac{7\pi}{3}$ (v) $\frac{10\pi}{9}$.

3. What must be the radius of a circular running path, around which an athlete must run 5 times in order to describe $1\ km$?
4. In a circle of diameter $40\ cm$, a chord is of length $20\ cm$. Find the length of the minor arc of the chord.
5. Find the degree measure of the angle subtended at the centre of circle of radius $100\ cm$ by an arc of length $22\ cm$.
6. What is the length of the arc intercepted by a central angle of measure $41°$ in a circle of radius $10\ ft$?
7. If in two circles, arcs of the same length subtend angles $60°$ and $75°$ at the centre, find the ratio of their radii.

8. The perimeter of a certain sector of a circle is equal to the length of the arc of a semi-circle having the same radius. Express the angle of the sector in degrees, minutes and seconds.
9. An airplane propeller rotates 1000 times per minute. Find the number of degrees that a point on the edge of the propeller will rotate in 1 second.
10. A train is moving on a circular track of $1500\ m$ radius at the rate of $66\ km/hr$. What angle will it turn in 20 seconds?
11. A circular metallic plate of radius $8\ cm$ and thickness $6\ mm$ is melted and molded into a pie (a sector of the circle with thickness) of radius $16\ cm$ and thickness $4\ mm$. Find the angle of the sector.

3.4 Trigonometric functions and their properties

3.4.1 Trigonometric Functions of any angle in terms of Cartesian coordinates

We have studied the principles of trigonometric ratios in the lower classes using acute angles . But we come across many angles which are not acute. We shall extend the acute angle idea and define trigonometric functions for any angle. The trigonometric ratios to any angle in terms of radian measure are called trigonometric functions.

Let $P(x, y)$ be a point other than the origin on the terminal side of an angle θ in standard position . Let $OP = r$. Thus, $r = \sqrt{x^2 + y^2}$

The six trigonometric functions of θ are defined as follows:

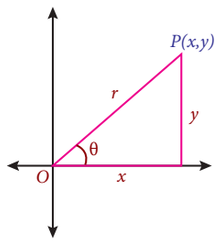


Figure 3.6

$$\sin\theta = \frac{y}{r} \text{ and } \cos\theta = \frac{x}{r}.$$

Using this, we have $\tan\theta = \dfrac{y}{x}, x \neq 0$; $\cot\theta = \dfrac{x}{y}, y \neq 0$; $\text{cosec}\ \theta = \dfrac{r}{y}, y \neq 0$; $\sec\theta = \dfrac{r}{x}, x \neq 0$.

(i) Since $|x| \leq r, |y| \leq r$, we have $|\sin\theta| \leq 1$ and $|\cos\theta| \leq 1$.
(ii) In the case of acute angle, the above definitions are equivalent to our earlier definitions using right triangle.
(iii) The trigonometric functions have positive or negative values depending on the quadrant in which the point $P(x, y)$ on the terminal side of θ lies.
(iv) The above definitions of trigonometric functions is independent of the points on the terminal side of the angle. (verify!)

Trigonometric ratios of Quadrantal angles

Let us recall that, an angle in its standard position for which the terminal side coincides with one of the axes, is a Quadrantal angle. We shall find the trigonometric ratios for the quadrantal angles.

Consider the unit circle $x^2 + y^2 = 1$. Let $P(x, y)$ be a point on the unit circle where the terminal side of the angle θ intersects the unit circle.

Then $\cos\theta = \dfrac{x}{1} = x$ (x-coordinate of P) and

$\sin\theta = \dfrac{y}{1} = y$ (y-coordinate of P)

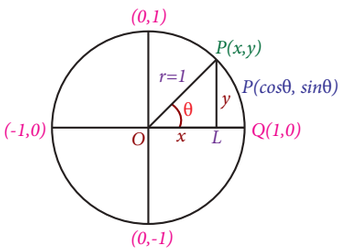


Figure 3.7

Thus, the coordinates of any point $P(x, y)$ on the unit circle is $P(\cos\theta, \sin\theta)$. In this way, the angle measure θ is associated with a point on the unit circle.

The following table illustrates how trigonometric function values are determined for a Quadrantal angles using the above explanation.

Exact values of trigonometric functions of quadrantal angles.

Quadrantal angle	Corresponding point on the Unit circle $P(x, y) = P(\cos\theta, \sin\theta)$	cosine value $\cos\theta$	sine value $\sin\theta$
$\theta = 0^\circ$	$(1, 0) = (\cos 0^\circ, \sin 0^\circ)$	$\cos 0^\circ = 1$	$\sin 0^\circ = 0$
$\theta = 90^\circ$	$(0, 1) = (\cos 90^\circ, \sin 90^\circ)$	$\cos 90^\circ = 0$	$\sin 90^\circ = 1$
$\theta = 180^\circ$	$(-1, 0) = (\cos 180^\circ, \sin 180^\circ)$	$\cos 180^\circ = -1$	$\sin 180^\circ = 0$
$\theta = 270^\circ$	$(0, -1) = (\cos 270^\circ, \sin 270^\circ)$	$\cos 270^\circ = 0$	$\sin 270^\circ = -1$
$\theta = 360^\circ$	$(1, 0) = (\cos 360^\circ, \sin 360^\circ)$	$\cos 360^\circ = 1$	$\sin 360^\circ = 0$

(i) Observe that x and y coordinates of all points on the unit circle lie between -1 and 1. Hence, $-1 \leq \cos\theta \leq 1$, $-1 \leq \sin\theta \leq 1$, no matter whatever be the value of θ.

(ii) When $\theta = 360^\circ$, we have completed one full rotation. Thus, the terminal side coincides with positive x- axis. Hence, sine has equal values at 0° and at 360°. Cosine and other trigonometric functions also follow it.

(iii) If two angles differ by an integral multiple of 360° or 2π, then each trigonometric function will have equal values at both angles.

(iv) Using the values of sine and cosine at quadrantal angles, we have the following generalization geometrically.

At Quadrantal angle	Justification	Generalization
$\sin 0 = 0$ $\sin \pi = 0$	$\sin(0 + 2n\pi) = 0; n \in \mathbb{Z}$ $\sin(\pi + 2n\pi) = 0; n \in \mathbb{Z}$	$\sin\theta = 0 \Rightarrow \theta = n\pi; n \in \mathbb{Z}$
$\cos\frac{\pi}{2} = 0$ $\cos\frac{3\pi}{2} = 0$	$\cos\left(\frac{\pi}{2} + 2n\pi\right) = 0; n \in \mathbb{Z}$ $\cos\left(\frac{3\pi}{2} + 2n\pi\right) = 0; n \in \mathbb{Z}$	$\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}; n \in \mathbb{Z}$
$\tan 0 = 0$ $\tan \pi = 0$	$\tan(0 + 2n\pi) = 0; n \in \mathbb{Z}$ $\tan(\pi + 2n\pi) = 0; n \in \mathbb{Z}$	$\tan\theta = 0 \Rightarrow \theta = n\pi; n \in \mathbb{Z}$

(v) $\tan\theta$ is not defined when $\cos\theta = 0$ and so, $\tan\theta$ is not defined when $\theta = (2n+1)\dfrac{\pi}{2}$, $n \in \mathbb{Z}$.

3.4.2 Trigonometric Functions of real numbers

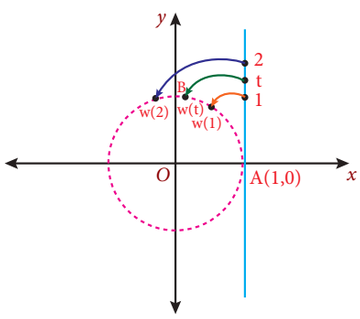


Figure 3.8

For applications of trigonometry to the problems in higher mathematics including calculus and to problems in physics and chemistry, scientists required trigonometric functions of real numbers. This was skillfully done by exhibiting a correspondence between an angle and an arc length denoting a real number on a unit circle.

Consider a unit circle with the centre at the origin. Let the angle zero (in radian measure) be associated with the point $A(1, 0)$ on the unit circle. Draw a tangent to the unit circle at the point $A(1, 0)$. Let t be a real number such that t is y- coordinate of a point on the tangent line.

For each real number t, identify a point $B(x, y)$ on the unit circle such that the arc length AB is equal to t. If t is positive, choose the point $B(x, y)$ in the anticlockwise direction, otherwise choose it in the clockwise direction. Let θ be the angle subtended by the arc AB at the centre. In this way, we have a function w(t) associating a real number t to a point on the unit circle. Such a function is called a **wrapping function** . Then $s = r\theta$ gives arc length $t = \theta$.

Now, define $\sin t = \sin\theta$ and $\cos t = \cos\theta$.

Clearly, $\sin t = \sin\theta = y$ and $\cos t = \cos\theta = x$.

Using $\sin t$ and $\cos t$, other trigonometric functions can be defined as functions of real numbers.

(i) $B(x, y) = B(\cos t, \sin t)$ is a point on the unit circle.
Thus, $-1 \leq \cos t \leq 1$ and $-1 \leq \sin t \leq 1$ for any real number t.

(ii) Wrapping function w(t) is analogous to wrapping a line around a circle.

(iii) The value of a trigonometric function of a real number t is its value at the angle t radians.

(iv) Trigonometric functions of real numbers are used to model phenomena like waves, oscillations, that occur in regular intervals.

Example 3.8 The terminal side of an angle θ in standard position passes through the point $(3, -4)$. Find the six trigonometric function values at an angle θ.

Solution:

Let $B(x, y) = B(3, -4)$, OA be the initial side and OB be the terminal side of the angle θ in the standard position.

Then $\angle AOB$ is the angle θ and θ lies in the IV quadrant. Also,

$$OB = r,\ r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-4)^2} = 5$$

$$x = 3,\ \ y = -4 \text{ and } r = 5, \text{ we have}$$

$$\sin\theta = \frac{y}{r} = -\frac{4}{5};\quad \cos\theta = \frac{x}{r} = \frac{3}{5};\quad \tan\theta = \frac{y}{x} = -\frac{4}{3};$$

$$\text{cosec}\,\theta = \frac{r}{y} = -\frac{5}{4};\quad \sec\theta = \frac{r}{x} = \frac{5}{3};\quad \cot\theta = \frac{x}{y} = -\frac{3}{4}.$$

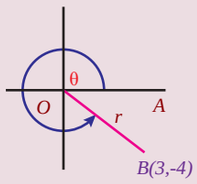


Figure 3.9

Signs of Trigonometric functions

Consider a unit circle with centre at the origin. Let θ be in standard position. Let $P(x, y)$ be the point on the unit circle corresponding to the angle θ. Then, $\cos\theta = x$, $\sin\theta = y$ and $\tan\theta = \frac{y}{x}$. The values of x and y are positive or negative depending on the quadrant in which P lies.

In the first quadrant:

$\cos\theta = x > 0$ (positive); $\sin\theta = y > 0$ (positive)

Thus, $\cos\theta$ and $\sin\theta$ and hence all trigonometric functions are positive in the first quadrant.

In the second quadrant:

$\cos\theta = x < 0$ (negative); $\sin\theta = y > 0$ (positive)

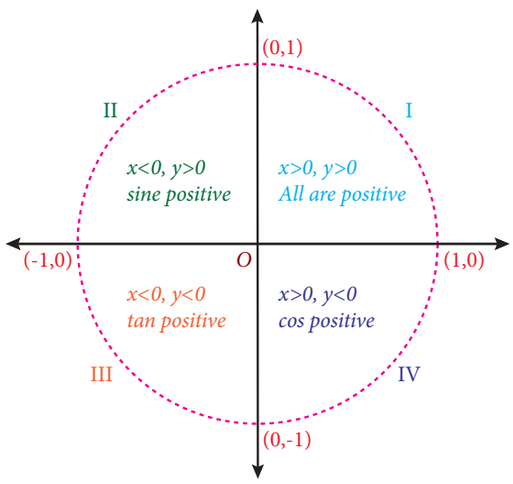


Figure 3.10

Thus, $\sin\theta$ and cosec θ are positive and others are negative.
Similarly, we can find the sign of trigonometric functions in other two quadrants.
Let us illustrate the above discussions in Figure 3.10.

Signs of trigonometric functions in various quadrants can be remembered with the slogan

"All Students Take Chocolate" (ASTC rule)

Example 3.9 If $\sin\theta = \frac{3}{5}$ and the angle θ is in the second quadrant, then find the values of other five trigonometric functions.

Solution:

We know that $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\sqrt{1-\frac{9}{25}} = \pm\frac{4}{5}$

Thus, $\cos\theta = -\frac{4}{5}$ as $\cos\theta$ is negative in the second quadrant.

$$\sin\theta = \frac{3}{5} \Rightarrow \text{cosec}\,\theta = \frac{5}{3}; \quad \cos\theta = -\frac{4}{5} \Rightarrow \sec\theta = -\frac{5}{4}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{3}{4}; \quad \cot\theta = -\frac{4}{3}$$

If $\sin\theta$ and $\cos\theta$ are known, then the reciprocal identities and quotient identities can be used to find the other four trigonometric values. The Pythagorean identities can be used to find trigonometric values when one trigonometric value and the quadrant are known.

3.4.3 Allied Angles

Two angles are said to be allied if their sum or difference is a multiple of $\frac{\pi}{2}$ radians.

Thus, any two angles of θ such as, $\pm\theta, \frac{\pi}{2}\pm\theta, \pi\pm\theta, \frac{3\pi}{2}\pm\theta, \ldots$, are all allied angles .

Now we shall find the trigonometric ratios involving allied angles θ and $-\theta$.

Trigonometric ratios of $-\theta$ in terms of θ

Let $\angle AOL = \theta$ and $\angle AOM = -\theta$. Let $P(a,b)$ be a point on OL. Choose a point P' on OM such that $OP = OP'$.

Draw PN perpendicular to OA intersecting OM at P'. Since $\angle AOP = \angle AOP'$ and $\angle PON = \angle P'ON$ $\triangle PON$ and $\triangle P'ON$ are congruent.

Thus, $PN = P'N$ and hence the point P' is given by $P'(a,-b)$

Now, by the definition of trigonometric functions

$$\sin\theta = \frac{b}{OP}, \ \cos\theta = \frac{a}{OP}, \ \tan\theta = \frac{b}{a}$$

Thus, $\sin(-\theta) = \frac{-b}{OP'} = \frac{-b}{OP} = -\sin\theta$

and $\cos(-\theta) = \frac{a}{OP'} = \frac{a}{OP} = \cos\theta$

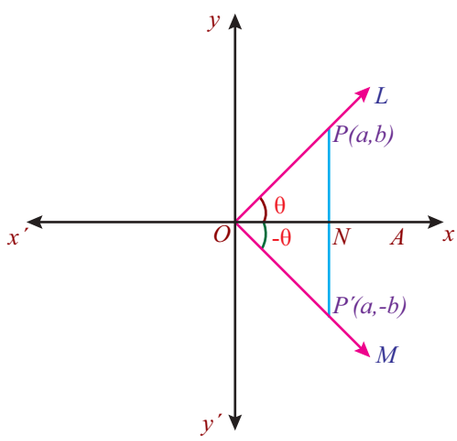


Figure 3.11

Then, it is easy to get

$$\tan(-\theta) = -\tan\theta,\ \text{cosec}(-\theta) = -\text{cosec}\,\theta,\ \sec(-\theta) = \sec\theta,\ \cot(-\theta) = -\cot\theta.$$

(i) $\sin(-\theta) = -\sin\theta$, and $\cos(-\theta) = \cos\theta$. These facts follow from the symmetry of the unit circle about the x-axis. The angle $-\theta$ is the same as angle θ except it is on the other side of the x- axis. Flipping a point (x, y) to the other side of the x- axis makes the point into $(x, -y)$, so the y-coordinate is negated and hence the sine is negated, but the x-coordinate remains the same and therefore the cosine is unchanged.

(ii) The negative-angle identities can be used to determine if a trigonometric function is an odd function or an even function .

Example 3.10 Find the values of (i) $\sin(-45^\circ)$ (ii) $\cos(-45^\circ)$ (iii) $\cot(-45^\circ)$

Solution:

(i) $\sin(-45^\circ) = -\sin(45^\circ) = -\dfrac{1}{\sqrt{2}}$.

(ii) $\cos(-45^\circ) = \dfrac{1}{\sqrt{2}}$ and $(iii)\ \cot(-45^\circ) = -1$

We have already learnt the trigonometric ratios of the angle $(90^\circ - \theta)$, $\left(0 < \theta < \dfrac{\pi}{2}\right)$ in the lower class. Let us recall the trigonometric ratios of angle $(90^\circ - \theta)$;

$$\sin(90^\circ - \theta) = \cos\theta,\quad \cos(90^\circ - \theta) = \sin\theta,\quad \tan(90^\circ - \theta) = \cot\theta$$
$$\text{cosec}(90^\circ - \theta) = \sec\theta,\quad \sec(90^\circ - \theta) = \text{cosec}\,\theta,\quad \cot(90^\circ - \theta) = \tan\theta.$$

Now, we will establish the corresponding trigonometric ratios for an angle of the form $(90^\circ + \theta)$.

Trigonometric ratios of an angle of the form $(90^\circ + \theta)$, $\left(0 < \theta < \dfrac{\pi}{2}\right)$ in terms of θ .

Let $\angle AOL = \theta$ and $\angle AOR = (90^\circ + \theta)$. Let $P(a, b)$ be a point on OL and choose a point P' on OR such that $OP = OP'$.

Draw perpendiculars PM and $P'N$ from P and P' on Ox and Ox' respectively.

Now, $\angle AOP' = 90^\circ + \theta$.

Clearly, $\triangle OPM$ and $\triangle P'ON$ are congruent.

$$ON = MP \text{ and } NP' = OM$$

Hence, the coordinates of P and P' are $P(a, b)$ and $P'(-b, a)$, respectively. Now

$$\sin(90^\circ + \theta) = \frac{y\text{-coordinate of } P'}{OP'} = \frac{a}{OP} = \cos\theta,$$
$$\cos(90^\circ + \theta) = \frac{x\text{-coordinate of } P'}{OP'} = \frac{-b}{OP} = -\sin\theta,$$

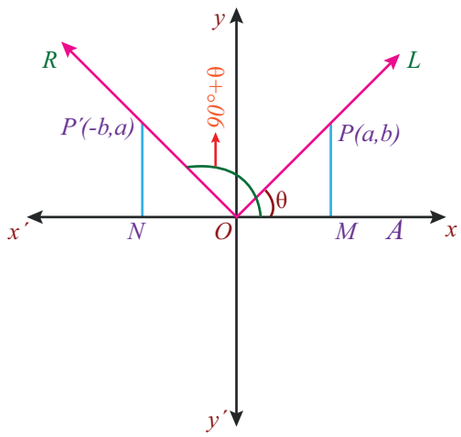


Figure 3.12

Thus, $\tan(90^\circ+\theta) = -\cot\theta$, $\text{cosec}(90^\circ+\theta) = \sec\theta$, $\sec(90^\circ+\theta) = -\text{cosec}\theta$, $\cot(90^\circ+\theta) = -\tan\theta$.

The trigonometric function of other allied angles $\pi \pm \theta$, $\dfrac{3\pi}{2} \pm \theta$, $2\pi \pm \theta$ can be obtained in a similar way.

The above results can be summarized in the following table: $\left(\text{Here } 0 < \theta < \dfrac{\pi}{2}\right)$

	$-\theta$	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$	$2\pi+\theta$
sine	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
cosine	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tangent	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$

(i) The corresponding reciprocal ratios can be written using the above table.

(ii) If the allied angles are $\pm\theta, \pi\pm\theta, 2\pi\pm\theta$, that is, angles of the form $2n\frac{\pi}{2}\pm\theta, n\in\mathbb{Z}$, then, the form of trigonometric ratio is unaltered (i.e., sine remains sine, cosine remains cosine etc.,)

(iii) If the allied angles are $\frac{\pi}{2}\pm\theta, \frac{3\pi}{2}\pm\theta$, that is, angles of the form $(2n+1)\frac{\pi}{2}\pm\theta, n\in\mathbb{Z}$ then, the form of trigonometric ratio is altered to its complementary ratio. $i.e.$, it is to add the prefix "co" if it is absent and remove the prefix "co" if it is already present ($i.e.$, sine becomes cosine, cosine become sine etc.,)

(iv) For determining the sign, first find out the quadrant and then attach the appropriate sign (+ or $-$) according to the quadrant rule "ASTC".

Example 3.11 Find the value of (i) $\sin 150^\circ$ (ii) $\cos 135^\circ$ (iii) $\tan 120^\circ$.

Solution:

(i) $\sin 150^\circ = \sin(90^\circ+60^\circ) = \cos(60^\circ) = \frac{1}{2}$

(or) $\sin 150^\circ = \sin(180^\circ-30^\circ) = \sin(30^\circ) = \frac{1}{2}$

(ii) $\cos 135^\circ = \cos(90^\circ+45^\circ) = -\sin(45^\circ) = -\frac{1}{\sqrt{2}}$

(or) $\cos 135^\circ = \cos(180^\circ-45^\circ) = -\cos(45^\circ) = -\frac{1}{\sqrt{2}}$

(iii) $\tan 120^\circ = \tan(180^\circ-60^\circ) = -\tan(60^\circ) = -\sqrt{3}$

(or) write $\tan 120^\circ$ as $\tan(90^\circ+30^\circ)$ and find the value.

Example 3.12 Find the value of:

(i) $\sin(765^\circ)$ (ii) cosec (-1410°) (iii) $\cot\left(\frac{-15\pi}{4}\right)$.

Solution:

(i) $\sin 765^\circ = \sin(2\times 360^\circ+45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$

(ii) cosec $(-1410^\circ) = -$cosec $(1410^\circ) = -$cosec $(4\times 360^\circ-30^\circ) =$ cosec $30^\circ = 2$

(iii) $\cot\left(\frac{-15\pi}{4}\right) = -\cot\left(\frac{15\pi}{4}\right) = -\cot\left(4\pi-\frac{\pi}{4}\right) = \cot\frac{\pi}{4} = 1.$

Example 3.13 Prove that $\tan(315^\circ)\cot(-405^\circ)+\cot(495^\circ)\tan(-585^\circ) = 2$

Solution:

$$\begin{aligned}\text{L.H.S} &= \tan(360^\circ-45^\circ)[-\cot(360^\circ+45^\circ)] + \cot(360^\circ+135^\circ)[-\tan(360^\circ+225^\circ)]\\ &= [-\tan 45^\circ][-\cot 45^\circ] + [-\tan 45^\circ][-\tan 45^\circ] = (-1)(-1)+(-1)(-1) = 2.\end{aligned}$$

3.4.4 Some Characteristics of Trigonometric Functions

Trigonometric functions have some nice properties. For example,

(i) Sine and cosine functions are complementary to each other in the sense that $\sin(90^\circ - \theta) = \cos\theta$ and $\cos(90^\circ - \theta) = \sin\theta$.

(ii) As $\cos\theta$ and $\sin\theta$ are obtained as coordinates of a point on the unit circle, they satisfy the inequalities $-1 \leq \cos\theta \leq 1$ and $-1 \leq \sin\theta \leq 1$. Hence, $\cos\theta, \sin\theta \in [-1, 1]$

(iii) Trigonometric function repeats its values in regular intervals.

(iv) Sine and cosine functions have an interesting property that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

Let us discuss the last two properties.

Periodicity of Trigonometric Functions

We know that a function f is said to be a periodic function with period p, if there exists a smallest positive number p such that $f(x+p) = f(x)$ for all x in the domain.

For example, $\sin(x + 2n\pi) = \sin x, n \in \mathbb{Z}$.

$i.e., \sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi) = \ldots = \sin x$

Thus, $\sin x$ is a periodic function with period 2π.

Similarly, $\cos x$, cosec$x$ and $\sec x$ are periodic functions with period 2π.

But $\tan x$ and $\cot x$ are periodic functions with period π.

The periodicity of $\sin x$ and $\cos x$ can be viewed best using their graphs.

(i) **The graph of the sine function**

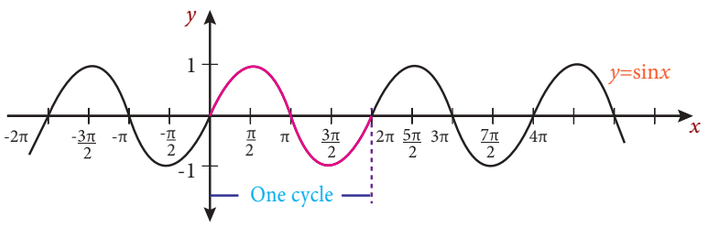


Figure 3.13: y =sinx

Here x represents a variable angle. Take the horizontal axis to be the x-axis and vertical axis to be the y-axis. Graph of the function $y = \sin x$ is shown in the Figure 3.13. First, note that it is periodic of period 2π. Geometrically it means that if you take the curve and slide it 2π either left or right, then the curve falls back on itself. Second, note that the graph is within one unit of the y-axis. The graph increases and decreases periodically. For instance, increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and decreases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

(ii) **The graph of the cosine function**

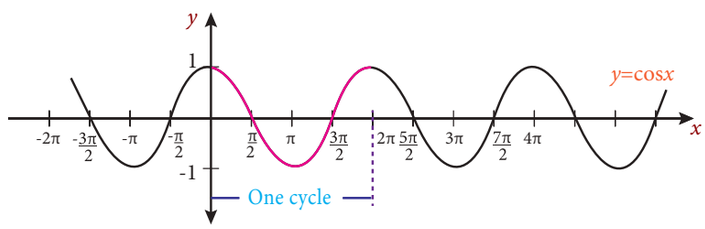


Figure 3.14: y = cosx

Observe that the graph of $y = \cos x$ looks just like the graph of $y = \sin x$ except it is being translated to the left by $\frac{\pi}{2}$. This is because of the identity $\cos x = \sin\left(\frac{\pi}{2} + x\right)$. It easily follows from the graph that $\cos x = \cos(-x) = \sin\left(\frac{\pi}{2} + x\right)$

(i) The sine and cosine functions are useful for one very important reason, since they repeat in a regular pattern ($i.e.$, they are periodic). There are a vast array of things in and around us that repeat periodically. For example, the rising and setting of the sun, the motion of a spring up and down, the tides of the ocean and so on, are repeating at regular intervals of time. All periodic behaviour can be studied through combinations of the sine and cosine functions.

(ii) Periodic functions are used throughout science to describe oscillations, waves and other phenomena that occur periodically.

Odd and Even trigonometric functions

Even and odd functions are functions satisfying certain symmetries. A real valued function $f(x)$ is an even function if it satisfies $f(-x) = f(x)$ for all real number x and an odd function if it satisfies $f(-x) = -f(x)$ for all real number x.

Basic trigonometric functions are examples of non-polynomial even and odd functions

Because $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$ for all x, it follows that $\cos x$ is an even function and $\sin x$ is an odd function.

Also note that $\sec x$ is an even function while $\tan x$, cosec x and $\cot x$ are all odd functions.

However, $f(t) = t - \cos t$ is neither even function nor odd function (why ?)

Example 3.14 Determine whether the following functions are even, odd or neither.

(i) $\sin^2 x - 2\cos^2 x - \cos x$ (ii) $\sin(\cos(x))$ (iii) $\cos(\sin(x))$ (iv) $\sin x + \cos x$

Solution:

(i) Let $f(x) = \sin^2 x - 2\cos^2 x - \cos x$

$f(-x) = f(x)$ [since $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$]

Thus, $f(x)$ is even.

(ii) Let $f(x) = \sin(\cos(x))$

$f(-x) = f(x)$, $f(x)$ is an even function.

(iii) $f(x) = \cos(\sin(x))$, $f(-x) = f(x)$, Thus, $f(x)$ is an even function.

(iv) Let $f(x) = \sin x + \cos x$

$f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

Thus, $f(x) = \sin x + \cos x$ is neither even nor odd.

(i) In general, a function is an even function if its graph is unchanged under reflection about the y-axis. A function is odd if its graph is symmetric about the origin.

(ii) The properties of even and odd functions are useful in analyzing trigonometric functions particularly in the sum and difference formula.

(iii) The properties of even and odd functions are useful in evaluating some definite integrals, which we will see in calculus.

Exercise - 3.3

1. Find the values of (i) $\sin(480^\circ)$ (ii) $\sin(-1110^\circ)$ (iii) $\cos(300^\circ)$ (iv) $\tan(1050^\circ)$ (v) $\cot(660^\circ)$ (vi) $\tan\left(\frac{19\pi}{3}\right)$ (vii) $\sin\left(-\frac{11\pi}{3}\right)$.
2. $\left(\frac{5}{7}, \frac{2\sqrt{6}}{7}\right)$ is a point on the terminal side of an angle θ in standard position. Determine the six trigonometric function values of angle θ.
3. Find the values of other five trigonometric functions for the following:
   (i) $\cos\theta = -\frac{1}{2}$, θ lies in the III quadrant.
   (ii) $\cos\theta = \frac{2}{3}$, θ lies in the I quadrant.
   (iii) $\sin\theta = -\frac{2}{3}$, θ lies in the IV quadrant.
   (iv) $\tan\theta = -2$, θ lies in the II quadrant.
   (v) $\sec\theta = \frac{13}{5}$, θ lies in the IV quadrant.
4. Prove that $\dfrac{\cot(180^\circ+\theta)\sin(90^\circ-\theta)\cos(-\theta)}{\sin(270^\circ+\theta)\tan(-\theta)\text{cosec}(360^\circ+\theta)} = \cos^2\theta\cot\theta$.
5. Find all the angles between 0° and 360° which satisfy the equation $\sin^2\theta = \frac{3}{4}$.
6. Show that $\sin^2\frac{\pi}{18} + \sin^2\frac{\pi}{9} + \sin^2\frac{7\pi}{18} + \sin^2\frac{4\pi}{9} = 2$.

3.5 Trigonometric Identities

3.5.1 Sum and difference identities or compound angles formulas

Now, compound angles are algebraic sum of two or more angles. Trigonometric functions do not satisfy the functional relations like $f(x+y) = f(x) + f(y)$ and $f(kx) = kf(x)$, k is a real number. For example, $\cos(\alpha+\beta) \neq \cos\alpha + \cos\beta$, $\sin(2\alpha) \neq 2\sin\alpha$, $\tan 3\alpha \neq 3\tan\alpha, \ldots$. Thus, we need to derive formulas for $\sin(\alpha+\beta), \cos(\alpha+\beta), \ldots$ and use them in calculations of application problems.

Music is made up of vibrations that create pressure on ear-drums. Musical tones can be modeled with sinusoidal graphs (graphs looks like that of $y = \sin x$ or $y = \cos x$). When more than one tone is played, the resulting pressure is equal to the sum of the individual pressures. In this context sum and difference trigonometric identities are used as an important application. Also, sum and difference trigonometric identities are helpful in the analysis of waves.

First we shall prove the identity for the cosine of the sum of two angles and extend it to prove all other sum or difference identities.

Identity 3.1: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Proof. Consider the unit circle with centre at O. Let $P = P(1,0)$.

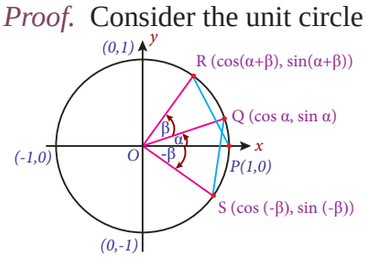


Figure 3.15

Let Q, R and S be points on the unit circle such that $\angle POQ = \alpha$, $\angle POR = \alpha+\beta$ and $\angle POS = -\beta$ as shown in the Figure 3.15. Clearly, angles $\alpha, \alpha+\beta$ and $-\beta$ are in standard positions. Now, the points Q, R and S are given by $Q(\cos\alpha, \sin\alpha)$, $R(\cos(\alpha+\beta), \sin(\alpha+\beta))$ and $S(\cos(-\beta), \sin(-\beta))$.

Since $\triangle POR$ and $\triangle SOQ$ are congruent. So, $PR = SQ$ which gives $PR^2 = SQ^2$

Thus, $[\cos(\alpha+\beta)-1]^2 + \sin^2(\alpha+\beta) = [\cos\alpha - \cos(-\beta)]^2 + [\sin\alpha - \sin(-\beta)]^2$

$$-2\cos(\alpha+\beta) + 2 = 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$$

Hence, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

(i) In the above proof, $PR = SQ$ says that the distance between two points on a circle is determined by the radius and the central angle .

(ii) Arc lengths PR and SQ, subtends angles $\alpha+\beta$ and $\alpha+(-\beta)$ respectively at the center. Thus, PR = SQ. Thus, distance between the points $(\cos\alpha, \sin\alpha)$ and $(\cos(-\beta), \sin(-\beta))$ is same as the distance between the points $(\cos(\alpha+\beta), \sin(\alpha+\beta))$ and $(1, 0)$.

(iii) In the above derivations, $0 \le \alpha < 2\pi,\ 0 \le \beta < 2\pi$. Because of periodicity of sine and cosine, the result follows for any α and β.

Identity 3.2: $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Proof.

We know that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$\begin{aligned}
\text{Now,}\quad \cos(\alpha-\beta) &= \cos[\alpha+(-\beta)] \\
&= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) \\
\text{Hence, } \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.
\end{aligned}$$

(i) If $\alpha = \beta$, the above identity is reduced to $\cos^2\alpha + \sin^2\alpha = 1$.

(ii) If $\alpha = 0$ and $\beta = x$, then $\cos(-x) = \cos x$, which shows that $\cos x$ is an even function.

Identity 3.3: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Proof. This formula may be proved by writing $\sin(\alpha+\beta) = \cos\left[\frac{\pi}{2} - (\alpha+\beta)\right] = \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right]$ and by using Identity 3.2.

If $\alpha+\beta = \frac{\pi}{2}$, the above identity is reduced to $\cos^2\alpha + \sin^2\alpha = 1$.

Identity 3.4: $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Proof. This formula may be proved by writing $\sin(\alpha-\beta) = \sin[\alpha+(-\beta)]$ and by using Identity 3.3.

The sum and difference formulas for sine and cosine can be written in the matrix form

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}.$$

Identity 3.5: $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

Proof.

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

$$= \frac{\dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

Hence, $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

Identity 3.6: $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Proof. This result may be proved by writing $\tan(\alpha-\beta) = \tan[\alpha + (-\beta)]$ and using the Identity 3.5.

(i) Ptolemy (CE 100-170) treated the chord of an angle as his basic trigonometric function and proved the theorem: In a cyclic quadrilateral, the product of diagonals equals the sum of the products of opposite sides. That is, in a cyclic quadrilateral $ABCD$,

$$(AC)(BD) = (AB)(CD) + (AD)(BC).$$

Using this theorem, one can prove Sum and Difference identities. Hence, these identites are known as Ptolemy's sum and difference formulas.

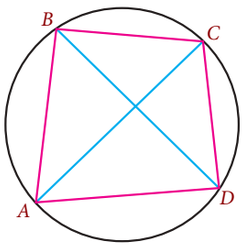


Figure 3.16

(ii) Observe that $\cos(\alpha \pm \beta) \neq \cos\alpha \pm \cos\beta$. Similarly we can observe for other trigonometric functions also.

(iii) When $\alpha = \beta$, $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ implies $\sin 0 = 0$, which we have already established.

(iv) When $\alpha = \dfrac{\pi}{2}$ and $\beta = \theta$, $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ gives $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$, which we have already proved.

(v) We can find the trigonometric function values of a given angle, if we can break it up into sum or difference of two of the special angles. For example, we can evaluate $\tan 75^\circ$ as $\tan(45^\circ + 30^\circ)$ and $\cos 135^\circ$ as $\cos(180^\circ - 45^\circ)$.

Example 3.15 Find the values of (i) $\cos 15^\circ$ and (ii) $\tan 165^\circ$.

Solution:

(i) Now, $\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Also, note that $\sin 75^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ [try yourself]

(ii) Now, $\tan 165^\circ = \tan(120^\circ + 45^\circ) = \dfrac{\tan 120^\circ + \tan 45^\circ}{1 - \tan 120^\circ \tan 45^\circ}$

But, $\tan 120^\circ = \tan(90^\circ + 30^\circ) = -\cot 30^\circ = -\sqrt{3}$ and $\tan 45^\circ = 1$

Thus, $\tan 165^\circ = \dfrac{1-\sqrt{3}}{1+\sqrt{3}}$.

Example 3.16 If $\sin x = \frac{4}{5}$ (in I quadrant) and $\cos y = \frac{-12}{13}$ (in II quadrant), then find (i) $\sin(x-y)$, (ii) $\cos(x-y)$.

Solution:
Given that $\sin x = \frac{4}{5}$.

$$\cos^2 x + \sin^2 x = 1 \text{ gives } \cos x = \pm\sqrt{1-\sin^2 x} = \pm\sqrt{1-\frac{16}{25}} = \pm\frac{3}{5}$$

In the first quadrant, $\cos x$ is always positive. Thus, $\cos x = \frac{3}{5}$.
Also, given that $\cos y = -\frac{12}{13}$ in the II quadrant. We have

$$\sin y = \pm\sqrt{1-\cos^2 y} = \pm\sqrt{1-\frac{144}{169}} = \pm\frac{5}{13}.$$

In the second quadrant, $\sin y$ is always positive. Thus, $\sin y = \frac{5}{13}$.

(i) $\sin(x-y) = \sin x\cos y - \cos x \sin y = \frac{4}{5}\left(\frac{-12}{13}\right) - \frac{3}{5}\left(\frac{5}{13}\right) = -\frac{63}{65}$.

(ii) $\cos(x-y) = \cos x\cos y + \sin x \sin y = \frac{3}{5}\left(\frac{-12}{13}\right) + \frac{4}{5}\left(\frac{5}{13}\right) = -\frac{16}{65}$.

Example 3.17 Prove that $\cos\left(\frac{3\pi}{4}+x\right) - \cos\left(\frac{3\pi}{4}-x\right) = -\sqrt{2}\sin x$

Solution:

$$\begin{aligned}\text{L.H.S} &= \cos\frac{3\pi}{4}\cos x - \sin\frac{3\pi}{4}\sin x - \cos\frac{3\pi}{4}\cos x - \sin\frac{3\pi}{4}\sin x \\ &= -2\sin\left(\pi-\frac{\pi}{4}\right)\sin x = -2\left(\frac{1}{\sqrt{2}}\right)\sin x = -\sqrt{2}\sin x.\end{aligned}$$

Observe that $\cos(A+x) - \cos(A-x) = -2\sin A \sin x$

Example 3.18 Point $A(9,12)$ rotates around the origin O in a plane through 60° in the anticlockwise direction to a new position B. Find the coordinates of the point B.

Solution:
Let $A(9,12) = A(r\cos\theta, r\sin\theta)$, where $r = OA$. Then $r\cos\theta = 9$ and $r\sin\theta = 12$.
Thus, $r^2 = 81 + 144 = 225 \Rightarrow r = 15$.
Hence, the point A is given by $A\,(15\cos\theta, 15\sin\theta)$.
Now, the point B is given by $B\,(15\cos(\theta+60^\circ), 15\sin(\theta+60^\circ))$.

$$\begin{aligned}15\cos(\theta+60^\circ) &= 15\,(\cos\theta\cos 60^\circ - \sin\theta\sin 60^\circ) \\ &= (15\cos\theta)\cos 60^\circ - (15\sin\theta)\sin 60^\circ = 9\times\frac{1}{2} - 12\times\frac{\sqrt{3}}{2} = \frac{3}{2}(3-4\sqrt{3})\end{aligned}$$

Similarly, $15\sin(\theta + 60^\circ) = \frac{3}{2}(4 + 3\sqrt{3})$. Hence, the point B is given by

$$B\left(\frac{3}{2}(3 - 4\sqrt{3}), \frac{3}{2}(4 + 3\sqrt{3})\right).$$

Example 3.19 A ripple tank demonstrates the effect of two water waves being added together. The two waves are described by $h = 8\cos t$ and $h = 6\sin t$, where $t \in [0, 2\pi)$ is in seconds and h is the height in millimeters above still water. Find the maximum height of the resultant wave and the value of t at which it occurs.

Solution:
Let H be the height of the resultant wave at time t. Then H is given by

$$H = 8\cos t + 6\sin t$$

Let $8\cos t + 6\sin t = k\cos(t - \alpha) = k(\cos t\cos\alpha + \sin t\sin\alpha)$

Hence, $k = 10$ and $\tan\alpha = \frac{3}{4}$, so that

$H = 10\cos(t - \alpha)$

Thus, the maximum of $H = 10$ mm. The maximum occurs when $t = \alpha$, where $\tan\alpha = \frac{3}{4}$.

Example 3.20 Expand (i) $\sin(A + B + C)$ (ii) $\tan(A + B + C)$

Solution:

$$\begin{aligned}
\text{(i) } \sin(A+B+C) &= \sin[A + (B + C)] \\
&= \sin A\cos(B + C) + \cos A\sin(B + C) \\
&= \sin A\cos B\cos C + \cos A\sin B\cos C \\
&\quad + \cos A\cos B\sin C - \sin A\sin B\sin C \\
\text{(ii) } \tan(A+B+C) &= \tan[A + (B + C)] \\
&= \frac{\tan A + \tan(B + C)}{1 - \tan A\tan(B + C)} \\
&= \frac{\tan A + \dfrac{\tan B + \tan C}{1 - \tan B\tan C}}{1 - \tan A\dfrac{\tan B + \tan C}{1 - \tan B\tan C}} \\
&= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}
\end{aligned}$$

(i) If A+B+C = 0 or π, we have $\tan(A + B + C) = 0$ so that

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C.$$

This result is also true in the case of oblique triangles.

(ii) $\tan(x - y) + \tan(y - z) + \tan(z - x) = \tan(x - y)\tan(y - z)\tan(z - x)$

(iii) $\tan 3A - \tan 2A - \tan A = \tan 3A + \tan(-2A) + \tan(-A) = \tan 3A \tan 2A \tan A$

(iv) If $A + B + C = \frac{\pi}{2}$, then $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$ (How!).

Exercise - 3.4

1. If $\sin x = \frac{15}{17}$ and $\cos y = \frac{12}{13}$, $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$,
   find the value of (i) $\sin(x + y)$ (ii) $\cos(x - y)$ (iii) $\tan(x + y)$.
2. If $\sin A = \frac{3}{5}$ and $\cos B = \frac{9}{41}$, $0 < A < \frac{\pi}{2}$, $0 < B < \frac{\pi}{2}$,
   find the value of (i) $\sin(A + B)$ (ii) $\cos(A - B)$.
3. Find $\cos(x - y)$, given that $\cos x = -\frac{4}{5}$ with $\pi < x < \frac{3\pi}{2}$ and $\sin y = -\frac{24}{25}$ with $\pi < y < \frac{3\pi}{2}$.
4. Find $\sin(x - y)$, given that $\sin x = \frac{8}{17}$ with $0 < x < \frac{\pi}{2}$ and $\cos y = -\frac{24}{25}$ with $\pi < y < \frac{3\pi}{2}$.
5. Find the value of (i) $\cos 105^\circ$ (ii) $\sin 105^\circ$ (iii) $\tan \frac{7\pi}{12}$.
6. Prove that (i) $\cos(30^\circ + x) = \frac{\sqrt{3}\cos x - \sin x}{2}$ (ii) $\cos(\pi + \theta) = -\cos\theta$
   (iii) $\sin(\pi + \theta) = -\sin\theta$.
7. Find a quadratic equation whose roots are $\sin 15^\circ$ and $\cos 15^\circ$.
8. Expand $\cos(A + B + C)$. Hence prove that
   $\cos A \cos B \cos C = \sin A \sin B \cos C + \sin B \sin C \cos A + \sin C \sin A \cos B$,
   $if\ A + B + C = \frac{\pi}{2}$.
9. Prove that
   (i) $\sin(45^\circ + \theta) - \sin(45^\circ - \theta) = \sqrt{2}\sin\theta$.
   (ii) $\sin(30^\circ + \theta) + \cos(60^\circ + \theta) = \cos\theta$.
10. If $a\cos(x + y) = b\cos(x - y)$, show that $(a + b)\tan x = (a - b)\cot y$.
11. Prove that $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$.
12. Prove that $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$.
13. Show that $\tan 75^\circ + \cot 75^\circ = 4$.
14. Prove that $\cos(A + B)\cos C - \cos(B + C)\cos A = \sin B \sin(C - A)$.
15. Prove that $\sin(n + 1)\theta \sin(n - 1)\theta + \cos(n + 1)\theta \cos(n - 1)\theta = \cos 2\theta,\ n \in \mathbb{Z}$.
16. If $x\cos\theta = y\cos\left(\theta + \frac{2\pi}{3}\right) = z\cos\left(\theta + \frac{4\pi}{3}\right)$, find the value of $xy + yz + zx$.
17. Prove that
    (i) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$
    (ii) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
    (iii) $\sin^2(A + B) - \sin^2(A - B) = \sin 2A \sin 2B$
    (iv) $\cos 8\theta \cos 2\theta = \cos^2 5\theta - \sin^2 3\theta$
18. Show that $\cos^2 A + \cos^2 B - 2\cos A \cos B \cos(A + B) = \sin^2(A + B)$.
19. If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$, then prove that
    $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$.

20. Show that

(i) $\tan(45^\circ + A) = \dfrac{1+\tan A}{1-\tan A}$ (ii) $\tan(45^\circ - A) = \dfrac{1-\tan A}{1+\tan A}$.

21. Prove that $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$.

22. If $\tan x = \dfrac{n}{n+1}$ and $\tan y = \dfrac{1}{2n+1}$, find $\tan(x+y)$.

23. Prove that $\tan\left(\dfrac{\pi}{4}+\theta\right)\tan\left(\dfrac{3\pi}{4}+\theta\right) = -1$.

24. Find the values of $\tan(\alpha+\beta)$, given that $\cot\alpha = \dfrac{1}{2}, \alpha \in \left(\pi, \dfrac{3\pi}{2}\right)$ and $\sec\beta = -\dfrac{5}{3}, \beta \in \left(\dfrac{\pi}{2}, \pi\right)$.

25. If $\theta + \phi = \alpha$ and $\tan\theta = k\tan\phi$, then prove that $\sin(\theta-\phi) = \dfrac{k-1}{k+1}\sin\alpha$.

3.5.2 Multiple angle identities and submultiple angle identities

In 1831, Michael Faraday discovered that when a wire is passed near a magnet, a small electric current is produced in the wire. This property is used to generate electric current for houses, institutions and business establishments throughout the world. By rotating thousands of wires near large electromagnets, massive amount of electricity can be produced.

Voltage is a quantity that can be modeled by sinusoidal graphs and functions. To model electricity and other phenomena, trigonometric functions and identities involving multiple angles or sub multiple angles are used.

If A is an angle, then $2A, 3A, \ldots$ are called **multiple angles** of A and the angle $\dfrac{A}{2}, \dfrac{A}{3}, \ldots$ are called **sub-multiple angles** of A. Now we shall discuss the trigonometric ratio of multiple angles and sub-multiple angles and derive some identities.

Double Angle Identities

Let us take up the sum and difference identities and examine some of the consequences that come from them. Double angle identities are a special case of the sum identities. That is, when the two angles are equal, the sum identities are reduced to double angle identities. They are useful in solving trigonometric equations and also in the verification of trigonometric identities. Further double angle identities can be used to derive the reduction identities (power reducing identities). Also double angle identities are used to find maximum or minimum values of trigonometric expressions.

Identity 3.7: $\sin 2A = 2\sin A\cos A$

Proof.

We know that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Taking $\alpha = \beta = A$, we have $\sin(A+A) = \sin A\cos A + \sin A\cos A$

Thus, $\sin 2A = 2\sin A\cos A$.

(i) $y = \sin 2x$ and $y = 2\sin x$ are different. Draw their graphs and identify the difference.

(ii) Application of $\sin 2A = 2\sin A\cos A$: When an object is projected with speed u at an angle α to the horizontal over level ground, the horizontal distance (Range) it travels before striking the ground is given by the formula $R = \dfrac{u^2\sin 2\alpha}{g}$.

Clearly maximum of R is $\dfrac{u^2}{g}$, when $\alpha = \dfrac{\pi}{4}$.

(iii) $|\sin A\cos A| = \left|\dfrac{\sin 2A}{2}\right| \le \dfrac{1}{2}$.

Thus, $-\frac{1}{2} \leq \sin A \cos A \leq \frac{1}{2}$.

From this, we infer that the maximum value of $\sin A \cos A$ is $\frac{1}{2}$ when $A = \frac{\pi}{4}$.

Example 3.21 A foot ball player can kick a football from ground level with an initial velocity of 80 ft/second. Find the maximum horizontal distance the football travels and at what angle? (Take $g = 32$).

Solution:
The formula for horizontal distance R is given by

$$R = \frac{u^2 \sin 2\alpha}{g} = \frac{(80 \times 80) \sin 2\alpha}{32} = 10 \times 20 \sin 2\alpha.$$

Thus, the maximum distance is 200 ft.
Hence, he has to kick the football at an angle of $\alpha = 45°$ to reach the maximum distance.

Identity 3.8: $\cos 2A = \cos^2 A - \sin^2 A$

Proof.

We know that $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

Take $\alpha = \beta = A$. We have $\cos(A + A) = \cos A \cos A - \sin A \sin A$

$$\cos 2A = \cos^2 A - \sin^2 A.$$

From the identity $\cos 2A = \cos^2 A - \sin^2 A$, we also have

$$\cos 2A = \cos^2 A - \left(1 - \cos^2 A\right) = 2\cos^2 A - 1 \text{ and}$$
$$\cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A.$$

Identity 3.9: $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Proof.

Now, $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$

Take $\alpha = \beta = A$. We have, $\tan(A + A) = \dfrac{\tan A + \tan A}{1 - \tan A \tan A}$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}.$$

Identity 3.10: $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$

Proof.

we know that $\sin 2A = 2\sin A \cos A = \dfrac{2\sin A \cos A}{\sin^2 A + \cos^2 A}$

$$= \frac{\dfrac{2\sin A \cos A}{\cos^2 A}}{\dfrac{\sin^2 A + \cos^2 A}{\cos^2 A}} = \frac{2\tan A}{1 + \tan^2 A}.$$

Identity 3.11: $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

Proof.

$$\begin{aligned}
\text{We know that } \cos 2A &= \cos^2 A - \sin^2 A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} \\
&= \frac{\dfrac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}} \\
\text{Thus, } \cos 2A &= \frac{1-\tan^2 A}{1+\tan^2 A}.
\end{aligned}$$

Power reducing identities or Reduction identities

Power reducing identities for sine, cosine and tangent can be derived using the double-angle identities. For example,

$$\cos 2A = 2\cos^2 A - 1 \Rightarrow \cos^2 A = \frac{1+\cos 2A}{2}$$

The following table is the list of power reducing identities.

Power Reducing Identities
$\sin^2 A = \dfrac{1-\cos 2A}{2}$, $\cos^2 A = \dfrac{1+\cos 2A}{2}$, $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$

(i) In the power reducing identities, we have reduced the square power on one side to power 1 on the other side.

(ii) Power reducing identities allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine.

For example, using power reducing identities one can easily prove that

$$\cos^4 x = \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8} \text{ and } \sin^4 x = \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8} \text{ (Try it!).}$$

(iii) Power reducing formulas are important in higher level mathematics.

Triple-Angle Identities

Using double angle identities, we can derive triple angle identities.

Identity 3.12: $\sin 3A = 3\sin A - 4\sin^3 A$

Proof.

$$\begin{aligned}
\text{We have, } \sin 3A &= \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A \\
&= 2\sin A \cos^2 A + \left(1 - 2\sin^2 A\right)\sin A \\
&= 2\sin A\left(1 - \sin^2 A\right) + \left(1 - 2\sin^2 A\right)\sin A \\
&= 3\sin A - 4\sin^3 A
\end{aligned}$$

Identity 3.13: $\cos 3A = 4\cos^3 A - 3\cos A$

Proof.

$$\begin{aligned}
\text{We have, } \cos 3A &= \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A \\
&= \left(2\cos^2 A - 1\right)\cos A - 2\sin A \cos A \sin A \\
&= \left(2\cos^2 A - 1\right)\cos A - 2\cos A\left(1 - \cos^2 A\right) \\
&= 4\cos^3 A - 3\cos A.
\end{aligned}$$

Identity 3.14: $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Proof.

$$\begin{aligned}
\text{We have, } \tan 3A &= \tan(2A + A) \\
&= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} \\
&= \frac{\dfrac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2\tan A}{1 - \tan^2 A}\tan A} \\
&= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.
\end{aligned}$$

Double and Triple angle identities are given below:

sine	cosine	Tangent
$\sin 2A = 2\sin A \cos A$	$\cos 2A = \cos^2 A - \sin^2 A$	$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$
$\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$	$\cos 2A = 2\cos^2 A - 1$	$\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
$\sin 3A = 3\sin A - 4\sin^3 A$	$\cos 2A = 1 - 2\sin^2 A$	
	$\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$	
	$\cos 3A = 4\cos^3 A - 3\cos A$	

Half-Angle Identities

Half angle identities are closely related to the double angle identities. We can use half angle identities when we have an angle that is half the size of a special angle. For example, $\sin 15^\circ$ can be computed by writing $\sin 15^\circ = \sin \dfrac{30^\circ}{2}$. Also one can find exact values for some angles using half-angle identities. If we put $2A = \theta$ or $A = \dfrac{\theta}{2}$ in the double angle identities, we get new identities in terms of angle $\dfrac{\theta}{2}$.

Let us list out the half angle identities in the following table:

Double angle identity	Half-angle identity
$\sin 2A = 2\sin A\cos A$	$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
$\cos 2A = \cos^2 A - \sin^2 A$	$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$
$\cos 2A = 2\cos^2 A - 1$	$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$
$\cos 2A = 1 - 2\sin^2 A$	$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$
$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$	$\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$
$\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$	$\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$
$\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$	$\cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(i) The half angle identities are often used to replace a squared trigonometric function by a non squared trigonometric function.

(ii) Half angle identities allow us to find the value of the sine and cosine of half the angle if we know the value of the cosine of the original angle.

Example 3.22 Find the value of $\sin\left(22\frac{1}{2}^\circ\right)$

Solution:

We know that $\cos\theta = 1 - 2\sin^2\frac{\theta}{2} \Rightarrow \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$. Take $\theta = 45^\circ$

we get $\sin\frac{45^\circ}{2} = \pm\sqrt{\frac{1-\cos 45^\circ}{2}}$, (taking positive sign only, since $22\frac{1}{2}^\circ$ lies in the first quadrant)

Thus, $\sin 22\frac{1}{2}^\circ = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$.

Example 3.23 Find the value of $\sin 2\theta$, when $\sin\theta = \frac{12}{13}$, θ lies in the first quadrant.

Solution:

Using a right triangle, we can easily find that $\cos\theta = \frac{5}{13}$

$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right) = \frac{120}{169}$.

Instead of constructing the triangle, we can also find the value of $\cos\theta$ using $\cos\theta = \pm\sqrt{1-\sin^2\theta}$ formula.

Example 3.24 Prove that $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$

Solution:

$$\begin{aligned} 4\sin A\cos^3 A - 4\cos A\sin^3 A &= 4\sin A\cos A\left(\cos^2 A - \sin^2 A\right) \\ &= 4\sin A\cos A\cos 2A = 2(2\sin A\cos A)\cos 2A \\ &= 2(\sin 2A)\cos 2A = \sin 4A. \end{aligned}$$

Example 3.25 Prove that $\sin x = 2^{10}\sin\left(\frac{x}{2^{10}}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\ldots\cos\left(\frac{x}{2^{10}}\right)$

Solution:

$$\begin{aligned} \sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\times 2\times\sin\frac{x}{2^2}\cos\frac{x}{2^2}\cos\frac{x}{2} \\ &= 2^2\sin\frac{x}{2^2}\cos\frac{x}{2}\cos\frac{x}{2^2} \end{aligned}$$

Applying repeatedly the half angle sine formula, we get

$$\sin x = 2^{10}\sin\left(\frac{x}{2^{10}}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\ldots\cos\left(\frac{x}{2^{10}}\right).$$

The above result can be extended to any finite number of times.

Example 3.26 Prove that $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

Solution:

We have $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta} = \dfrac{\sin\theta+2\sin\theta\cos\theta}{\cos\theta+(1+\cos 2\theta)} = \dfrac{\sin\theta(1+2\cos\theta)}{\cos\theta(1+2\cos\theta)} = \tan\theta.$

Example 3.27 Prove that $1-\frac{1}{2}\sin 2x = \dfrac{\sin^3 x+\cos^3 x}{\sin x+\cos x}$

Solution:

We have $$\begin{aligned} \frac{\sin^3 x+\cos^3 x}{\sin x+\cos x} &= \frac{(\sin x+\cos x)\left(\sin^2 x-\sin x\cos x+\cos^2 x\right)}{\sin x+\cos x} \\ &= 1-\sin x\cos x = 1-\frac{1}{2}\sin 2x. \end{aligned}$$

Example 3.28 Find x such that $-\pi \le x \le \pi$ and $\cos 2x = \sin x$

Solution:

We have $\cos 2x = \sin x$ which gives

$2\sin^2 x + \sin x - 1 = 0$. The roots of the equation are

$\sin x = \dfrac{-1 \pm 3}{4} = -1$ or $\dfrac{1}{2}$

Now, $\sin x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Also $\sin x = -1 \Rightarrow x = -\dfrac{\pi}{2}$

Thus, $x = -\dfrac{\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$.

Example 3.29 Find the values of (i) $\sin 18^\circ$ (ii) $\cos 18^\circ$ (iii) $\sin 72^\circ$ (iv) $\cos 36^\circ$ (v) $\sin 54^\circ$

Solution:

(i) Let $\theta = 18^\circ$. Then $5\theta = 90^\circ$

$3\theta + 2\theta = 90^\circ \Rightarrow 2\theta = 90^\circ - 3\theta$

$\sin 2\theta = \sin(90^\circ - 3\theta) = \cos 3\theta$

$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$. Since $\cos\theta = \cos 18^\circ \ne 0$, we have

$2\sin\theta = 4\cos^2\theta - 3 = 4(1 - \sin^2\theta) - 3$

$4\sin^2\theta + 2\sin\theta - 1 = 0$

$\sin\theta = \dfrac{-2 \pm \sqrt{4 - 4(4)(-1)}}{2(4)} = \dfrac{-1 \pm \sqrt{5}}{4}$

Thus, $\sin 18^\circ = \dfrac{\sqrt{5} - 1}{4}$ (positive sign is taken. Why ?)

(ii) $\cos 18^\circ = \sqrt{1 - \sin^2 18^\circ}$

$= \sqrt{1 - \left[\dfrac{\sqrt{5} - 1}{4}\right]^2} = \dfrac{1}{4}\sqrt{16 - (5 + 1 - 2\sqrt{5})} = \dfrac{1}{4}\sqrt{10 + 2\sqrt{5}}$

(iii) $\sin 72^\circ = \sin(90^\circ - 18^\circ) = \cos 18^\circ = \dfrac{1}{4}\sqrt{10 + 2\sqrt{5}}$

(iv) $\cos 36^\circ = 1 - 2\sin^2 18^\circ = 1 - 2\left[\dfrac{\sqrt{5} - 1}{4}\right]^2 = \dfrac{\sqrt{5} + 1}{4}$

(v) $\sin 54^\circ = \sin(90^\circ - 36^\circ) = \cos 36^\circ = \dfrac{\sqrt{5} + 1}{4}$.

Observe that $\sin 18^\circ = \cos 72^\circ$, $\cos 18^\circ = \sin 72^\circ$ and $\cos 36^\circ = \sin 54^\circ$

Example 3.30 If $\tan\dfrac{\theta}{2} = \sqrt{\dfrac{1 - a}{1 + a}}\ \tan\dfrac{\phi}{2}$, then prove that $\cos\phi = \dfrac{\cos\theta - a}{1 - a\cos\theta}$.

Solution:
By the half-angle identity, we have

$$\cos\phi = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}} = \frac{1-\left(\frac{1+a}{1-a}\right)\tan^2\frac{\theta}{2}}{1+\left(\frac{1+a}{1-a}\right)\tan^2\frac{\theta}{2}}$$

$$= \frac{\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)-a}{1-a\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)} = \frac{\cos\theta - a}{1-a\cos\theta}.$$

Example 3.31 Find the value of $\sqrt{3}$ cosec $20^\circ - \sec 20^\circ$

Solution:

We have $\sqrt{3}$ cosec $20^\circ - \sec 20^\circ = \dfrac{\sqrt{3}}{\sin 20^\circ} - \dfrac{1}{\cos 20^\circ} = 4\left[\dfrac{\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ}{2\sin 20^\circ \cos 20^\circ}\right]$

$$= 4\left[\frac{\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ}{\sin 40^\circ}\right] = 4.$$

Example 3.32 Prove that $\cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A = \dfrac{\sin 2^n A}{2^n \sin A}$

Solution:

$$\begin{aligned}
\text{L.H.S.} &= \cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A \\
&= \frac{1}{2\sin A} 2\sin A \cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A \\
&= \frac{1}{2\sin A} \sin 2A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A \\
&= \frac{1}{2^2 \sin A} \sin 4A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A
\end{aligned}$$

Continuing the process, we get

$$= \frac{\sin 2^n A}{2^n \sin A}.$$

Exercise - 3.5

1. Find the value of $\cos 2A$, A lies in the first quadrant, when
   (i) $\cos A = \dfrac{15}{17}$ (ii) $\sin A = \dfrac{4}{5}$ (iii) $\tan A = \dfrac{16}{63}$.

2. If θ is an acute angle, then find

   (i) $\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$, when $\sin\theta=\frac{1}{25}$.

   (ii) $\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, when $\sin\theta=\frac{8}{9}$.

3. If $\cos\theta=\frac{1}{2}\left(a+\frac{1}{a}\right)$, show that $\cos 3\theta=\frac{1}{2}\left(a^3+\frac{1}{a^3}\right)$.

4. Prove that $\cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$.

5. Prove that $\sin 4\alpha=4\tan\alpha\frac{1-\tan^2\alpha}{\left(1+\tan^2\alpha\right)^2}$.

6. If $A+B=45^\circ$, show that $(1+\tan A)(1+\tan B)=2$.

7. Prove that $(1+\tan 1^\circ)(1+\tan 2^\circ)(1+\tan 3^\circ)\ldots(1+\tan 44^\circ)$ is a multiple of 4.

8. Prove that $\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)=2\tan 2\theta$.

9. Show that $\cot\left(7\frac{1}{2}^\circ\right)=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$.

10. Prove that $(1+\sec 2\theta)(1+\sec 4\theta)\ldots(1+\sec 2^n\theta)=\tan 2^n\theta\ \cot\theta$.

11. Prove that $32(\sqrt{3})\sin\frac{\pi}{48}\cos\frac{\pi}{48}\cos\frac{\pi}{24}\cos\frac{\pi}{12}\cos\frac{\pi}{6}=3$.

3.5.3 Product to Sum and Sum to Product Identities

Some applications of trigonometric functions demand that a product of trigonometric functions be written as sum or difference of trigonometric functions. The sum and difference identities for the cosine and sine functions look amazingly like each other except for the sign in the middle. So, we tend to combine them to get nice identities. Thus, we use them to derive several identities that make it possible to rewrite a product as a sum or a sum as a product.

We know that

$$\sin(A+B) = \sin A\cos B+\cos A\sin B \tag{3.1}$$
$$\sin(A-B) = \sin A\cos B-\cos A\sin B \tag{3.2}$$
$$\cos(A+B) = \cos A\cos B-\sin A\sin B \tag{3.3}$$
$$\cos(A-B) = \cos A\cos B+\sin A\sin B \tag{3.4}$$

From the above identities, we can easily derive the following Product to Sum identities.

$$\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)] \tag{3.5}$$
$$\cos A\sin B = \frac{1}{2}[\sin(A+B)-\sin(A-B)] \tag{3.6}$$
$$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)] \tag{3.7}$$
$$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)] \tag{3.8}$$

The above identities are very important whenever need arises to transform the product of sine and cosine into sum. This idea is very much useful in evaluation of some integrals.

To get Sum to Product identities, let us introduce the substitutions $A+B=C$ and $A-B=D$ or equivalently $A=\frac{C+D}{2}, B=\frac{C-D}{2}$ in the product to sum identities (3.5) to (3.8). Then, we have the following Sum to Product identities

$$\sin C+\sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2} \tag{3.9}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2} \tag{3.10}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2} \tag{3.11}$$

$$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2} \tag{3.12}$$

Identity 3.15: Prove that $\sin(60^\circ - A)\sin A\sin(60^\circ + A) = \frac{1}{4}\sin 3A$

Proof.

$$\begin{aligned}\sin(60^\circ - A)\sin A\sin(60^\circ + A) &= \sin(60^\circ - A)\sin(60^\circ + A)\sin A\\ &= \frac{1}{2}\left[\cos 2A - \cos 120^\circ\right]\sin A\\ &= \frac{1}{2}\left[\cos 2A\sin A + \frac{1}{2}\sin A\right]\\ &= \frac{1}{2}\left[\frac{1}{2}\sin 3A\right] = \frac{1}{4}\sin 3A\end{aligned}$$

Similarly we can prove the following two important identities

Identity 3.16: $\cos(60^\circ - A)\cos A\cos(60^\circ + A) = \frac{1}{4}\cos 3A$

Identity 3.17: $\tan(60^\circ - A)\tan A\tan(60^\circ + A) = \tan 3A$

Example 3.33 Express each of the following product as a sum or difference

(i) $\sin 40^\circ\cos 30^\circ$ (ii) $\cos 110^\circ\sin 55^\circ$ (iii) $\sin\frac{x}{2}\cos\frac{3x}{2}$

Solution:

(i) We know that $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

Take $A = 40^\circ$and $B = 30^\circ$. We get,

$2\sin 40^\circ\cos 30^\circ = \sin(40^\circ + 30^\circ) + \sin(40^\circ - 30^\circ) = \sin 70^\circ + \sin 10^\circ$.

Thus, $\sin 40^\circ\cos 30^\circ = \frac{1}{2}\left[\sin 70^\circ + \sin 10^\circ\right]$.

(ii) We know that $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

Take $A = 110^\circ$ and $B = 55^\circ$. We get,

$2\cos 110^\circ\sin 55^\circ = \sin(110^\circ + 55^\circ) - \sin(110^\circ - 55^\circ)$.

Thus, $\cos 110^\circ\sin 55^\circ = \frac{1}{2}\left[\sin 165^\circ - \sin 55^\circ\right]$

(iii) We know that $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

Take $A = \frac{x}{2}$ and $B = \frac{3x}{2}$

We get, $2\sin\frac{x}{2}\cos\frac{3x}{2} = \sin\left(\frac{x}{2} + \frac{3x}{2}\right) + \sin\left(\frac{x}{2} - \frac{3x}{2}\right)$

Thus, $\sin\frac{x}{2}\cos\frac{3x}{2} = \frac{1}{2}\left[\sin 2x - \sin x\right]$.

Example 3.34 Express each of the following sum or difference as a product

(i) $\sin 50^\circ + \sin 20^\circ$ (ii) $\cos 6\theta + \cos 2\theta$ (iii) $\cos \frac{3x}{2} - \cos \frac{9x}{2}$.

Solution:

(i) We know that $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$

Take $C = 50^\circ$ and $D = 20^\circ$. We have

$$\sin 50^\circ + \sin 20^\circ = 2 \sin \frac{50^\circ + 20^\circ}{2} \cos \frac{50^\circ - 20^\circ}{2} = 2 \sin 35^\circ \cos 15^\circ$$

(ii) We know that $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$

Take $C = 6\theta$ and $D = 2\theta$. We have

$$\cos 6\theta + \cos 2\theta = 2 \cos \frac{6\theta + 2\theta}{2} \cos \frac{6\theta - 2\theta}{2} = 2 \cos 4\theta \cos 2\theta$$

(iii) We know that $\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$

Take $C = \frac{3x}{2}$ and $D = \frac{9x}{2}$. We have

$$\cos \frac{3x}{2} - \cos \frac{9x}{2} = 2 \sin \frac{\frac{3x}{2} + \frac{9x}{2}}{2} \sin \frac{\frac{9x}{2} - \frac{3x}{2}}{2} = 2 \sin 3x \sin \frac{3x}{2}.$$

Example 3.35 Find the value of $\sin 34^\circ + \cos 64^\circ - \cos 4^\circ$.

Solution:

We have
$$\begin{aligned} \sin 34^\circ + \cos 64^\circ - \cos 4^\circ &= \sin 34^\circ - 2 \sin \left(\frac{64^\circ + 4^\circ}{2} \right) \sin \left(\frac{64^\circ - 4^\circ}{2} \right) \\ &= \sin 34^\circ - 2 \sin 34^\circ \sin 30^\circ = 0. \end{aligned}$$

Example 3.36 Show that $\cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ = \frac{1}{16}$.

Solution:

$$\begin{aligned} \text{L.H.S.} &= \cos 36^\circ \cos (90^\circ - 18^\circ) \cos (90^\circ + 18^\circ) \cos (180^\circ - 36^\circ) \\ &= \sin^2 18^\circ \cos^2 36^\circ \\ &= \left(\frac{\sqrt{5} - 1}{4} \right)^2 \left(\frac{\sqrt{5} + 1}{4} \right)^2 = \frac{1}{16}. \end{aligned}$$

Example 3.37 Simplify $\dfrac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ}$

Solution:

We have
$$\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{2\cos\left(\dfrac{75^\circ + 15^\circ}{2}\right)\sin\left(\dfrac{75^\circ - 15^\circ}{2}\right)}{2\cos\left(\dfrac{75^\circ + 15^\circ}{2}\right)\cos\left(\dfrac{75^\circ - 15^\circ}{2}\right)}$$
$$= \frac{2\cos 45^\circ \sin 30^\circ}{2\cos 45^\circ \cos 30^\circ} = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

Try to solve using $\sin 75^\circ = \cos 15^\circ$ and $\cos 75^\circ = \sin 15^\circ$

Example 3.38 Show that $\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \dfrac{3}{16}$.

Solution:

We know that $\cos(60^\circ - A)\cos A \cos(60^\circ + A) = \dfrac{1}{4}\cos 3A$

$$\begin{aligned}\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ &= \cos 30^\circ\,[\cos 10^\circ \cos 50^\circ \cos 70^\circ]\\ &= \cos 30^\circ\,[\cos(60^\circ - 10^\circ)\cos 10^\circ \cos(60^\circ + 10^\circ)]\\ &= \frac{\sqrt{3}}{2}\left[\frac{1}{4}\cos 30^\circ\right] = \frac{\sqrt{3}}{2}\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{16}\end{aligned}$$

Exercise - 3.6

1. Express each of the following as a sum or difference

   (i) $\sin 35^\circ \cos 28^\circ$ (ii) $\sin 4x \cos 2x$ (iii) $2\sin 10\theta \cos 2\theta$ (iv) $\cos 5\theta \cos 2\theta$ (v) $\sin 5\theta \sin 4\theta$.
2. Express each of the following as a product

   (i) $\sin 75^\circ - \sin 35^\circ$ (ii) $\cos 65^\circ + \cos 15^\circ$ (iii) $\sin 50^\circ + \sin 40^\circ$ (iv) $\cos 35^\circ - \cos 75^\circ$.
3. Show that $\sin 12^\circ \sin 48^\circ \sin 54^\circ = \dfrac{1}{8}$.
4. Show that $\cos\dfrac{\pi}{15}\cos\dfrac{2\pi}{15}\cos\dfrac{3\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{5\pi}{15}\cos\dfrac{6\pi}{15}\cos\dfrac{7\pi}{15} = \dfrac{1}{128}$.
5. Show that $\dfrac{\sin 8x \cos x - \sin 6x \cos 3x}{\cos 2x \cos x - \sin 3x \sin 4x} = \tan 2x$.
6. Show that $\dfrac{(\cos\theta - \cos 3\theta)(\sin 8\theta + \sin 2\theta)}{(\sin 5\theta - \sin\theta)(\cos 4\theta - \cos 6\theta)} = 1$.
7. Prove that $\sin x + \sin 2x + \sin 3x = \sin 2x\,(1 + 2\cos x)$.
8. Prove that $\dfrac{\sin 4x + \sin 2x}{\cos 4x + \cos 2x} = \tan 3x$.
9. Prove that $1 + \cos 2x + \cos 4x + \cos 6x = 4\cos x \cos 2x \cos 3x$.

10. Prove that $\sin\frac{\theta}{2}\sin\frac{7\theta}{2}+\sin\frac{3\theta}{2}\sin\frac{11\theta}{2}=\sin 2\theta\sin 5\theta$.
11. Prove that $\cos(30^\circ-A)\cos(30^\circ+A)+\cos(45^\circ-A)\cos(45^\circ+A)=\cos 2A+\frac{1}{4}$.
12. Prove that $\dfrac{\sin x+\sin 3x+\sin 5x+\sin 7x}{\cos x+\cos 3x+\cos 5x+\cos 7x}=\tan 4x$.
13. Prove that $\dfrac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)}=\tan(A+B)$.
14. Show that $\cot(A+15^\circ)-\tan(A-15^\circ)=\dfrac{4\cos 2A}{1+2\sin 2A}$.

3.5.4 Conditional Trigonometric Identities

We know that trigonometric identities are true for all admissible values of the angle involved. There are some trigonometric identities which satisfy the given additional conditions. Such identities are called conditional trigonometric identities.

In this section, we shall make use of the relations obtained in the earlier sections to establish some conditional identities based on some relationship.

Example 3.39 If $A+B+C=\pi$, prove the following

(i) $\cos A+\cos B+\cos C=1+4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

(ii) $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\le\frac{1}{8}$

(iii) $1<\cos A+\cos B+\cos C\le\frac{3}{2}$

Solution:

(i)
$$\begin{aligned}
\cos A+\cos B+\cos C &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C\\
&= 2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\left(\frac{A}{2}-\frac{B}{2}\right)+\cos C \qquad \left(\frac{A+B+C}{2}=\frac{\pi}{2}\right)\\
&= 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A}{2}-\frac{B}{2}\right)+1-2\sin^2\left(\frac{C}{2}\right)\\
&= 1+2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\sin\left(\frac{C}{2}\right)\right]\\
&= 1+2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\right]\\
&= 1+2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\cos\left(\frac{A}{2}+\frac{B}{2}\right)\right]\\
&= 1+4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)
\end{aligned}$$

(ii) Let
$$\begin{aligned}
u &= \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\\
&= -\frac{1}{2}\left[\cos\left(\frac{A+B}{2}\right)-\cos\left(\frac{A-B}{2}\right)\right]\sin\frac{C}{2}\\
&= -\frac{1}{2}\left[\cos\left(\frac{A+B}{2}\right)-\cos\left(\frac{A-B}{2}\right)\right]\cos\frac{A+B}{2}
\end{aligned}$$

$\cos^2\dfrac{A+B}{2} - \cos\dfrac{A-B}{2}\cos\dfrac{A+B}{2} + 2u = 0$, which is a quadratic in $\cos\dfrac{A+B}{2}$.

Since $\cos\dfrac{A+B}{2}$ is real number, the above equation has a solution.

Thus, the discriminant $b^2 - 4ac \geq 0$, which gives

$\cos^2\dfrac{A-B}{2} - 8u \geq 0 \Rightarrow u \leq \dfrac{1}{8}\cos^2\dfrac{A-B}{2} \leq \dfrac{1}{8}$

Hence, $\sin\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right) \leq \dfrac{1}{8}$

(iii) From (i) and (ii), we have $\cos A + \cos B + \cos C > 1$ and

$\cos A + \cos B + \cos C \leq 1 + 4 \times \dfrac{1}{8}$

Thus, we get $1 < \cos A + \cos B + \cos C \leq \dfrac{3}{2}$.

$\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} > 0$, if $A + B + C = \pi$

Example 3.40 Prove that

$\sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2} = 1 + 4\sin\left(\dfrac{\pi - A}{4}\right)\sin\left(\dfrac{\pi - B}{4}\right)\sin\left(\dfrac{\pi - C}{4}\right)$, if $A + B + C = \pi$

Solution:

$$\begin{aligned}
\text{L.H.S.} &= \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) + \cos\left(\frac{\pi}{2} - \frac{B}{2}\right) + \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) \\
&= \left[2\cos\left(\frac{\pi}{2} - \frac{A+B}{4}\right)\cos\left(\frac{B-A}{4}\right)\right] + \left[1 - 2\sin^2\left(\frac{\pi}{4} - \frac{C}{4}\right)\right] \\
&= 1 + 2\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)\cos\left(\frac{B-A}{4}\right) - 2\sin^2\left(\frac{\pi}{4} - \frac{C}{4}\right) \\
&= 1 + 2\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)\left[\cos\left(\frac{B-A}{4}\right) - \sin\left(\frac{\pi}{4} - \frac{C}{4}\right)\right] \\
&= 1 + 2\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)\left[\cos\left(\frac{B-A}{4}\right) - \sin\left(\frac{A+B}{4}\right)\right] \\
&= 1 + 2\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)\left[\cos\left(\frac{B-A}{4}\right) - \cos\left(\frac{\pi}{2} - \frac{A+B}{4}\right)\right] \\
&= 1 + 2\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)\left[2\sin\left(\frac{\frac{B-A}{4} + \frac{\pi}{2} - \frac{A+B}{4}}{2}\right)\sin\left(\frac{\frac{\pi}{2} - \frac{A+B}{4} - \frac{B-A}{4}}{2}\right)\right] \\
&= 1 + 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)
\end{aligned}$$

Example 3.41 If $A+B+C=\pi$, prove that $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$.

Solution:

$$\begin{aligned}
\cos^2 A+\cos^2 B+\cos^2 C &= \frac{1}{2}\left[2\cos^2 A+2\cos^2 B+2\cos^2 C\right]\\
&= \frac{1}{2}\left[(1+\cos 2A)+(1+\cos 2B)+(1+\cos 2C)\right]\\
&= \frac{3}{2}+\frac{1}{2}\left[(\cos 2A+\cos 2B)+\cos 2C\right]\\
&= \frac{3}{2}+\frac{1}{2}\left[2\cos(A+B)\cos(A-B)+(2\cos^2 C-1)\right]\\
&= \frac{3}{2}+\frac{1}{2}\left[-2\cos C\cos(A-B)+2\cos^2 C-1\right] \quad (A+B=\pi-C)\\
&= \frac{3}{2}-\frac{1}{2}+\frac{1}{2}\left[-2\cos C\left(\cos(A-B)-\cos C\right)\right]\\
&= 1-\cos C\left[\cos(A-B)-\cos C\right]\\
&= 1-\cos C\left[\cos(A-B)+\cos(A+B)\right]\\
&= 1-\cos C\left[2\cos A\cos B\right]\\
&= 1-2\cos A\cos B\cos C
\end{aligned}$$

Exercise - 3.7

1. If $A+B+C=180^\circ$, prove that
   (i) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$
   (ii) $\cos A+\cos B-\cos C=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
   (iii) $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C$
   (iv) $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C$
   (v) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$
   (vi) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
   (vii) $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)=4\sin A\sin B\sin C$.
2. If $A+B+C=2s$, then prove that $\sin(s-A)\sin(s-B)+\sin s\sin(s-C)=\sin A\sin B$.
3. If $x+y+z=xyz$, then prove that $\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\frac{2y}{1-y^2}\frac{2z}{1-z^2}$.
4. If $A+B+C=\frac{\pi}{2}$, prove the following
   (i) $\sin 2A+\sin 2B+\sin 2C=4\cos A\cos B\cos C$
   (ii) $\cos 2A+\cos 2B+\cos 2C=1+4\sin A\sin B\sin C$.
5. If $\triangle ABC$ is a right triangle and if $\angle A=\frac{\pi}{2}$, then prove that
   (i) $\cos^2 B+\cos^2 C=1$
   (ii) $\sin^2 B+\sin^2 C=1$
   (iii) $\cos B-\cos C=-1+2\sqrt{2}\cos\frac{B}{2}\sin\frac{C}{2}$.

3.6 Trigonometric equations

The equations containing trigonometric functions of unknown angles are known as trigonometric equations. A solution of trigonometric equation is the value of unknown angle that satisfies the equation. Unless the domain is restricted, the trigonometric equations have infinitely many solutions, a fact due to the periodicity of trigonometric functions. Some of the equations may not have a solution at all.

For example, the equation $\sin\theta = \frac{3}{2}$, does not have solution, since $-1 \leq \sin\theta \leq 1$.

The equation $\sin\theta = 0$ has infinitely many solutions given by $\theta = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ and note that these solutions occur periodically.

General Solution

The solution of a trigonometric equation giving all the admissible values obtained with the help of periodicity of a trigonometric function is called the **general solution** of the equation.

Principal Solution

The smallest numerical value of unknown angle satisfying the equation in the interval $[0, 2\pi]$ (or) $[-\pi, \pi]$ is called a **principal solution.** We shall take the interval $[-\pi, \pi]$ for defining the principal solution. Further, in this interval we may have two solutions. Even though both are valid solutions, we take only the numerically smaller one. This helps us to define the principal domain of the trigonometric functions in order to have their inverses.

Principal value of sine function lies in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and hence lies in I quadrant or IV quadrant. Principal value of cosine function is in $[0, \pi]$ and hence in I quadrant or II quadrant. Principal value of tangent function is in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and hence in I quadrant or IV quadrant.

(i) Trigonometric equations are different from trigonometric identities, since trigonometric identities are true for all admissible values of unknown angle θ. But trigonometric equations are valid only for particular values of unknown angle.

(ii) There is no general method for solving trigonometric equations. However, one may notice that some equations may be factorisable; some equations may be expressed in terms of single function; some equations may be squared.

(iii) To find solutions to trigonometric equations, some times one may go for the technique of squaring both sides. One has to take care as it can also produce false solutions (**extraneous solutions**).

For example, to find solution for $\sin x - \cos x = 1$ in $0 \leq x < 360°$, we do squaring on both sides to get $(\sin x - 1)^2 = \cos^2 x$, which gives $2\sin x(\sin x - 1) = 0$. So, we get $x = 0, \frac{\pi}{2}, \pi$. Clearly $x = 0$ is a false solution. Thus, we have to check for correct solutions, in the squaring process.

(iv) Mostly we write the solutions of trigonometric equations in radians.

Now, we find the solutions to different forms of trigonometrical equations.

(i) **To solve an equation of the form** $\sin\theta = k \ \ (-1 \le k \le 1)$:

Let α be the numerically smallest angle such that $\sin\alpha = k$. Thus,

$$\begin{aligned} \sin\theta &= \sin\alpha \\ \sin\theta - \sin\alpha &= 0 \\ 2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} &= 0 \text{ , which gives either} \\ \cos\frac{\theta+\alpha}{2} = 0 \text{ or } \sin\frac{\theta-\alpha}{2} &= 0. \end{aligned}$$

Now, $\cos\frac{\theta+\alpha}{2} = 0$	Now, $\sin\frac{\theta-\alpha}{2} = 0$,
$\frac{\theta+\alpha}{2} = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$	$\frac{\theta-\alpha}{2} = n\pi, n \in \mathbb{Z}$
$\theta = (2n+1)\pi - \alpha, n \in \mathbb{Z}$ ...(i)	$\theta = 2n\pi + \alpha, n \in \mathbb{Z}$ ...(ii)

Combining (i) and (ii), we have $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha, n \in \mathbb{Z}.$ (3.13)

(ii) **To solve an equation of the form** $\cos\theta = k \ \ (-1 \le k \le 1)$:

Let α be the numerically smallest angle such that $\cos\alpha = k$. Thus,

$$\begin{aligned} \cos\theta &= \cos\alpha \\ \cos\theta - \cos\alpha &= 0 \\ 2\sin\frac{\theta+\alpha}{2}\sin\frac{\alpha-\theta}{2} &= 0 \\ \sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} &= 0 \text{ , which gives either} \\ \sin\frac{\theta+\alpha}{2} = 0 \text{ or } \sin\frac{\theta-\alpha}{2} &= 0 \end{aligned}$$

$\sin\frac{\theta+\alpha}{2} = 0$ gives	$\sin\frac{\theta-\alpha}{2} = 0$ gives
$\frac{\theta+\alpha}{2} = n\pi, n \in \mathbb{Z}$	$\frac{\theta-\alpha}{2} = n\pi, n \in \mathbb{Z}$
$\theta = 2n\pi - \alpha, n \in \mathbb{Z}.$ ...(i)	$\theta = 2n\pi + \alpha, n \in \mathbb{Z}$ ...(ii)

Combining (i) and (ii), we have $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha, n \in \mathbb{Z}.$ (3.14)

(iii) **To solve an equation of the form** $\tan\theta = k \ \ (-\infty < k < \infty)$:

Let α be the numerically smallest angle such that $\tan\alpha = k$. Thus,

$$\begin{aligned} \tan\theta &= \tan\alpha \\ \frac{\sin\theta}{\cos\theta} &= \frac{\sin\alpha}{\cos\alpha} \Rightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0 \\ \sin(\theta-\alpha) &= 0 \Rightarrow \theta - \alpha = n\pi \\ \theta &= n\pi + \alpha, n \in \mathbb{Z} \\ \text{Thus, } \tan\theta &= \tan\alpha \Rightarrow \theta = n\pi + \alpha, n \in \mathbb{Z} \end{aligned} \quad (3.15)$$

(iv) **To solve an equation of the form** $a\cos\theta + b\sin\theta = c$:

$$\begin{aligned}
\text{Take } a &= r\cos\alpha,\ \ b = r\sin\alpha. \text{ Then } r = \sqrt{a^2+b^2};\ \ \tan\alpha = \frac{b}{a},\ a \neq 0 \\
a\cos\theta + b\sin\theta &= c \Rightarrow r\cos\alpha\cos\theta + r\sin\alpha\sin\theta = c \\
r\cos(\theta-\alpha) &= c \\
\cos(\theta-\alpha) &= \frac{c}{r} = \frac{c}{\sqrt{a^2+b^2}} = \cos\phi \text{ (say)} \\
\theta - \alpha &= 2n\pi \pm \phi, n \in \mathbb{Z} \\
\theta &= 2n\pi + \alpha \pm \phi, n \in \mathbb{Z}.
\end{aligned}$$

The above method can be used only when $c \leq \sqrt{a^2+b^2}$.
If $c > \sqrt{a^2+b^2}$, then the equation $a\cos\theta + b\sin\theta = c$ has no solution.

Now we summarise the general solution of trigonometric equations.

Trigonometric Equation	General solution
$\sin\theta = 0$	$\theta = n\pi; n \in \mathbb{Z}$
$\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}; n \in \mathbb{Z}$
$\tan\theta = 0$	$\theta = n\pi; n \in \mathbb{Z}$
$\sin\theta = \sin\alpha$, where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\theta = n\pi + (-1)^n\alpha, n \in \mathbb{Z}$
$\cos\theta = \cos\alpha$, where $\alpha \in [0, \pi]$	$\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}$
$\tan\theta = \tan\alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\theta = n\pi + \alpha, n \in \mathbb{Z}$

Example 3.42 Find the principal solution of (i) $\sin\theta = \frac{1}{2}$ (ii) $\sin\theta = -\frac{\sqrt{3}}{2}$
(iii) cosec$\theta = -2$ (iv) $\cos\theta = \frac{1}{2}$

Solution:

(i) $\sin\theta = \frac{1}{2} > 0$ so principal value lies in the I quadrant.

$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

Thus, $\theta = \frac{\pi}{6}$ is the principal solution.

(ii) $\sin\theta = -\frac{\sqrt{3}}{2}$

We know that principal value of $\sin\theta$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Since, $\sin\theta = -\frac{\sqrt{3}}{2} < 0$, the principal value of $\sin\theta$ lies in the IV quadrant.

$\sin\theta = -\frac{\sqrt{3}}{2} = -\sin\left(\frac{\pi}{3}\right) = \sin\left(-\frac{\pi}{3}\right)$.

Hence, $\theta = -\frac{\pi}{3}$ is the principal solution.

(iii) $\operatorname{cosec}\theta = -2$

$\operatorname{cosec}\theta = -2 \Rightarrow \frac{1}{\sin\theta} = -2 \Rightarrow \sin\theta = -\frac{1}{2}$

Since $\sin\theta = -\frac{1}{2} < 0$, the principal value of $\sin\theta$ lies in the IV quadrant.

$\sin\theta = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(-\frac{\pi}{6}\right)$

Thus, $\theta = -\frac{\pi}{6}$ is the principal solution.

(iv) $\cos\theta = \frac{1}{2}$

Principal value of $\cos\theta$ lies in the I and II quadrant.

Since $\cos\theta = \frac{1}{2} > 0$, the principal value of $\cos\theta$ lies in the interval $\left[0, \frac{\pi}{2}\right]$.

$\cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$

Thus, $\theta = \frac{\pi}{3}$ is the principal solution.

Example 3.43 Find the general solution of $\sin\theta = -\frac{\sqrt{3}}{2}$

Solution:

The general solution of $\sin\theta = \sin\alpha, \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is $\theta = n\pi + (-1)^n\alpha, n \in \mathbb{Z}$

$$\sin\theta = -\frac{\sqrt{3}}{2} = \sin\left(-\frac{\pi}{3}\right),$$

Thus the general solution is

$$\theta = n\pi + (-1)^n\left(-\frac{\pi}{3}\right) = n\pi + (-1)^{n+1}\left(\frac{\pi}{3}\right); n \in \mathbb{Z} \quad \text{... (i)}$$

In arriving at the above general solution, we took the principal value as $-\left(\frac{\pi}{3}\right)$ with our convention that principal value is the numerically smallest one in the interval $[-\pi, \pi]$. Now through this example, we shall justify that the principal value may also be taken in $[0, 2\pi]$,

as we mentioned in the definition of principle solution. If we take the principal solution in the interval $[0, 2\pi]$, then the principal solution is $\theta = \frac{4\pi}{3}$ and the general solution is

$$\theta = n\pi + (-1)^n \left(\frac{4\pi}{3}\right), n \in \mathbb{Z}, \frac{4\pi}{3} \in [0, 2\pi] \quad \text{... (ii)}$$

From (ii), for $n = 0, -1, 1, -2, 2, \ldots$
the corresponding solutions are $\frac{4\pi}{3}, \frac{-7\pi}{3}, \frac{-\pi}{3}, \frac{-2\pi}{3}, \frac{10\pi}{3}, \ldots$
From (i), for $n = 0, -1, 1, -2, 2, \ldots$
the corresponding solutions are $\frac{-\pi}{3}, \frac{-2\pi}{3}, \frac{4\pi}{3}, \frac{-7\pi}{3}, \frac{5\pi}{3}, \ldots$

In both the cases, we get the same set of solutions, but in different order. Thus, we have justified that the principal solution can be taken either in $[0, 2\pi]$ or in $[-\pi, \pi]$.

Example 3.44 Find the general solution of
(i) $\sec\theta = -2$ (ii) $\tan\theta = \sqrt{3}$

Solution:

(i) $\sec\theta = -2$

$\sec\theta = -2 \Rightarrow \cos\theta = -\frac{1}{2}$

We know that the general solution of $\cos\theta = \cos\alpha, \alpha \in [0, \pi]$, is $\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}$

Let us find $\alpha \in [0, \pi]$ such that $\cos\alpha = -\frac{1}{2} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$.

so, $\alpha = \frac{2\pi}{3}$.

Thus, the general solution is $\theta = 2n\pi \pm \frac{2\pi}{3}, n \in \mathbb{Z}$.

(ii) $\tan\theta = \sqrt{3}$

$\tan\theta = \sqrt{3} = \tan\frac{\pi}{3}$

We know that the general solution of $\tan\theta = \tan\alpha, \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is $\theta = n\pi + \alpha, n \in \mathbb{Z}$

Thus, $\theta = n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$, is the general solution.

Example 3.45 Solve $3\cos^2\theta = \sin^2\theta$

Solution:

$3\cos^2\theta = 1 - \cos^2\theta \Rightarrow \cos^2\theta = \frac{1}{4}$

$\frac{\cos 2\theta + 1}{2} = \frac{1}{4} \Rightarrow \cos 2\theta = -\frac{1}{2} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$

$2\theta = 2n\pi \pm \frac{2\pi}{3}, n \in \mathbb{Z} \Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$.

Try to solve by writing $\tan^2\theta = 3$.

Example 3.46 Solve $\sin x + \sin 5x = \sin 3x$

Solution:

$\sin x + \sin 5x = \sin 3x \Rightarrow 2\sin 3x \cos 2x = \sin 3x$

$\sin 3x\,(2\cos 2x - 1) = 0$

Thus, either $\sin 3x = 0$ (or) $\cos 2x = \dfrac{1}{2}$

If $\sin 3x = 0$, then $3x = n\pi \Rightarrow x = \dfrac{n\pi}{3}, n \in \mathbb{Z}$ ... (i)

If $\cos 2x = \dfrac{1}{2} \Rightarrow \cos 2x = \cos\dfrac{\pi}{3}$ ... (ii)

$2x = 2n\pi \pm \dfrac{\pi}{3} \Rightarrow x = n\pi \pm \dfrac{\pi}{6}, n \in \mathbb{Z}$

From (i) and (ii), we have the general solution $x = \dfrac{n\pi}{3}$ (or) $x = n\pi \pm \dfrac{\pi}{6}, n \in \mathbb{Z}$.

Example 3.47 Solve $\cos x + \sin x = \cos 2x + \sin 2x$

Solution:

$\cos x + \sin x = \cos 2x + \sin 2x \Rightarrow \cos x - \cos 2x = \sin 2x - \sin x$

$2\sin\left(\dfrac{x+2x}{2}\right)\sin\left(\dfrac{2x-x}{2}\right) = 2\cos\left(\dfrac{2x+x}{2}\right)\sin\left(\dfrac{2x-x}{2}\right)$

$2\sin\left(\dfrac{3x}{2}\right)\sin\left(\dfrac{x}{2}\right) = 2\cos\left(\dfrac{3x}{2}\right)\sin\left(\dfrac{x}{2}\right)$

$\sin\left(\dfrac{x}{2}\right)\left[\sin\left(\dfrac{3x}{2}\right) - \cos\left(\dfrac{3x}{2}\right)\right] = 0$

Thus, either $\sin\left(\dfrac{x}{2}\right) = 0$ (or) $\sin\left(\dfrac{3x}{2}\right) - \cos\left(\dfrac{3x}{2}\right) = 0.$

When $\sin\left(\dfrac{x}{2}\right) = 0$	When $\sin\left(\dfrac{3x}{2}\right) - \cos\left(\dfrac{3x}{2}\right) = 0$
$\dfrac{x}{2} = n\pi \Rightarrow x = 2n\pi, n \in \mathbb{Z}.$	$\Rightarrow \tan\left(\dfrac{3x}{2}\right) = 1 = \tan\left(\dfrac{\pi}{4}\right)$
	$\dfrac{3x}{2} = n\pi + \dfrac{\pi}{4} \Rightarrow x = \dfrac{2n\pi}{3} + \dfrac{\pi}{6}.$

Thus, the general solution is $x = 2n\pi$ (or) $x = \dfrac{2n\pi}{3} + \dfrac{\pi}{6}, n \in \mathbb{Z}$.

If $\sin\theta = \cos\theta$, then $\theta \neq (2n+1)\dfrac{\pi}{2}, n \in \mathbb{Z}$. So, we have $\dfrac{\sin\theta}{\cos\theta} = 1$

Example 3.48 Solve the equation $\sin 9\theta = \sin\theta$

Solution:

$\sin 9\theta = \sin\theta \Rightarrow \sin 9\theta - \sin\theta = 0$

$2\cos 5\theta \sin 4\theta = 0$

Either $\cos 5\theta = 0$ (or) $\sin 4\theta = 0$

When $\cos 5\theta = 0 \Rightarrow 5\theta = (2n+1)\dfrac{\pi}{2}$
$\Rightarrow \theta = (2n+1)\dfrac{\pi}{10}, n \in \mathbb{Z}$

When, $\sin 4\theta = 0 \Rightarrow 4\theta = n\pi$
$\Rightarrow \theta = \dfrac{n\pi}{4}, n \in \mathbb{Z}$

Thus, the general solution of the given equation is $\theta = (2n+1)\dfrac{\pi}{10}, \quad \theta = \dfrac{n\pi}{4}, n \in \mathbb{Z}$.

Example 3.49 Solve $\tan 2x = -\cot\left(x + \dfrac{\pi}{3}\right)$

Solution:

$$\tan 2x = -\cot\left(x+\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{2}+x+\frac{\pi}{3}\right) = \tan\left(\frac{5\pi}{6}+x\right)$$
$$2x = n\pi + \frac{5\pi}{6} + x, n \in \mathbb{Z} \Rightarrow x = n\pi + \frac{5\pi}{6}, n \in \mathbb{Z}.$$

Example 3.50 Solve $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

Solution:

$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$
$\sin 3x + \sin x - 3\sin 2x = \cos 3x + \cos x - 3\cos 2x$
$2\sin 2x\cos x - 3\sin 2x = 2\cos 2x\cos x - 3\cos 2x$
$(\sin 2x - \cos 2x)(2\cos x - 3) = 0$
Then, $\sin 2x - \cos 2x = 0$ since $2\cos x - 3 \neq 0$
$\sin 2x = \cos 2x \Rightarrow \tan 2x = 1 \Rightarrow x = \dfrac{n\pi}{2} + \dfrac{\pi}{8}, n \in \mathbb{Z}.$

Example 3.51 Solve $\sin x + \cos x = 1 + \sin x\cos x$

Solution:

Let $\sin x + \cos x = t$
$\Rightarrow 1 + 2\sin x\cos x = t^2 \Rightarrow \sin x\cos x = \dfrac{t^2-1}{2}$

Thus, the given equation becomes $t^2 - 2t + 1 = 0 \Rightarrow t = 1$

Hence, $\sin x + \cos x = 1$
$\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\sin x + \dfrac{1}{\sqrt{2}}\cos x\right) = 1$
$\sqrt{2}\cos\left(\dfrac{\pi}{4} - x\right) = 1 \Rightarrow \cos\left(\dfrac{\pi}{4} - x\right) = \dfrac{1}{\sqrt{2}}$
Thus, we have $x - \dfrac{\pi}{4} = \pm\dfrac{\pi}{4} + 2n\pi, n \in \mathbb{Z}$
$x = \dfrac{\pi}{2} + 2n\pi,$ or $x = 2n\pi,\ n \in \mathbb{Z}.$

Example 3.52 Solve $2\sin^2 x + \sin^2 2x = 2$

Solution:

$$2\sin^2 x + \sin^2 2x = 2 \Rightarrow 2\sin^2 x + (2\sin x\cos x)^2 = 2$$
$$\cos^2 x\left(2\sin^2 x - 1\right) = 0$$

Either $\cos x = 0$ or $\sin^2 x = \dfrac{1}{2}$

$\cos x = 0$	$\sin^2 x = \dfrac{1}{2} = \sin^2\dfrac{\pi}{4}$
$\Rightarrow x = (2n+1)\dfrac{\pi}{2}, n \in \mathbb{Z}$	$x = n\pi \pm \dfrac{\pi}{4}, n \in \mathbb{Z}$ How

Thus the solution is $x = (2n+1)\dfrac{\pi}{2}, \quad x = n\pi \pm \dfrac{\pi}{4}, n \in \mathbb{Z}$.

Example 3.53 Prove that for any a and b , $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$

Solution:

Now, $a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\left[\dfrac{a}{\sqrt{a^2+b^2}}\sin\theta + \dfrac{b}{\sqrt{a^2+b^2}}\cos\theta\right]$

$= \sqrt{a^2+b^2}\,[\cos\alpha\sin\theta + \sin\alpha\cos\theta]$

where $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}, \quad \sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$

$= \sqrt{a^2+b^2}\ \ \sin(\alpha+\theta)$

Thus, $|a\sin\theta + b\cos\theta| \le \sqrt{a^2+b^2}$

Hence, $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$.

Example 3.54 Solve $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$

Solution:

$\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$

Here $a = -1$; $b = \sqrt{3}$; $c = \sqrt{2}$; $r = \sqrt{a^2+b^2} = 2$.

Thus, the given equation can be rewritten as

$$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{1}{\sqrt{2}}$$
$$\sin\theta\cos\frac{\pi}{6} - \cos\theta\sin\frac{\pi}{6} = \sin\frac{\pi}{4}$$
$$\sin\left(\theta - \frac{\pi}{6}\right) = \sin\frac{\pi}{4}$$
$$\theta - \frac{\pi}{6} = n\pi \pm (-1)^n\frac{\pi}{4}, n \in \mathbb{Z}$$

Thus, $\theta = n\pi + \dfrac{\pi}{6} \pm (-1)^n\dfrac{\pi}{4}, n \in \mathbb{Z}$.

Example 3.55 Solve $\sqrt{3}\tan^2\theta + \left(\sqrt{3}-1\right)\tan\theta - 1 = 0$

Solution:

$$\sqrt{3}\tan^2\theta + \left(\sqrt{3}-1\right)\tan\theta - 1 = 0$$
$$\sqrt{3}\tan^2\theta + \sqrt{3}\tan\theta - \tan\theta - 1 = 0$$
$$\left(\sqrt{3}\tan\theta - 1\right)(\tan\theta + 1) = 0$$

Thus, either $\sqrt{3}\tan\theta - 1 = 0$ (or) $\tan\theta + 1 = 0$

If $\sqrt{3}\tan\theta - 1 = 0$, then	If $\tan\theta + 1 = 0$ then
$\tan\theta = \dfrac{1}{\sqrt{3}} = \tan\dfrac{\pi}{6}$	$\tan\theta = -1 = \tan\left(\dfrac{-\pi}{4}\right)$
$\Rightarrow \theta = n\pi + \dfrac{\pi}{6}, n \in \mathbb{Z}$...(i)	$\Rightarrow \theta = n\pi - \dfrac{\pi}{4}, n \in \mathbb{Z}$...(ii)

From (i) and (ii) we have the general solution.

Exercise - 3.8

1. Find the principal solution and general solutions of the following:
   (i) $\sin\theta = -\dfrac{1}{\sqrt{2}}$ (ii) $\cot\theta = \sqrt{3}$ (iii) $\tan\theta = -\dfrac{1}{\sqrt{3}}$.
2. Solve the following equations for which solutions lies in the interval $0^\circ \le \theta < 360^\circ$
   (i) $\sin^4 x = \sin^2 x$
   (ii) $2\cos^2 x + 1 = -3\cos x$
   (iii) $2\sin^2 x + 1 = 3\sin x$
   (iv) $\cos 2x = 1 - 3\sin x$.
3. Solve the following equations:
   (i) $\sin 5x - \sin x = \cos 3x$
   (ii) $2\cos^2\theta + 3\sin\theta - 3 = 0$
   (iii) $\cos\theta + \cos 3\theta = 2\cos 2\theta$
   (iv) $\sin\theta + \sin 3\theta + \sin 5\theta = 0$
   (v) $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$
   (vi) $\sin\theta + \cos\theta = \sqrt{2}$
   (vii) $\sin\theta + \sqrt{3}\cos\theta = 1$
   (viii) $\cot\theta + \text{cosec}\theta = \sqrt{3}$
   (ix) $\tan\theta + \tan\left(\theta + \dfrac{\pi}{3}\right) + \tan\left(\theta + \dfrac{2\pi}{3}\right) = \sqrt{3}$
   (x) $\cos 2\theta = \dfrac{\sqrt{5}+1}{4}$
   (xi) $2\cos^2 x - 7\cos x + 3 = 0$

3.7 Properties of Triangle

One important use of trigonometry is to solve practical problems that can be modeled by a triangle. Determination of all the sides and angles of a triangle is referred as solving the triangle. In any triangle, the three sides and three angles are called basic elements of a triangle. Pythagorean theorem plays a vital role in finding solution of the right triangle. The law of sines and the law of cosines are important tools that can be used effectively in solving an oblique triangle (a triangle with no right angle). In this section, we shall discuss the relationship between the sides and angles of a triangle and derive the law of sines and the law of cosines.

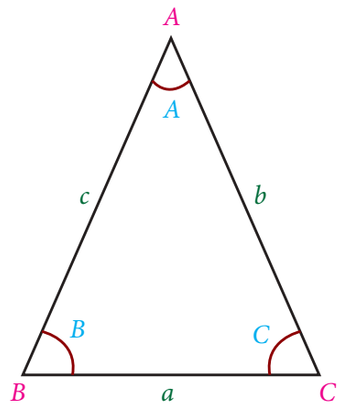


Figure 3.17

Notation: Let ABC be a triangle. The angles of $\triangle ABC$ corresponding to the vertices A, B, C are denoted by A, B, C themselves. The sides opposite to the angles A, B, C are denoted by a, b, c respectively. Also we use the symbol $\triangle$ to denote the area of a triangle.

The circle which passes through the three vertices of a triangle is called circumcircle of the triangle. The centre and the radius R of the circumcircle are known as circumcentre and circumradius respectively.

In $\triangle ABC$, we have $A + B + C = \pi$ and $a + b > c, \quad b + c > a, \quad a + c > b$.

The Law of Sines or Sine Formula

3.7.1 Law of Sines

The Law of Sines is a relationship between the angles and the sides of a triangle. While solving a triangle, the law of sines can be effectively used in the following situations:

(i) To find an angle if two sides and one angle which is not included, by them are given.
(ii) To find a side, if two angles and one side which is opposite to one of given angles, are given.

Theorem 3.1 (Law of Sines)**:** In any triangle, the lengths of the sides are proportional to the sines of the opposite angles. That is, in $\triangle ABC$, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ where R is the circumradius of the triangle.

Proof. The angle A of the $\triangle ABC$ is either acute or right or obtuse. Let O be the centre of the circumcircle of $\triangle ABC$ and R, its radius.

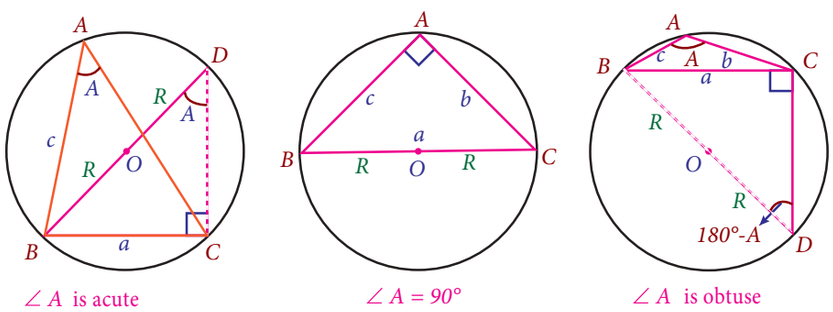


Figure 3.18

Case I: $\angle A$ is acute.

Produce BO to meet the circle at D.

$\angle BDC = \angle BAC = A$

$\angle BCD = 90^\circ$

$\sin \angle BDC = \dfrac{BC}{BD}$ or $\sin A = \dfrac{a}{2R} \Rightarrow \dfrac{a}{\sin A} = 2R$

Case II: $\angle A$ is right angle.

In this case O must be on the side BC of the$\triangle ABC$.

Now, $\dfrac{a}{\sin A} = \dfrac{BC}{\sin 90^\circ} = \dfrac{2R}{1} = 2R \Rightarrow \dfrac{a}{\sin A} = 2R$

Case III: $\angle A$ is obtuse .

Produce BO to meet the circle at D.

$\angle BDC + \angle BAC = 180^\circ$

$\angle BDC = 180^\circ - \angle BAC = 180^\circ - A$

$\angle BCD = 90^\circ$

$\sin \angle BDC = \dfrac{BC}{BD}$ or $\sin(180^\circ - A) = \sin A = \dfrac{a}{2R} \Rightarrow \dfrac{a}{\sin A} = 2R$

In each case, we have $\dfrac{a}{\sin A} = 2R$

Similarly, by considering angles B and C, we can prove that $\dfrac{b}{\sin B} = 2R$ and $\dfrac{c}{\sin C} = 2R$ respectively.

Thus, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R.$

(i) The Law of Sines can be written as a collection of three equations

$$\frac{a}{b} = \frac{\sin A}{\sin B}; \qquad \frac{a}{c} = \frac{\sin A}{\sin C}; \qquad \frac{b}{c} = \frac{\sin B}{\sin C};$$

(ii) The Law of Sines says that the sides of a triangle are proportional to the sines of their opposite angles.

(iii) Using the Law of Sines, it is impossible to find the solution to a triangle given two sides and the included angle.

(iv) An interesting geometric consequence of the Law of Sines is that the largest side of any triangle is opposite to the largest angle. (Prove)

Napier's Formula

Theorem 3.2: In $\triangle ABC$, we have

(i) $\tan \dfrac{A-B}{2} = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$

(ii) $\tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c} \cot \dfrac{A}{2}$

(iii) $\tan \dfrac{C-A}{2} = \dfrac{c-a}{c+a} \cot \dfrac{B}{2}$

Proof.

We know the sine formula: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

Now,
$$\begin{aligned}
\frac{a-b}{a+b}\cot\frac{C}{2} &= \frac{2R\sin A - 2R\sin B}{2R\sin A + 2R\sin B}\cot\frac{C}{2}\\
&= \frac{\sin A - \sin B}{\sin A + \sin B}\cot\frac{C}{2}\\
&= \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}\cot\frac{C}{2}\\
&= \cot\frac{A+B}{2}\tan\frac{A-B}{2}\cot\frac{C}{2}\\
&= \cot\left(90^\circ - \frac{C}{2}\right)\tan\frac{A-B}{2}\cot\frac{C}{2}\\
&= \tan\frac{C}{2}\tan\frac{A-B}{2}\cot\frac{C}{2} = \tan\frac{A-B}{2}
\end{aligned}$$

Similarly we can prove the other two results.

3.7.2 Law of Cosines

When two sides and included angle or the three sides of a triangle are given, the triangle cannot be solved using the sine formula. In such a situation, the law of cosines can be used to solve the triangle. Also, the Law of Cosines is used to derive a formula for finding the area of a triangle given two sides and the included angle.

Theorem 3.3 (The Law of Cosines)**:** In $\triangle ABC$, we have
$$\cos A = \frac{b^2+c^2-a^2}{2bc};\ \cos B = \frac{c^2+a^2-b^2}{2ca};\ \cos C = \frac{a^2+b^2-c^2}{2ab}.$$

Proof. In $\triangle ABC$, draw $AD \perp BC$.
In $\triangle ABD$, we have $AB^2 = AD^2 + BD^2 \Rightarrow c^2 = AD^2 + BD^2$.
Now, we find the values of AD and BD in terms of the elements of $\triangle ABC$.

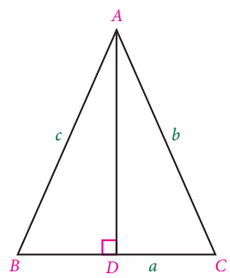


Figure 3.19

$$\begin{aligned}
&\frac{AD}{AC} = \sin C \Rightarrow AD = b\sin C\\
&BD = BC - DC = a - b\cos C\\
&c^2 = (b\sin C)^2 + (a - b\cos C)^2\\
&\quad = b^2\sin^2 C + a^2 + b^2\cos^2 C - 2ab\cos C\\
&\quad = a^2 + b^2\left(\sin^2 C + \cos^2 C\right) - 2ab\cos C\\
&\quad = a^2 + b^2 - 2ab\cos C
\end{aligned}$$

Thus, $c^2 = a^2 + b^2 - 2ab\cos C$ or $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$.

Similarly, we can prove the other two results, namely
$$a^2 = b^2 + c^2 - 2bc\cos A \ \text{ or } \ \cos A = \frac{b^2+c^2-a^2}{2bc}$$
$$b^2 = c^2 + a^2 - 2ca\cos B \ \text{ or } \ \cos B = \frac{c^2+a^2-b^2}{2ca}$$

(i) $a^2 = b^2 + c^2 - 2bc\cos A$ says that the square of a side is the sum of squares of other two sides diminished by twice the product of those two sides and the cosine of the included angle. Also one formula will give the other formula by cycling through the letters a, b, c.

(ii) The Laws of Cosine for right triangles reduce to Pythagorean theorem. Thus, the Law of cosines can be viewed as a generalisation of Pythagorean theorem.

(iii) The advantage of using law of cosines over law of sines is that unlike the sine function, the cosine function distinguishes between acute and obtuse angles. If cosine of an angle is positive, then it is acute. Otherwise, it is obtuse.

(iv) **The Law of Cosines says** : The direct route is the shortest. Let us explain this. In a $\triangle ABC$, $c^2 = a^2 + b^2 - 2ab\cos C$. Since $-\cos C < 1$ we have $c^2 < a^2 + b^2 + 2ab$. Thus, we have $c < a + b$. Hence, In $\triangle ABC$, we have $a < b + c,\ \ b < c + a,\ \ c < a + b$

(v) When using the law of cosines, it is always best to find the measure of the largest unknown angle first, since this will give us the obtuse angle of the triangle if there is one such angle.

3.7.3 Projection Formula

Theorem 3.4: In a $\triangle ABC$, we have

(i) $a = b\cos C + c\cos B$, (ii) $b = c\cos A + a\cos C$, (iii) $c = a\cos B + b\cos A$

Proof. In $\triangle ABC$, we have $a = BC$. Draw $AD \perp BC$.

$$\begin{aligned} a &= BC = BD + DC \\ &= \frac{BD}{AB}AB + \frac{DC}{AC}AC \\ &= (\cos B)c + (\cos C)b \\ a &= b\cos C + c\cos B \end{aligned}$$

Similarly, one can prove the other two projection formulas.

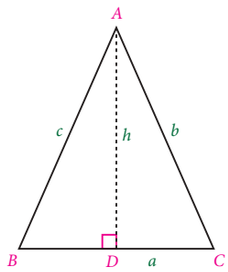


Figure 3.20

$a = b\cos C + c\cos B$ says that $a =$ projection of b on $a +$ projection of c on a
Thus, a side of triangle is equal to sum of the projections of other two sides on it.

3.7.4 Area of the Triangle

We shall use some elements of an oblique triangle and the sine function to find the area of the triangle. Recall that area formula for $\triangle ABC$ is $\frac{1}{2}bh$ where b is the base and h is the height. For oblique triangle, we must find h before using the area formula.

Theorem 3.5: In $\triangle ABC$, area of the triangle is

$$\triangle = \frac{1}{2}\,ab\sin C = \frac{1}{2}\,bc\sin A = \frac{1}{2}\,ac\sin B$$

Proof. In $\triangle ABC$, draw $AD \perp BC$

In $\triangle ADC$, $\dfrac{AD}{AC} = \sin C \Rightarrow AD = b\sin C$

Thus, $\triangle = \dfrac{1}{2} \times$ base $\times$ height $= \dfrac{1}{2}\,ab\sin C.$

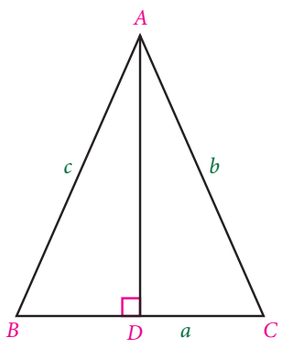


Figure 3.21

Similarly we can derive the other two results.

(i) The formula for the area of an oblique triangle says that the area is equal to one half of the product of two sides and the sine of their included angle.

(ii) The area formula is used to compute the area of the segment of a circle. Segment of a circle is the region between a chord and the arc it cuts off.

Let r be the radius of a circle and θ be the angle subtended by the chord AB at the centre.

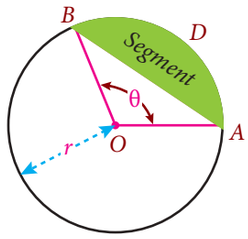


Figure 3.22

$$\begin{aligned}\text{Area of the segment } ABD &= \text{Area of the sector} - \text{Area of the } \triangle OAB \\ &= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta \\ &= \frac{1}{2}r^2(\theta - \sin\theta)\end{aligned}$$

(iii) The area formula of a triangle is viewed as generalisation of area formula of a right triangle.

(iv) In the above formula, it is clear that the measure of third side is not required in finding the area of the triangles. Also there is no need of finding the altitude of the triangle in order to find its area.

Example 3.56

The Government plans to have a circular zoological park of diameter $8\ km$. A separate area in the form of a segment formed by a chord of length $4\ km$ is to be allotted exclusively for a veterinary hospital in the park. Find the area of the segment to be allotted for the veterinary hospital.

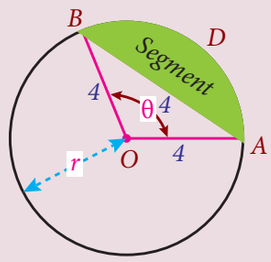


Figure 3.23

Solution:

Let AB be the chord and O be the centre of the circular park.

Let $\angle AOB = \theta$.

$$\begin{aligned}\text{Area of the segment} &= \text{Area of the sector} - \text{Area of} \triangle OAB. \\ &= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta \\ &= \left(\frac{1}{2}\times 4^2\right)[\theta - \sin\theta] = 8[\theta - \sin\theta] \qquad \text{...(i)}\end{aligned}$$

But $\cos\theta = \dfrac{4^2+4^2-4^2}{2(4)(4)} = \dfrac{1}{2}$

Thus, $\theta = \dfrac{\pi}{3}$

From (i), area of the the segment to be allotted for the veterinary hospital

$$= 8\left[\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right] = \frac{4}{3}\left[2\pi - 3\sqrt{3}\right]\ m^2$$

3.7.5 Half-Angle formula

Theorem 3.6: In $\triangle ABC$

(i) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, (ii) $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$,

(iii) $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, where s is the semi-perimeter of $\triangle ABC$ given by $s = \dfrac{a+b+c}{2}$

Proof. (i)

$$\begin{aligned}
\sin\frac{A}{2} &= +\sqrt{\sin^2\frac{A}{2}} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1}{2}\left(1-\frac{b^2+c^2-a^2}{2bc}\right)} \\
&= \sqrt{\frac{2bc-b^2-c^2+a^2}{4bc}} = \sqrt{\frac{a^2-(b-c)^2}{4bc}} \\
&= \sqrt{\frac{(a+b-c)(a-b+c)}{4bc}} = \sqrt{\frac{(a+b+c-2c)(a+b+c-2b)}{4bc}} \\
&= \sqrt{\frac{(2s-2b)(2s-2c)}{4bc}} = \sqrt{\frac{(s-b)(s-c)}{bc}}
\end{aligned}$$

Thus, $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$.

Similarly, we can prove the other two results.

The other half-angle formulas are

$$\begin{aligned}
\sin\frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{ac}}, & \sin\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}} \\
\cos\frac{B}{2} &= \sqrt{\frac{s(s-b)}{ac}}, & \cos\frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}} \\
\tan\frac{B}{2} &= \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, & \tan\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}},
\end{aligned}$$

Corollary: $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2} = 2\sqrt{\dfrac{(s-b)(s-c)}{bc}}\sqrt{\dfrac{s(s-a)}{bc}}$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

Area of a triangle (Heron's Formula)

Heron's formula is named after Hero of Alexendria, a Greek Engineer and Mathematician in (CE 10 - 70). This formula is used only when all the three sides are known.

Theorem 3.7: In $\triangle ABC$, $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$ where s is the semi-perimeter of $\triangle ABC$.

Proof.

$$\begin{aligned}
\triangle &= \frac{1}{2}ab\sin C = \frac{1}{2}ab\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right) \\
&= ab\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{s(s-c)}{ab}} = \sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}$$

(i) Using Heron's formula , Pythagorean theorem can be proved for right triangle and conversely, using Pythagorean theorem for right triangle one can establish Heron's area formula.

(ii) If area of a triangle is given as an integer, then Heron's formula is useful in finding triangles with integer sides.

(iii) If the perimeter of a triangle is fixed, then Heron's formula is useful for finding triangles having integer area and integer sides.

For example, if the perimeter of a triangle is $100m$, then there is a triangle with sides $32\ m, 34\ m, 34\ m$ and area $480\ m^2$.

Example 3.57 In a $\triangle ABC$, prove that $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A$.

Solution:

The Sine formula is, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

Thus, $a = 2R \sin A; \quad b = 2R \sin B; \quad c = 2R \sin C$

$$\begin{aligned}
b^2 \sin 2C + c^2 \sin 2B &= 4R^2 \sin^2 B \sin 2C + 4R^2 \sin^2 C \sin 2B \\
&= 4R^2 \left(2 \sin^2 B \sin C \cos C + 2 \sin^2 C \sin B \cos B\right) \\
&= 8R^2 \sin B \sin C \left(\sin B \cos C + \sin C \cos B\right) \\
&= 8R^2 \sin B \sin C \sin(B + C) \\
&= 8R^2 \sin B \sin C \sin(\pi - A) = 8R^2 \sin B \sin C \sin A \\
&= 8R^2 \left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right) \sin A = 2bc \sin A.
\end{aligned}$$

Example 3.58 In a $\triangle ABC$, prove that $\sin\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{a} \cos \dfrac{A}{2}$.

Solution:

The Sine formula is, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

Now,
$$\begin{aligned}
\frac{b-c}{a} \cos \frac{A}{2} &= \frac{2R \sin B - 2R \sin C}{2R \sin A} \cos \frac{A}{2} \\
&= \frac{2 \sin\left(\dfrac{B-C}{2}\right) \cos\left(\dfrac{B+C}{2}\right)}{2 \sin \dfrac{A}{2} \cos \dfrac{A}{2}} \cos \frac{A}{2} \\
&= \frac{\sin\left(\dfrac{B-C}{2}\right) \cos\left(90^\circ - \dfrac{A}{2}\right)}{\sin \dfrac{A}{2}} \\
&= \frac{\sin\left(\dfrac{B-C}{2}\right) \sin\left(\dfrac{A}{2}\right)}{\sin \dfrac{A}{2}} \\
&= \sin\left(\frac{B-C}{2}\right).
\end{aligned}$$

Example 3.59 If the three angles in a triangle are in the ratio $1 : 2 : 3$, then prove that the corresponding sides are in the ratio $1 : \sqrt{3} : 2$.

Solution:
Let the angles be $\theta, 2\theta, 3\theta$.

Then $\theta + 2\theta + 3\theta = 180^\circ$

Thus, $\theta = 30^\circ$

Using the sine formula, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, we have,

$$\frac{a}{\sin 30^\circ} = \frac{b}{\sin 60^\circ} = \frac{c}{\sin 90^\circ}$$

$$a : b : c = \sin 30^\circ : \sin 60^\circ : \sin 90^\circ$$

$$= \frac{1}{2} : \frac{\sqrt{3}}{2} : 1 = 1 : \sqrt{3} : 2$$

Example 3.60 In a $\triangle ABC$, prove that

$$(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a + b + c$$

Solution:

$$\begin{aligned} \text{L.H.S} &= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C \\ &= b\cos C + c\cos B + c\cos A + a\cos C + b\cos A + a\cos B \\ &= a + b + c \quad \text{[by projection formula]} \end{aligned}$$

Example 3.61 In a triangle ABC, prove that $\dfrac{a^2+b^2}{a^2+c^2} = \dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}$

Solution:
The sine formula is, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$$\begin{aligned} \text{L.H.S} &= \frac{a^2+b^2}{a^2+c^2} = \frac{(2R\sin A)^2 + (2R\sin B)^2}{(2R\sin A)^2 + (2R\sin C)^2} \\ &= \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C} = \frac{1 - \cos^2 A + \sin^2 B}{1 - \cos^2 A + \sin^2 C} \\ &= \frac{1 - \left(\cos^2 A - \sin^2 B\right)}{1 - \left(\cos^2 A - \sin^2 C\right)} = \frac{1 - \cos(A+B)\cos(A-B)}{1 - \cos(A+C)\cos(A-C)} \\ &= \frac{1 + \cos(A-B)\cos C}{1 + \cos(A-C)\cos B}. \end{aligned}$$

Example 3.62 Derive cosine formula using the law of sines in a $\triangle ABC$.

Solution:
The law of Sines: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$$\begin{aligned}
\frac{b^2+c^2-a^2}{2bc} &= \frac{(2R\sin B)^2+(2R\sin C)^2-(2R\sin A)^2}{2\,(2R\sin B)\,(2R\sin C)} \\
&= \frac{\sin^2 B+\sin(C+A)\sin(C-A)}{2\sin B\sin C} \\
&= \frac{\sin B\,[\sin B+\sin(C-A)]}{2\sin B\sin C} \\
&= \frac{\sin(C+A)+\sin(C-A)}{2\sin C} = \cos A
\end{aligned}$$

Thus, $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$. Similarly we can derive other two cosine formulas.

Example 3.63 Using Heron's formula, show that the equilateral triangle has the maximum area for any fixed perimeter. [Hint: In $xyz \le k$, maximum occurs when $x = y = z$].

Solution:
Let ABC be a triangle with constant perimeter $2s$. Thus s is constant.
We know that $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$
Observe that $\triangle$ is maximum, when $(s-a)(s-b))(s-c)$ is maximum.

Now, $(s-a)(s-b)(s-c) \le \left(\dfrac{(s-a)+(s-b)+(s-c)}{3}\right)^3 = \dfrac{s^3}{27}$ $\qquad [G.M \le A.M.]$

Thus, we get $(s-a)(s-b)(s-c) \le \dfrac{s^3}{27}$

Equality occurs when $s-a = s-b = s-c$. That is, when $a = b = c$, maximum of $(s-a)(s-b))(s-c)$ is $\dfrac{s^3}{27}$

Thus, for a fixed perimeter $2s$, the area of a triangle is maximum when $a = b = c$.

Hence, for a fixed perimeter, the equilateral triangle has the maximum area and the maximum area is given by $\triangle = \sqrt{\dfrac{s\,(s^3)}{27}} = \dfrac{s^2}{3\sqrt{3}}$ sq.units.

Exercise - 3.9

1. In a $\triangle ABC$, if $\dfrac{\sin A}{\sin C} = \dfrac{\sin(A-B)}{\sin(B-C)}$, prove that a^2, b^2, c^2 are in Arithmetic Progression.
2. The angles of a triangle ABC, are in Arithmetic Progression and if $b : c = \sqrt{3} : \sqrt{2}$, find $\angle A$.

3. In a $\triangle ABC$, if $\cos C = \dfrac{\sin A}{2\sin B}$, show that the triangle is isosceles.
4. In a $\triangle ABC$, prove that $\dfrac{\sin B}{\sin C} = \dfrac{c - a\cos B}{b - a\cos C}$.
5. In a $\triangle ABC$, prove that $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$.
6. In a $\triangle ABC$, $\angle A = 60°$. Prove that $b + c = 2a\cos\left(\dfrac{B-C}{2}\right)$.
7. In a $\triangle ABC$, prove the following
   (i) $a\sin\left(\dfrac{A}{2} + B\right) = (b+c)\sin\dfrac{A}{2}$
   (ii) $a(\cos B + \cos C) = 2(b+c)\sin^2\dfrac{A}{2}$
   (iii) $\dfrac{a^2 - c^2}{b^2} = \dfrac{\sin(A-C)}{\sin(A+C)}$
   (iv) $\dfrac{a\sin(B-C)}{b^2 - c^2} = \dfrac{b\sin(C-A)}{c^2 - a^2} = \dfrac{c\sin(A-B)}{a^2 - b^2}$
   (v) $\dfrac{a+b}{a-b} = \tan\left(\dfrac{A+B}{2}\right)\cot\left(\dfrac{A-B}{2}\right)$.
8. In a $\triangle ABC$, prove that $(a^2 - b^2 + c^2)\tan B = (a^2 + b^2 - c^2)\tan C$.
9. An Engineer has to develop a triangular shaped park with a perimeter $120\ m$ in a village. The park to be developed must be of maximum area. Find out the dimensions of the park.
10. A rope of length $12\ m$ is given. Find the largest area of the triangle formed by this rope and find the dimensions of the triangle so formed.
11. Derive Projection formula from (i) Law of sines, (ii) Law of cosines.

3.8 Application to Triangle

Much of architecture and engineering relies on triangular support on any structure where stability is desired. Trigonometry helps to calculate the correct angle for the triangular support. Also trigonometry envisages the builders to correctly layout a curved structure. For a right triangle, any two information with atleast one side say SS, SA are sufficient to find the remaining elements of the triangle. But, to find the solution of an oblique triangle we need three elements with atleast one side. If any three elements with atleast one side of a triangle are given, then the Law of Sines, the Law of Cosines, the Projection formula can be used to find the other three elements.

Working Rule:

- In a right triangle, two sides determine the third side via the Pythagorean theorem and one acute angle determine the other by using the fact that acute angles in a right triangle are complementary.
- If all the sides of a triangle are given, then we can use either cosine formula or half-angle formula to calculate all the angles of the triangle.
- If any two angles and any one of the sides opposite to given angles are given, then we can use sine formula to calculate the other sides.
- If any two sides of a triangle and the included angle are given, we cannot use the Law of sines; but then we can use the law of cosines to calculate other side and other angles of the triangle. In this case we have a unique triangle.
- All methods of solving an oblique triangle require that the length of atleast one side must be provided.

Let us summarise the working rule to solve an oblique triangle in the following table

Oblique triangle (all angles are acute or one angle is obtuse	Given Information				
	SAA	SSA*	SAS	SSS	AAA
Details and Application for solutions	Law of sines	#(ambiguity arises)	The given angle must be included angle; Either law of Cosines or tangents	Law of Cosines; First find the largest angle	Infinite number of triangles

Angle is not included; We may have more than one triangle;
Application of law of sines yields three cases: No solution or one triangle or two triangles.
Suppose a, b, A are known. Let $h = b \sin A$
If $a < h$, there is no triangle, If $a = h$, then it is right triangle.
If $a > h$ and $a < b$, we have two triangles.
If $a \geq b$, we have only one triangle.

* SSA means Side, Side and Angle

Example 3.64 In a $\triangle ABC$, $a = 3, b = 5$ and $c = 7$.
Find the values of $\cos A$, $\cos B$ and $\cos C$.

Solution:
By Cosine formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{5^2 + 7^2 - 3^2}{2(5)(7)} = \frac{13}{14}$$

Similarly, $\cos B = \frac{11}{14}$, $\cos C = -\frac{1}{2}$.

Example 3.65 In $\triangle ABC$, $A = 30°, B = 60°$ and $c = 10$, Find a and b.

Solution:
Given that $A = 30°, B = 60°, C = 180° - (A + B) = 90°$
Using sine formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 30°} = \frac{b}{\sin 60°} = \frac{10}{\sin 90°}$$

$$a = \frac{10 \sin 30°}{\sin 90°} = \frac{10\left(\frac{1}{2}\right)}{1} = 5$$

$$b = \frac{10 \sin 60°}{\sin 90°} = \frac{10\left(\frac{\sqrt{3}}{2}\right)}{1} = 5\sqrt{3}.$$

Example 3.66 In a $\triangle ABC$, if $a = 2\sqrt{2}, b = 2\sqrt{3}$ and $C = 75°$, find the other side and the angles.

Solution:
Given that $a = 2\sqrt{2}, b = 2\sqrt{3}$ and $C = 75°$.

Using cosine formula, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

$$\cos 75^\circ = \frac{8+12-c^2}{8\sqrt{6}} \Rightarrow \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{8+12-c^2}{8\sqrt{6}} \Rightarrow c = \sqrt{2}\left(\sqrt{3}+1\right)$$

Also, $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{1}{\sqrt{2}}$, Thus $A = 60^\circ,\ B = 180^\circ - (A+C) = 45^\circ$.

Example 3.67 Find the area of the triangle whose sides are $13\ cm, 14\ cm$ and $15\ cm$.

Solution:
We know that $s = \dfrac{a+b+c}{2} \Rightarrow s = \dfrac{13+14+15}{2} = 21\ cm.$

Area of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{21(21-13)(21-14)(21-15)} = 84$ sq.cm.

Example 3.68 In any $\triangle ABC$, prove that $a\cos A + b\cos B + c\cos C = \dfrac{8\triangle^2}{abc}$.

Solution:
We know that $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

Thus, $a\cos A + b\cos B + c\cos C = 2a\left(\dfrac{2\triangle}{ac}\right)\left(\dfrac{2\triangle}{ab}\right) = \dfrac{8\triangle^2}{abc}$

Example 3.69
Suppose that there are two cell phone towers within range of a cell phone. The two towers are located at $6\ km$ apart along a straight highway, running east to west and the cell phone is north of the highway. The signal is $5\ km$ from the first tower and $\sqrt{31}\ km$ from the second tower. Determine the position of the cell phone north and east of the first tower and how far it is from the highway.

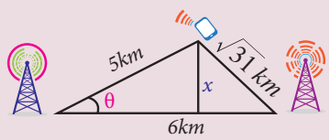


Figure 3.24

Solution:
Let θ be the position of the cell phone from north to east of the first tower.
Then, using the cosine formula, we have,

$$\begin{aligned}\left(\sqrt{31}\right)^2 &= 5^2+6^2-2\times 5\times 6\cos\theta\\ 31 &= 25+36-60\cos\theta\\ \cos\theta &= \frac{1}{2} \Rightarrow \theta = 60^\circ\end{aligned}$$

Let x be the distance of the cell phone's position from the highway.

Then, $\sin\theta = \dfrac{x}{5} \Rightarrow x = 5\sin\theta = 5\sin 60^\circ = \dfrac{5\times\sqrt{3}}{2}\ km.$

Example 3.70

Suppose that a boat travels $10\ km$ from the port towards east and then turns $60°$ to its left. If the boat travels further $8\ km$, how far from the port is the boat?

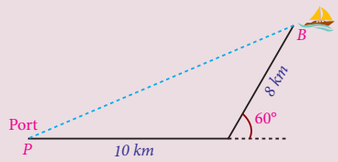


Figure 3.25

Solution:

Let BP be the required distance.

By using the cosine formula, we have,

$$BP^2 = 10^2 + 8^2 - 2 \times \times 10 \times 8 \times \cos 120° = 244km \Rightarrow BP = 2\sqrt{61}\ km$$

Example 3.71

Suppose two radar stations located $100km$ apart, each detect a fighter aircraft between them. The angle of elevation measured by the first station is $30°$, whereas the angle of elevation measured by the second station is $45°$. Find the altitude of the aircraft at that instant.

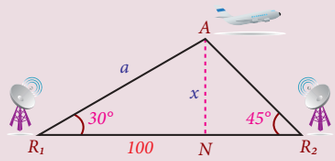


Figure 3.26

Solution:

Let R_1 and R_2 be two radar stations and A be the position of fighter aircraft at the time of detection.

Let x be the required altitude of the aircraft.

Draw $\perp AN$ from A to R_1R_2 meeting at N.

$$\angle A = 180° - (30° + 45°) = 105°$$

Thus, $\dfrac{a}{\sin 45°} = \dfrac{100}{\sin 105°} \Rightarrow a = \dfrac{100}{\dfrac{\sqrt{3}+1}{2\sqrt{2}}} \times \dfrac{1}{\sqrt{2}} = \dfrac{200\left(\sqrt{3}-1\right)}{2} = 100 \times \left(\sqrt{3}-1\right) km$

Now, $\sin 30° = \dfrac{x}{a} \Rightarrow x = 50 \times \left(\sqrt{3}-1\right) km.$

Exercise - 3.10

1. Determine whether the following measurements produce one triangle, two triangles or no triangle: $\angle B = 88°, a = 23, b = 2$. Solve if solution exists.

2. If the sides of a $\triangle ABC$ are $a = 4, b = 6$ and $c = 8$, then show that $4\cos B + 3\cos C = 2$.
3. In a $\triangle ABC$, if $a = \sqrt{3} - 1, b = \sqrt{3} + 1$ and $C = 60°$, find the other side and other two angles.
4. In any $\triangle ABC$, prove that the area $\triangle = \dfrac{b^2 + c^2 - a^2}{4\cot A}$.

5. In a $\triangle ABC$, if $a = 12\ cm$, $b = 8\ cm$ and $C = 30°$, then show that its area is $24\ sq.cm.$
6. In a $\triangle ABC$, if $a = 18\ cm$, $b = 24\ cm$ and $c = 30\ cm$, then show that its area is $216\ sq.cm.$
7. Two soldiers A and B in two different underground bunkers on a straight road, spot an intruder at the top of a hill. The angle of elevation of the intruder from A and B to the ground level in the eastern direction are $30°$ and $45°$ respectively. If A and B stand $5\ km$ apart, find the distance of the intruder from B.
8. A researcher wants to determine the width of a pond from east to west, which cannot be done by actual measurement. From a point P, he finds the distance to the eastern-most point of the pond to

be $8\ km$, while the distance to the western most point from P to be $6\ km$. If the angle between the two lines of sight is $60°$, find the width of the pond.

9. Two Navy helicopters A and B are flying over the Bay of Bengal at same altitude from the sea level to search a missing boat. Pilots of both the helicopters sight the boat at the same time while they are apart $10\ km$ from each other. If the distance of the boat from A is $6\ km$ and if the line segment AB subtends $60°$ at the boat, find the distance of the boat from B.
10. A straight tunnel is to be made through a mountain. A surveyor observes the two extremities A and B of the tunnel to be built from a point P in front of the mountain. If $AP = 3km$, $BP = 5\ km$ and $\angle APB = 120°$, then find the length of the tunnel to be built.
11. A farmer wants to purchase a triangular shaped land with sides $120 feet$ and $60 feet$ and the angle included between these two sides is $60°$. If the land costs ₹500 per sq.ft, find the amount he needed to purchase the land. Also find the perimeter of the land.
12. A fighter jet has to hit a small target by flying a horizontal distance. When the target is sighted, the pilot measures the angle of depression to be $30°$. If after $100\ km$, the target has an angle of depression of $60°$, how far is the target from the fighter jet at that instant?
13. A plane is $1\ km$ from one landmark and $2\ km$ from another. From the planes point of view the land between them subtends an angle of $45°$. How far apart are the landmarks?
14. A man starts his morning walk at a point A reaches two points B and C and finally back to A such that $\angle A = 60°$ and $\angle B = 45°,\ AC = 4km$ in the $\triangle ABC$. Find the total distance he covered during his morning walk.
15. Two vehicles leave the same place P at the same time moving along two different roads. One vehicle moves at an average speed of $60km/hr$ and the other vehicle moves at an average speed of $80\ km/hr$. After half an hour the vehicle reach the destinations A and B. If AB subtends $60°$ at the initial point P, then find AB.
16. Suppose that a satellite in space, an earth station and the centre of earth all lie in the same plane. Let r be the radius of earth and R be the distance from the centre of earth to the satellite. Let d be the distance from the earth station to the satellite. Let $30°$ be the angle of elevation from the earth station to the satellite. If the line segment connecting earth station and satellite subtends angle α at the centre of earth, then prove that $d = R\sqrt{1 + \left(\frac{r}{R}\right)^2 - 2\frac{r}{R}\cos\alpha}$.

3.9 Inverse Trigonometric Functions

A function $f(x)$ has inverse if and only if it is one-to-one and onto. Thus, inverse of a function cannot be defined if it fails to be one-to-one. However, if we restrict the domain suitably, we can make the function to be one-to-one in the restricted domain. For example, $y = x^2$ is not one-to-one for all real numbers. But $y = x^2$ is one-to-one and onto either for $x \geq 0$ or $x \leq 0$. Hence $y = x^2, x \geq 0$ has the inverse $f^{-1}(x) = \sqrt{x}, x \geq 0$. Now, owing to their periodicity, none of six trigonometric functions is one-to-one over their natural domains. We shall restrict their domains so that trigonometric functions are one-to-one enabling the existence of their inverse functions. This restriction can be done in many ways once again due to their periodicity. The conventional choices for the restricted domains are arbitrary but they have some important characteristics. Each restricted domain includes the number 0 and some positive angles and the image of restricted domain contains the entire range.

Let us define the inverse of sine function. Consider $f(x) = \sin x, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, $\sin x$ is one-to-one and onto in this restricted domain. Hence, the inverse of sine function exists. Note that $f^{-1}(y) = x$ if and only if $f(x) = y$. We write $f^{-1}(x) = \sin^{-1}(x)$. Thus, inverse of sine is defined as $\sin^{-1}(y) = x$ if and only if $\sin x = y$.

Clearly, $\sin x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ and $\sin^{-1} x : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, $\sin^{-1} t$ is an angle whose sine is equal to t and which is located in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Similarly we can define the other inverse trigonometric functions.

The inverse functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\text{cosec}^{-1}(x)$, $\sec^{-1}(x)$, $\cot^{-1}(x)$ are called inverse circular functions. For the function $y = \sin x$, there are infinitely many angles x which satisfy $\sin x = t$, $-1 \leq t \leq 1$. Of these infinite set of values, there is one which lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This angle is called the **principal angle** and denoted by $\sin^{-1} t$. The principal value of an inverse function is that value of the general value which is numerically least. It may be positive or negative. When there are two values, one is positive and the other is negative such that they are numerically equal, then the principal value is the positive one.

We shall illustrate below the restricted domains, ranges of trigonometric functions and the domains, ranges of the corresponding inverse functions.

(i) $\sin x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$; $\sin^{-1} x : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos x : [0, \pi] \to [-1, 1]$; $\cos^{-1} x : [-1, 1] \to [0, \pi]$

(iii) $\tan x : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$; $\tan^{-1} x : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\cot x : (0, \pi) \to (-\infty, \infty)$; $\cot^{-1} x : (-\infty, \infty) \to (0, \pi)$

(v) $\text{cosec} x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \to \mathbb{R} - (-1, 1)$; $\text{cosec}^{-1} x : \mathbb{R} - (-1, 1) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(vi) $\sec x : [0, \pi] - \left\{\frac{\pi}{2}\right\} \to \mathbb{R} - (-1, 1)$; $\sec^{-1} x : \mathbb{R} - (-1, 1) \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$

(i) $\sin^{-1} x$ does not mean $\frac{1}{\sin x}$.

(ii) Another notation for $\sin^{-1} x$ is $\arcsin x$ due to **Sir. John FW Herschel (1813)**.

(iii) While discussing the inverse of sine function, we confined to $y = \sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and
$x = \sin^{-1} y, -1 \leq y \leq 1$

(iv) The graph of inverse function f^{-1} is the reflexion of the graph of f about the line $y = x$. Thus, if $(a, b) \in f$ then $(b, a) \in f^{-1}$.

Principal values of inverse trigonometric functions are listed below:

Principal value for $x \geq 0$	Principal value for $x < 0$
$0 \leq \sin^{-1}(x) \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \sin^{-1}(x) < 0$
$0 \leq \cos^{-1}(x) \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1}(x) \leq \pi$
$0 \leq \tan^{-1}(x) < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1}(x) < 0$
$0 < \cot^{-1}(x) \leq \frac{\pi}{2}$	$-\frac{\pi}{2} < \cot^{-1}(x) < 0$
$0 \leq \sec^{-1}(x) < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1}(x) \leq \pi$
$0 < \text{cosec}^{-1}(x) \leq \frac{\pi}{2}$	$-\frac{\pi}{2} < \text{cosec}^{-1}(x) < 0$

(i) Properties, graphs, theorems on inverse trigonometric functions will be studied in higher secondary second year.

(ii) Inverse trigonometric functions are much useful in the evaluation of some integrals which will be studied later.

Example 3.72 Find the principal value of (i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$, (ii) $\text{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$, (iii) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$.

Solution:

(i) Let $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$, where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow \sin y = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3} \Rightarrow y = \frac{\pi}{3}$

Thus, the principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

(ii) Let $\text{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$, where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow \text{cosec}\, y = \frac{2}{\sqrt{3}} = \sin y = \frac{\sqrt{3}}{2}$

Thus, the principal value of $\text{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{3}$

(iii) Let $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = y$, where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\tan y = -\frac{1}{\sqrt{3}} \Rightarrow \tan y = \tan\left(-\frac{\pi}{6}\right) \Rightarrow y = -\frac{\pi}{6}$

Thus the principal value of $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$.

Exercise - 3.11

1. Find the principal value of (i) $\sin^{-1}\frac{1}{\sqrt{2}}$ (ii) $\cos^{-1}\frac{\sqrt{3}}{2}$ (iii) $\text{cosec}^{-1}(-1)$ (iv) $\sec^{-1}\left(-\sqrt{2}\right)$ (v) $\tan^{-1}\left(\sqrt{3}\right)$.
2. A man standing directly opposite to one side of a road of width x meter views a circular shaped traffic green signal of diameter a meter on the other side of the road. The bottom of the green signal is b meter height from the horizontal level of viewer's eye. If α denotes the angle subtended by the diameter of the green signal at the viewer's eye, then prove that

$$\alpha = \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right).$$

Exercise - 3.12

244X66

Choose the correct answer or the most suitable answer:

1. $\dfrac{1}{\cos 80^\circ} - \dfrac{\sqrt{3}}{\sin 80^\circ} =$

   (1) $\sqrt{2}$ (2) $\sqrt{3}$ (3) 2 (4) 4

2. If $\cos 28^\circ + \sin 28^\circ = k^3$, then $\cos 17^\circ$ is equal to

   (1) $\dfrac{k^3}{\sqrt{2}}$ (2) $-\dfrac{k^3}{\sqrt{2}}$ (3) $\pm\dfrac{k^3}{\sqrt{2}}$ (4) $-\dfrac{k^3}{\sqrt{3}}$

3. The maximum value of $4\sin^2 x + 3\cos^2 x + \sin\dfrac{x}{2} + \cos\dfrac{x}{2}$ is

   (1) $4+\sqrt{2}$ (2) $3+\sqrt{2}$ (3) 9 (4) 4

4. $\left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\dfrac{5\pi}{8}\right)\left(1+\cos\dfrac{7\pi}{8}\right) =$

   (1) $\dfrac{1}{8}$ (2) $\dfrac{1}{2}$ (3) $\dfrac{1}{\sqrt{3}}$ (4) $\dfrac{1}{\sqrt{2}}$

5. If $\pi < 2\theta < \dfrac{3\pi}{2}$, then $\sqrt{2+\sqrt{2+2\cos 4\theta}}$ equals to

   (1) $-2\cos\theta$ (2) $-2\sin\theta$ (3) $2\cos\theta$ (4) $2\sin\theta$

6. If $\tan 40^\circ = \lambda$, then $\dfrac{\tan 140^\circ - \tan 130^\circ}{1+\tan 140^\circ \tan 130^\circ} =$

   (1) $\dfrac{1-\lambda^2}{\lambda}$ (2) $\dfrac{1+\lambda^2}{\lambda}$ (3) $\dfrac{1+\lambda^2}{2\lambda}$ (4) $\dfrac{1-\lambda^2}{2\lambda}$

7. $\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \ldots + \cos 179^\circ =$

   (1) 0 (2) 1 (3) -1 (4) 89

8. Let $f_k(x) = \dfrac{1}{k}\left[\sin^k x + \cos^k x\right]$ where $x \in R$ and $k \geq 1$. Then $f_4(x) - f_6(x) =$

   (1) $\dfrac{1}{4}$ (2) $\dfrac{1}{12}$ (3) $\dfrac{1}{6}$ (4) $\dfrac{1}{3}$

9. Which of the following is not true?

   (1) $\sin\theta = -\dfrac{3}{4}$ (2) $\cos\theta = -1$ (3) $\tan\theta = 25$ (4) $\sec\theta = \dfrac{1}{4}$

10. $\cos 2\theta \cos 2\phi + \sin^2(\theta-\phi) - \sin^2(\theta+\phi)$ is equal to

    (1) $\sin 2(\theta+\phi)$ (2) $\cos 2(\theta+\phi)$ (3) $\sin 2(\theta-\phi)$ (4) $\cos 2(\theta-\phi)$

11. $\dfrac{\sin(A-B)}{\cos A\cos B} + \dfrac{\sin(B-C)}{\cos B\cos C} + \dfrac{\sin(C-A)}{\cos C\cos A}$ is

    (1) $\sin A + \sin B + \sin C$ (2) 1 (3) 0 (4) $\cos A + \cos B + \cos C$

12. If $\cos p\theta + \cos q\theta = 0$ and if $p \neq q$, then θ is equal to (n is any integer)

    (1) $\dfrac{\pi(3n+1)}{p-q}$ (2) $\dfrac{\pi(2n+1)}{p\pm q}$ (3) $\dfrac{\pi(n\pm 1)}{p\pm q}$ (4) $\dfrac{\pi(n+2)}{p+q}$

13. If $\tan\alpha$ and $\tan\beta$ are the roots of $x^2+ax+b=0$, then $\dfrac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$ is equal to

(1) $\dfrac{b}{a}$ (2) $\dfrac{a}{b}$ (3) $-\dfrac{a}{b}$ (4) $-\dfrac{b}{a}$

14. In a triangle ABC, $\sin^2 A+\sin^2 B+\sin^2 C=2$, then the triangle is

(1) equilateral triangle (2) isosceles triangle (3) right triangle (4) scalene triangle.

15. If $f(\theta)=|\sin\theta|+|\cos\theta|\,,\theta\in R$, then $f(\theta)$ is in the interval

(1) $[0,2]$ (2) $\left[1,\sqrt{2}\right]$ (3) $[1,2]$ (4) $[0,1]$

16. $\dfrac{\cos 6x+6\cos 4x+15\cos 2x+10}{\cos 5x+5\cos 3x+10\cos x}$ is equal to

(1) $cos2x$ (2) $\cos x$ (3) $\cos 3x$ (4) $2\cos x$

17. The triangle of maximum area with constant perimeter $12m$

(1) is an equilateral triangle with side $4m$ (2) is an isosceles triangle with sides $2m, 5m, 5m$
(3) is a triangle with sides $3m, 4m, 5m$ (4) Does not exist.

18. A wheel is spinning at 2 radians/second. How many seconds will it take to make 10 complete rotations?

(1) 10π seconds (2) 20π seconds (3) 5π seconds (4) 15π seconds

19. If $\sin\alpha+\cos\alpha=b$, then $\sin 2\alpha$ is equal to

(1) $b^2-1,$ if $b\le\sqrt{2}$ (2) $b^2-1,$ if $b>\sqrt{2}$ (3) $b^2-1,$ if $b\ge 1$ (4) $b^2-1,$ if $b\ge\sqrt{2}$

20. In a $\triangle ABC$, if

(i) $\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}>0$

(ii) $\sin A\sin B\sin C>0$ then

(1) Both (i) and (ii) are true (2) Only (i) is true
(3) Only (ii) is true (4) Neither (i) nor (ii) is true.

Summary

Sum and Difference Identities(Ptolemy Identities):

$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$; $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$

$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$; $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

Double, Triple and Half angle Identities

sine	cosine	Tangent
$\sin 2A = 2\sin A\cos A$	$\cos 2A = \cos^2 A - \sin^2 A$	$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
$\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$	$\cos 2A = 2\cos^2 A - 1$	$\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$
$\sin 3A = 3\sin A - 4\sin^3 A$	$\cos 2A = 1 - 2\sin^2 A$	
	$\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$	
	$\cos 3A = 4\cos^3 A - 3\cos A$	
$\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$	$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$	$\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$
$\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$	$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$	
	$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$	

Trigonometric Equation	General solution
$\sin\theta = \sin\alpha,$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\theta = n\pi + (-1)^n\alpha, n \in \mathbb{Z}$
$\cos\theta = \cos\alpha,$ where $\alpha \in [0, \pi]$	$\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}$
$\tan\theta = \tan\alpha,$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\theta = n\pi + \alpha, n \in \mathbb{Z}$

Law of sine	Law of cosine	Law of Tangent
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R.$	$\cos A = \dfrac{b^2+c^2-a^2}{2bc};$	$\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$
	$\cos B = \dfrac{a^2+c^2-b^2}{2ac};$	$\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$
	$\cos C = \dfrac{a^2+b^2-c^2}{2ab}$	$\tan\dfrac{C-A}{2} = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

In a $\triangle ABC$, we have

(i) $a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$ and $c = a\cos B + b\cos A$ [**Projection formula**].

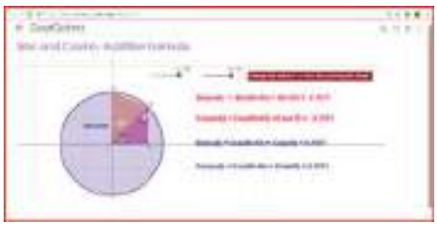
(ii) $\triangle = \dfrac{1}{2}ab\sin C = \dfrac{1}{2}bc\sin A = \dfrac{1}{2}ac\sin B$ [**Area of the triangle**].

(iii) $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$

(iv) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$, $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, where s is the semi-perimeter of $\triangle ABC$ given by $s = \dfrac{a+b+c}{2}$. [**Half Angle formula**].

ICT CORNER-3(a)

Expected Outcome⇒



Step–1

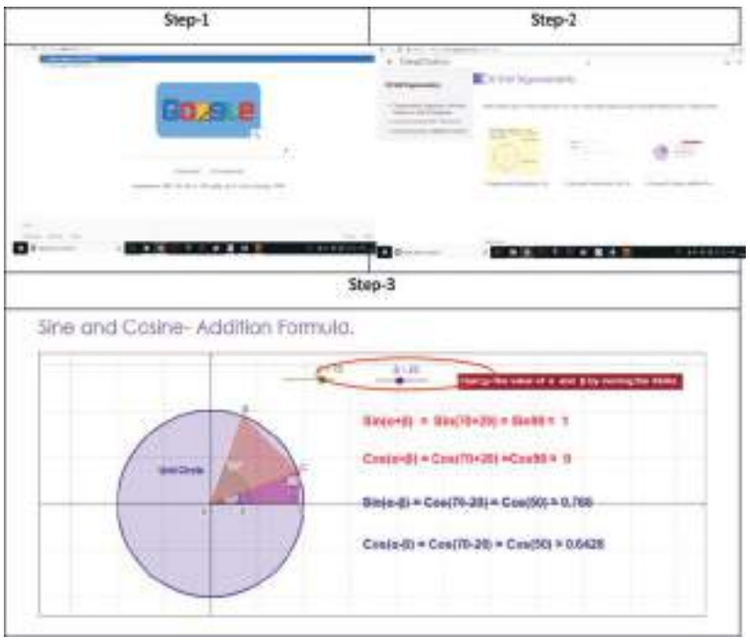
Open the Browser and type the URL Link given below (or) Scan the QR Code..

Step–2

GeoGebra Work book called “XI Std Trigonometry” will appear. In this several work sheets of different Trigonometry concepts are seen. Select one you want. For example, open “Sine and Cosine - Addition Formula”

Step–3

Move the Sliders to change α and β values. Angles in the Diagram changes and the Addition formula values change correspondingly.



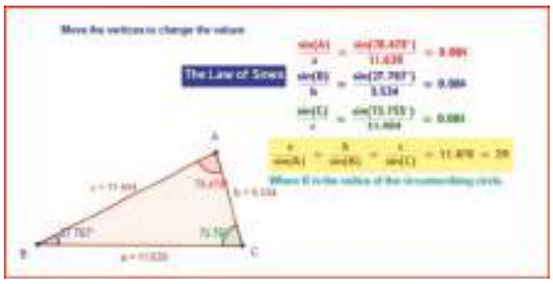
Observe for each value and calculate by formulae and check the results. Similarly Open and check other work sheets for Ratios and Unit circle etc..

*Pictures are only indicatives.

Browse in the link XI Std Trigonometry GeoGebra Work book: https://ggbm.at/k5vP7pv2

ICT CORNER 3(b)

Expected Outcome⇒



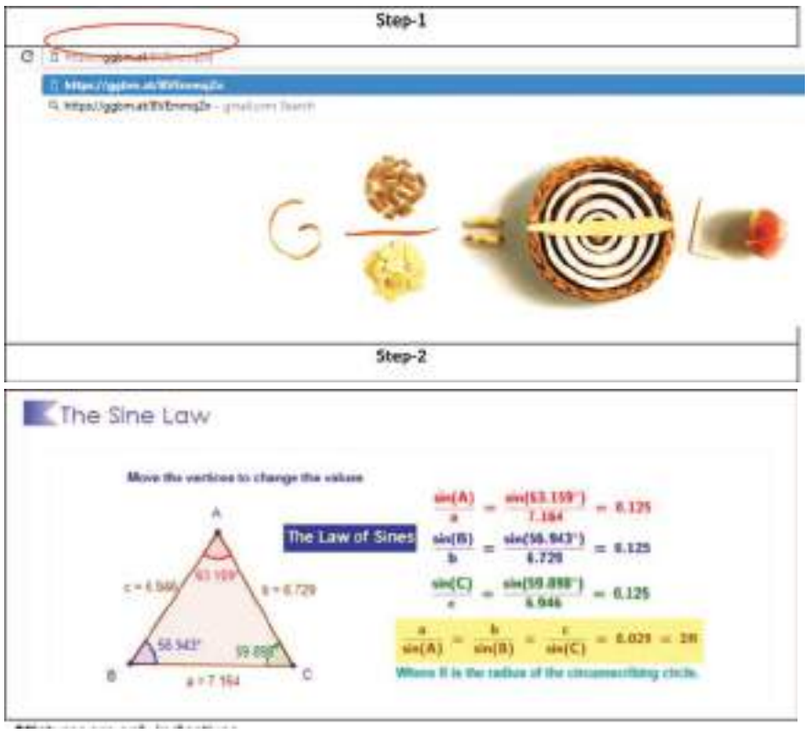
Step–1

Open the Browser andtype the URL Link given below (or) Scan the QR Code.

Step–2

GeoGebra Work sheet called “The Sine Law” will appear. In the worksheet a Triangle is seen with vertices A,B and C with sides a, b and c. Move the vertices to change the Triangle size.

For any triangle Sine Law is maintained and the working is shown on right-hand side-Observe.



*Pictures are only indicatives.

Browse in the link The Sine Law- GeoGebra Work sheet:
https://ggbm.at/BVEmmqZn

Chapter 4 Combinatorics and Mathematical Induction

2BUSXN

"No great discovery was ever made without a bold guess"

- Newton

4.1 Introduction

Combinatorics is the branch of mathematics which is related to counting. It deals with arrangements of objects as well as enumeration, that is, counting of objects with specific properties. The roots of the subject can be traced as far back as 2800 BC(BCE) when it was used to study magic squares and patterns within them.

English physicist and mathematician Sir Isaac Newton, most famous for his law of gravitation, was instrumental in the scientific revolution of the 17th century. Newton's belief in the "Persistance of patterns" led to his first significant mathematical discovery, the generalization of the expansion of binomial expressions.

Newton discovered Binomial Theorem which he claimed the easiest way to solve the quadratures of curves. This discovery is essential in understanding probability. The generalized version of the Binomial Theorem, the Multinomial Theorem, applies to multiple variables. It is widely used in Combinatorics and Statistics.

Newton (1643–1727)

He was the first to use fractional indices and to employ coordinate geometry to derive solutions to Diophantine equations. He approximated partial sums of the harmonic series by logarithms (a precursor to Euler's summation formula) and was the first to use power series with confidence and to revert power series. Newton's work on infinite series was inspired by Simon Stevin's decimals.

In 1705, he was knighted by Queen Anne of England, making him Sir Isaac Newton. Newton made discoveries in optics and theory of motions. Along with mathematician Leibnitz, Newton is credited for developing essential theories of calculus.

Combinatorics has many real life applications where counting of objects are involved. For example, we may be interested to know if there are enough mobile numbers to meet the demand or the number of allowable passwords in a computer system. It also deals with counting techniques and with optimisation methods, that is, methods related to finding the best possible solution among several possibilities in a real problem. In this chapter we shall study counting problems in terms of ordered or unordered arrangements of objects. These arrangements are referred to as permutations and combinations. Combinatorics are largely used in the counting problems of Network communications, Cryptography, Network Security and Probability theory. We shall explore their properties and apply them to counting problems.

Consider another situation: We all know that our electricity consumer card number is of the form A: B : C, where A denotes the electrical substation /larger capacity transformer number, B denotes the smaller capacity electricity transformer number and C denotes the consumer number. There may be conditions that to each substation certain maximal number of transformer can only be linked and with

a particular transformer certain maximal number of consumer connection can only be linked. Now the question of deciding, whether a new Transformer/Substation needs to be erected, can be made by the count of the number of consumer connections linked with a substation transformer. How to get that count? This count can be easily arrived by the use of counting principles.

In this Chapter, the art of counting is discussed starting with the Fundamental principles of counting, travelling through Permutation and Combinations.

Learning Objectives

On completion of this chapter, the students are expected to know

- the principles of counting and applying it to various situations.
- how to compute the number of ways in arranging a set of distinct objects.
- how to compute the number of arrangements from a set containing identical objects.
- how to compute and applying the strategies to find the number of combinations of a set of different objects.
- the applications of the principle of mathematical induction.

We shall start the chapter with the section on

4.2 Fundamental principles of counting

1. **The Sum Rule** Let us consider two tasks which need to be completed. If the first task can be completed in M different ways and the second in N different ways, and if these cannot be performed simultaneously, then there are $M + N$ ways of doing either task. This is the sum rule of counting.

Example 4.1 Suppose one girl or one boy has to be selected for a competition from a class comprising 17 boys and 29 girls. In how many different ways can this selection be made?

Solution:
The first task of selecting a girl can be done in 29 ways. The second task of selecting a boy can be done in 17 ways. It follows from the sum rule, that there are 17+29 = 46 ways of making this selection.

The sum rule may be extended to more than two tasks. Thus if there are n non-simultaneous tasks $T_1, T_2, T_3, \cdots, T_n$ which can be performed in $m_1, m_2, \cdots, m_n$ ways respectively, then the number of ways of doing one of these tasks is $m_1 + m_2 + \cdots + m_n$.

2. **The Product Rule** Let us suppose that a task comprises of two procedures. If the first procedure can be completed in M different ways and the second procedure can be done in N different ways after the first procedure is done, then the total number of ways of completing the task is $M \times N$

Example 4.2 Consider the 3 cities Chennai, Trichy and Tirunelveli. In order to reach Tirunelveli from Chennai, one has to pass through Trichy. There are 2 roads connecting Chennai with Trichy and there are 3 roads connecting Trichy with Tirunelveli. What are the total number of ways of travelling from Chennai to Tirunelveli?

Solution:
There are 2 roads connecting Chennai to Trichy. Suppose these are R_1 and R_2. Further there are 3 roads connecting Trichy to Tirunelveli . Let us name them as S_1, S_2 and S_3. Suppose a person chooses R_1 to travel from Chennai to Trichy and may further choose any of the 3 roads S_1, S_2 or S_3 to travel from Trichy to Tirunelveli. Thus the possible road choices are $(R_1, S_1), (R_1, S_2), (R_1, S_3)$. Similarly, if the person chooses R_2 to travel from Chennai to Trichy, the choices would be $(R_2, S_1), (R_2, S_2), (R_2, S_3)$.

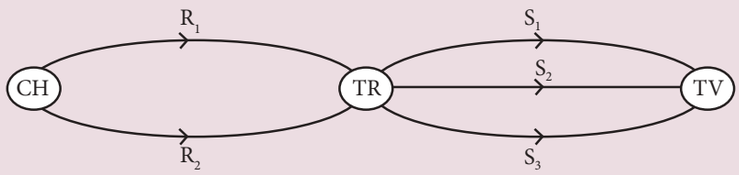


Figure 4.1

Thus there are $2 \times 3 = 6$ ways of travelling from Chennai to Tirunelveli.

An extension of the product rule may be stated as follows:
If a task comprises of n procedures $P_1, P_2, P_3, \cdots, P_n$ which can be performed in $m_1, m_2, \cdots, m_n$ ways respectively, and procedure P_i can be done after procedures $P_1, P_2, P_3, \cdots, P_{i-1}$ are done, then the number of ways of completing the task is $m_1 \times m_2 \times \cdots \times m_n$.

3. **The Inclusion-Exclusion Principle** Suppose two tasks A and B can be performed simultaneously. Let $n(A)$ and $n(B)$ represent the number of ways of performing the tasks A and B independent of each other. Also let $n(A \cap B)$ be the number of ways of performing the two tasks simultaneously. We cannot use the sum rule to count the number of ways of performing one of the tasks as that would lead to over counting. To obtain the correct number of ways we add the number of ways of performing each of the two tasks and then subtract the number of ways of doing both tasks simultaneously. This method is referred to as the principle of inclusion - exclusion. Using the notation of set theory we write it as

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Suppose we have to find the number of positive integers divisible by 2 or 7 (but not both), upto 1000. Let $n(A)$ denote the number of integers divisible by 2, $n(B)$ denote the number of integers divisible by 7 and $n(A \cap B)$ the number of integers divisible by both 2 and 7. Then the number of positive integers divisible by 2 or 7 is given by

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 500 + 142 - 71 = 571.$$

(Note that $n(A)$ will include all multiples of 2 upto 1000, $n(B)$ will include all multiples of 7 upto 1000 and so on.)

Tree Diagrams: Tree diagrams are often helpful in representing the possibilities in a counting problem. Typically in a tree the branches represent the various possibilities. For example, suppose a person wants to buy a Car for the family. There are two different branded cars and five colours are available for each brand. Each colour will have three different variant on it namely GL,SS,SL. Then the various choices for choosing a car can be represented through a tree diagram as follows:

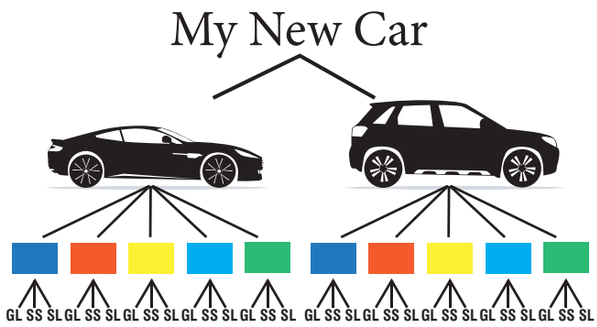


Figure 4.2

We shall now illustrate the different rules described above through examples

Example 4.3 A School library has 75 books on Mathematics, 35 books on Physics. A student can choose only one book. In how many ways a student can choose a book on Mathematics or Physics?

Solution:

(i) A student can choose a Mathematics book in "75" different ways.
(ii) A student can choose a Physics book in "35" different ways.
Hence applying the Rule of Sum, the number of ways a student can choose a book is $75 + 35 = 110$.

Now we shall discuss the problem stated in our introduction.

Example 4.4 If an electricity consumer has the consumer number say 238:110: 29, then describe the linking and count the number of house connections upto the 29th consumer connection linked to the larger capacity transformer number 238 subject to the condition that each smaller capacity transformer can have a maximal consumer link of say 100.

Solution:
The following figure illustrates the electricity distribution network.

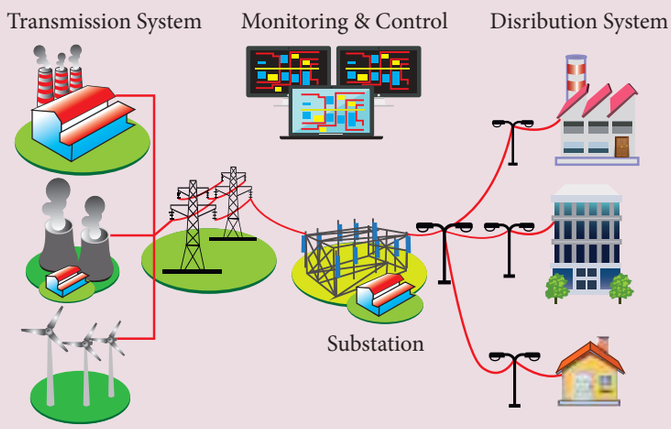


Figure 4.3

There are 110 smaller capacity transformers attached to a larger capacity transformer. As each smaller capacity transformer can be linked with only 100 consumers, we have for the 109 transformers, there will be $109 \times 100 = 10900$ links. For the 110th transformer there are only 29 consumers linked. Hence, the total number of consumers linked to the 238th larger capacity transformer is $10900 + 29 = 10929$.

Example 4.5 A person wants to buy a car. There are two brands of car available in the market and each brand has 3 variant models and each model comes in five different colours as in Figure 4.2 In how many ways she can choose a car to buy?

Solution:
A car can be bought by choosing a brand, then a variant model, and then a colour. A brand can be chosen in 2 ways; a model can be chosen in 3 ways and a colour can be chosen in 5 ways. By the rule of product the person can buy a car in $2 \times 3 \times 5 = 30$ different ways.

Example 4.6 A Woman wants to select one silk saree and one sungudi saree from a textile shop located at Kancheepuram. In that shop, there are 20 different varieties of silk sarees and 8 different varieties of sungudi sarees. In how many ways she can select her sarees?

Solution:
The work is done when she selects one silk saree and one sungudi saree. The Woman can select a silk saree in 20 ways and sungudi saree in 8 ways. By the rule of product, the total number of ways of selecting these 2 sarees is $20 \times 8 = 160$.

Example 4.7 In a village, out of the total number of people, 80 percentage of the people own Coconut groves and 65 percent of the people own Paddy fields. What is the minimum percentage of people own both?

Solution:
Let $n(C)$ denote the percentage of people who own the Coconut groves and $n(P)$ denote the percentage of people who own Paddy fields. We are given $n(C) = 80$ and $n(P) = 65$. By the rule of inclusion - exclusion $n(C \cap P) = n(C) + n(P) - n(C \cup P)$. The maximum value of $n(C \cup P)$ is 100. Therefore, the minimum value of $n(C \cap P)$ is $80 + 65 - 100 = 45$. That is, the minimum percentage of the people who own both is 45.

In the next problem, we use the notion of a 'string'. A string is formed by writing given letters one by one in a sequence. For instance, strings of length three formed out of the letters a,b,c $\&$ d are aaa, abb, bda, dca, cdd $\cdots$.

Example 4.8

(i) Find the number of strings of length 4, which can be formed using the letters of the word BIRD, without repetition of the letters.

(ii) How many strings of length 5 can be formed out of the letters of the word PRIME taking all the letters at a time without repetition.

Solution:

(i) There are as many strings as filling the 4 vacant places by the 4 letters, keeping in mind that repetition is not allowed. The first place can be filled in 4 different ways by any one of the letters B,I,R,D. Following which, the second place can be filled in by any one of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways, following which fourth place can be filled in 1 way.

Thus the number of ways in which the 4 places can be filled, by the rule of product is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of strings is 24.

(ii) There are 5 different letters with which 5 places are to be filled. The first place can be filled in 5 ways as any one of the five letters P,R,I,M,E can be placed there. Having filled the first place with any of the 5 letters, 4 letters are left to be placed in the second place, three letters are left for the third place and 2 letters are left to be put in the fourth place. The remaining 1 letter has to be placed in the fifth place.

Hence, the total number of ways filling up five places is $5 \times 4 \times 3 \times 2 \times 1 = 120$.

Observe the similarity between the above two cases.

Example 4.9 How many strings of length 6 can be formed using letters of the word FLOWER if

(i) either starts with F or ends with R?

(ii) neither starts with F nor ends with R?

Solution:

In any such string, each of the letters F,L,O,W,E,R is used exactly once.

(i) If such a string starts with F, then the other five positions are to be filled with the letters L, O, W, E, R.

As there cannot be any repetition of letters in the formation of the strings we can fill up the 2nd, 3rd, 4th, 5th and 6th places in 5, 4, 3, 2 and 1 ways.

F					
1 way	5 way	4 way	3 way	2 way	1 way

Figure 4.4

Hence, by the rule of product, the number of strings of length 6 starting with F is equal to $5 \times 4 \times 3 \times 2 \times 1 = 120$.

If such a string ends with R, then the other five positions are to be filled with the letters F,L,O,W,E.

					R
5 way	4 way	3 way	2 way	1 way	1 way

Figure 4.5

As in the previous case, we conclude that the number of strings of length ending with R is 120.

If a string starts with F and also ends with R, then the other 4 positions are to be filled with letters L, O, W, E.

F					R
1 way	4 way	3 way	2 way	1 way	1 way

Figure 4.6

As in the previous cases, the number of strings of length of 6 starting with F and ending with R is $4 \times 3 \times 2 \times 1 = 24$.

By the principle of inclusion - exclusion, the number of strings of length 6, either starting with F or ending with R is $120 + 120 - 24 = 216$.

(ii) A string that neither starts with F nor ends with R is one which has not been counted in (i). Together, they account for all possible strings of length 6 formed out of the letters, F,L,O,W,E,R, where no letter is repeated.

Now, the number of all such strings is formed by filling the first position by any of the 6 letters, the second by any of the remaining 5 letters and so on. That is, there are in total $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ such strings. The number of words neither starting with F nor ending with R is the same as the difference between total number of letter strings and the number of strings either starting with F or end with R which is $720 - 216 = 504$.

Example 4.10 How many licence plates may be made using either two distinct letters followed by four digits or two digits followed by 4 distinct letters where all digits or letters are distinct?

Solution:
Here we have two cases:

Case 1: The number of licence plates having two letters followed by four digits is
$26 \times 25 \times 10 \times 9 \times 8 \times 7 = 32,76,000$.

Case 2: The number of licence plates having two digits followed by four letters is
$10 \times 9 \times 26 \times 25 \times 24 \times 23 = 3,22,92,000$.
Since either case 1 or case 2 is possible, the total number of licence plates is
$(26 \times 25 \times 10 \times 9 \times 8 \times 7) + (10 \times 9 \times 26 \times 25 \times 24 \times 23) = 3,55,68,000$.

Example 4.11 Count the number of positive integers greater than 7000 and less than 8000 which are divisible by 5, provided that no digits are repeated.

Solution:
It should be a 4-digit number greater than 7000 and less than 8000. Then the 1000th place will be the digit 7. Further, as the number must be divisible by 5 the unit place should be either 0 or 5.

7			**0 or 5**
1 way	**8 way**	**7 way**	**2 way**

Figure 4.7

As repetition is not permitted, the 100th place can be filled in 8 ways using remaining numbers and 10th place can be filled in 7 ways.

Hence, the required number of numbers is $1 \times 8 \times 7 \times 2 = 112$.

Example 4.12 How many 4 - digit even numbers can be formed using the digits $0, 1, 2, 3$ and 4, if repetition of digits are not permitted?

Solution:
There are two conditions as follows:

1. It is 4-digit number and hence its 1000th place cannot be 0.
2. It is an even number and hence its unit place can be either 0, 2 or 4.

Two cases arise in this situation. Either 0 in the unit place or not.

Case 1. When the unit place is filled by 0, then the 1000th place can be filled in 4 ways, 100th place can be filled in 3 ways and 10th place in 2 ways. Therefore, number of 4-digit numbers having 0 at unit place is
$4 \times 3 \times 2 \times 1 = 24$.

			0
4 way	3 way	2 way	1 way

Figure 4.8

Case 2. When the unit place is filled with non-zero numbers, that is 2 or 4, the number of ways is 2, the number of ways of filling the 1000th place is in 3 ways (excluding '0'), 100th place in 3 ways and 10th place in 2 ways. Therefore, number of 4-digit numbers without 0 at unit place is $3 \times 3 \times 2 \times 2 = 36$.

$\overline{0}$			2/4
3 way	3 way	2 way	2 way

Figure 4.9

Hence, by the rule of sum, the required number of 4 digit even numbers is 24+36 = 60.

Example 4.13 Find the total number of outcomes when 5 coins are tossed once.

Solution:
When a coin is tossed, the outcomes are in two ways which are $\{Head, Tail\}$.
By the rule of product rule, the number of outcomes when 5 coins are tossed is
$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$.

More generally, if n coins are tossed then the number of outcomes is 2^n.

Example 4.14 In how many ways (i) 5 different balls be distributed among 3 boxes? (ii) 3 different balls be distributed among 5 boxes?

Solution:

(i) Each ball can be placed into any one of the three boxes in 3 different ways. Therefore, by rule of product, the number of ways of distributing 5 different balls among three boxes is $3 \times 3 \times 3 \times 3 \times 3 = 3^5 = 243$.

(ii) Each ball can be placed into any one of the five boxes in 5 different ways. Therefore, by rule of product, the number of ways of distributing 3 different balls among five boxes is $5 \times 5 \times 5 = 5^3 = 125$.

In order to avoid confusions, take the objects(balls) and distribute them in places(boxes). More generally, if n different objects are to be placed in m places, then the number of ways of placing is m^n.

Example 4.15 There are 10 bulbs in a room. Each one of them can be operated independently. Find the number of ways in which the room can be illuminated.

Solution:
Each of the 10 bulbs are operated independently means that each bulb can be operated in two ways. That is in off mode or on mode. The total number of doing this are 2^{10} which includes the case in which 10 bulbs are off. Keeping all 10 bulbs in "off" mode, the room cannot be illuminated. Hence, the total number of ways are $2^{10} - 1 = 1024 - 1 = 1023$.

Another concept which is an essential tool in a counting process which is stated as follows:

The Pigeonhole Principle:

Suppose a flock of pigeons fly into a set of pigeonholes. If there are more pigeons than pigeonholes then there must be at least one pigeonhole with at least two pigeons in it. A generalised form of this may be applied to other objects and situations as well.
If $k + 1$ or more objects are placed in k boxes, then there is at least one box containing two or more of the objects.
Here are some examples.

1. In any group of 27 English words, there must be at least two words which begin with the same letter (since there are only 26 letters in the English alphabet).
2. If six meetings are held on weekdays only, then there must be at least two meetings held on the same day.

In order to understand the Permutation and Combinations we need a concept called "Factorials" which will be discussed in the next section.

4.3 Factorials

Factorial of a natural number n is the product of the first n natural numbers. It is denoted by $n!$. That is,

$$n! = 1 \times 2 \times 3 \times \cdots \times n.$$

We read this symbol as "n factorial" or "factorial of n". The notation $n!$ was introduced by the French mathematician Christian Kramp in the year 1808. Note that for a positive integer n

$$\begin{aligned} n! &= n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 \\ &= n(n-1)! \quad \text{for} \quad n > 1 \\ &= n(n-1)(n-2)! \quad \text{for} \quad n > 2 \\ &= n(n-1)(n-2)(n-3)! \quad \text{for} \quad n > 3 \quad \text{and so on.} \end{aligned}$$

Observe that,

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \times 1 = 2 \\ 3! &= 3 \times 2 \times 1 = 6 \\ 4! &= 4 \times 3 \times 2 \times 1 = 24 \\ 5! &= 5 \times 4 \times 3 \times 2 \times 1 = 120 \\ \ldots &= \ldots \\ 22! &= 22 \times 21 \times 20 \times \cdots \times 3 \times 2 \times 1 \quad = 1124000727777607680000 \end{aligned}$$

The number 22 (the Birth date of Ramanujan) has a special place with respect to factorial that, it is the least integer N greater than 1 whose factorial has exactly N digits.

It will be a good exercise for both students and teachers to find the next number N such that $N!$ has exactly N digits.

Note that $0! = 1$ is evident by substituting $n = 0$ in the equation $(n+1)! = (n+1) \times n!$ as $1! = (0+1) \times 0! \implies 0! = \frac{1!}{1} = 1$. This way, we talk of factorial for non-negative integers. Note that factorials can be extended to certain negative numbers and also to complex numbers, which are beyond the scope of this book.

We shall now discuss certain examples in order to familiarise the computation of factorials.

Example 4.16 Find the value of

$$\text{(i)}\ 5! \quad \text{(ii)}\ 6! - 5! \quad \text{(iii)}\ \frac{8!}{5! \times 2!}.$$

Solution:

(i) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$

(ii) $6! - 5! = 6 \times 5! - 5! = (6-1) \times 5! = 5 \times 120 = 600.$

(iii) $\dfrac{8!}{5! \times 2!} = \dfrac{8 \times 7 \times 6 \times 5!}{5! \times 2!} = \dfrac{8 \times 7 \times 6}{2} = 168.$

Example 4.17 Simplify $\frac{7!}{2!}$

Solution:

$$\frac{7!}{2!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2!}{2!} = 7 \times 6 \times 5 \times 4 \times 3 = 2520.$$

Example 4.18 Evaluate $\frac{n!}{r!(n-r)!}$ when (i) $n = 7, r = 5$ (ii) $n = 50, r = 47$ (iii) For any n with $r = 3$.

Solution:

(i) When $n = 7, r = 5$

$$\frac{n!}{r!(n-r)!} = \frac{7!}{5!(7-5)!} = \frac{7 \times 6 \times 5!}{5! \times 2!} = \frac{7 \times 6}{1 \times 2} = 21.$$

(ii) When $n = 50, r = 47$

$$\frac{n!}{r!(n-r)!} = \frac{50!}{47!(50-47)!} = \frac{50 \times 49 \times 48 \times 47!}{47! \times 3!} = \frac{50 \times 49 \times 48}{1 \times 2 \times 3} = 19600.$$

(iii) For any n and $r = 3$

$$\frac{n!}{r!(n-r)!} = \frac{n!}{3!(n-3)!} = \frac{n \times (n-1) \times (n-2)(n-3)!}{1 \times 2 \times 3 \times (n-3)!} = \frac{n(n-1)(n-2)}{6}.$$

Example 4.19 Let N denote the number of days. If the value of $N!$ is equal to the total number of hours in N days then find the value of N?

Solution:
We need to solve the equation $N! = 24 \times N$.
For $N = 1, 2, 3, 4, \quad N! < 24 \times N$.
For $N = 5$, we have $N! = 5! = 4! \times 5 = 24N$.
For $N > 5$, we have $N! \geq 5!N > 24 \times N$. Hence $N = 5$.

Example 4.20 If $\frac{6!}{n!} = 6$, then find the value of n.

Solution:

$\frac{6!}{n!} = \frac{1.2.3.4.5.6.}{1.2.3...n} = 6$. As $n < 6$ we get, $n = 5$.

Example 4.21 If $n! + (n-1)! = 30$, then find the value of n.

Solution:
Now, $30 = 6 \times 5$. As $n! + (n-1)! = (n+1)(n-1)!$, equating $(n-1)! = 6 = 3!$, we get $n = 4$.

Example 4.22 What is the unit digit of the sum $2! + 3! + 4! + ... + 22!$?

Solution:
From $5!$ onwards for all $n!$ the unit digit is zero and hence the contribution to the unit digit is through

$2! + 3! + 4!$ only. which is $2 + 6 + 24 = 32$. Therefore the required unit digit is 2.

Example 4.23 If $\frac{1}{7!} + \frac{1}{8!} = \frac{A}{9!}$ then find the value of A.

Solution:
We have, $\frac{A}{9 \times 8 \times 7!} = \frac{1}{7!} + \frac{1}{8 \times 7!}$

Therefore, $\frac{1}{7!} \times \frac{A}{9 \times 8} = \frac{1}{7!} \times \left[1 + \frac{1}{8}\right]$ equivalently, $\frac{A}{72} = \frac{9}{8}$, which imply $A = 81$.

Example 4.24 Prove that $\frac{(2n)!}{n!} = 2^n(1.3.5 \cdots (2n-1))$.

Solution:

$$\begin{aligned}
\frac{(2n)!}{n!} &= \frac{1.2.3.4 \cdots (2n-2).(2n-1)2n}{n!} \\
&= \frac{(1.3.5. \cdots (2n-1))\,(2.4.6. \cdots (2n-2).2n)}{n!} \text{(Grouping the odd and even numbers separately)} \\
&= \frac{(1.3.5. \cdots (2n-1)) \times 2^n \times (1.2.3. \cdots (n-1).n)}{n!} \text{(taking out the 2' s)} \\
&= \frac{(1.3.5. \cdots (2n-1)) \times 2^n \times n!}{n!} \\
&= 2^n\,(1.3.5. \cdots (2n-1))\,.
\end{aligned}$$

Exercise - 4.1

1. (i) A person went to a restaurant for dinner. In the menu card, the person saw 10 Indian and 7 Chinese food items. In how many ways the person can select either an Indian or a Chinese food?

   (ii) There are 3 types of toy car and 2 types of toy train available in a shop. Find the number of ways a baby can buy a toy car and a toy train?

   (iii) How many two-digit numbers can be formed using 1,2,3,4,5 without repetition of digits?

(iv) Three persons enter in to a conference hall in which there are 10 seats. In how many ways they can take their seats?

(v) In how many ways 5 persons can be seated in a row?

2. (i) A mobile phone has a passcode of 6 distinct digits. What is the maximum number of attempts one makes to retrieve the passcode?

   (ii) Given four flags of different colours, how many different signals can be generated if each signal requires the use of three flags, one below the other?

3. Four children are running a race.

   (i) In how many ways can the first two places be filled?

   (ii) In how many different ways could they finish the race?

4. Count the number of three-digit numbers which can be formed from the digits 2,4,6,8 if

   (i) repetitions of digits is allowed. (ii) repetitions of digits is not allowed

5. How many three-digit numbers are there with 3 in the unit place? (i) with repetition (ii) without repetition.

6. How many numbers are there between 100 and 500 with the digits 0, 1, 2, 3, 4, 5 ? if (i) repetition of digits allowed (ii) the repetition of digits is not allowed.

7. How many three-digit odd numbers can be formed by using the digits 0, 1, 2, 3, 4, 5 ? if (i) the repetition of digits is not allowed (ii) the repetition of digits is allowed.

8. Count the numbers between 999 and 10000 subject to the condition that there are (i) no restriction. (ii) no digit is repeated. (iii) at least one of the digits is repeated.

9. How many three-digit numbers, which are divisible by 5, can be formed using the digits 0, 1, 2, 3, 4, 5 if (i) repetition of digits are not allowed? (ii) repetition of digits are allowed?

10. To travel from a place A to place B, there are two different bus routes B_1, B_2, two different train routes T_1, T_2 and one air route A_1. From place B to place C there is one bus route say $B_1^{'}$, two different train routes say $T_1^{'}, T_2^{'}$ and one air route $A_1^{'}$. Find the number of routes of commuting from place A to place C via place B without using similar mode of transportation.

11. How many numbers are there between 1 and 1000 (both inclusive) which are divisible neither by 2 nor by 5?

12. How many strings can be formed using the letters of the word LOTUS if the word

    (i) either starts with L or ends with S? (ii) neither starts with L nor ends with S?

13. (i) Count the total number of ways of answering 6 objective type questions, each question having 4 choices.

    (ii) In how many ways 10 pigeons can be placed in 3 different pigeon holes ?

    (iii) Find the number of ways of distributing 12 distinct prizes to 10 students?

14. Find the value of

    (i) $6!$ (ii) $4! + 5!$ (iii) $3! - 2!$ (iv) $3! \times 4!$ (v) $\dfrac{12!}{9! \times 3!}$ (vi) $\dfrac{(n+3)!}{(n+1)!}$

15. Evaluate $\dfrac{n!}{r!(n-r)!}$ when

    (i) $n = 6, r = 2$ (ii) $n = 10, r = 3$ (iii) For any n with $r = 2$.

16. Find the value of n if

    (i) $(n+1)! = 20(n-1)!$ (ii) $\dfrac{1}{8!} + \dfrac{1}{9!} = \dfrac{n}{10!}$.

Factorials can be generalised as double factorial as follows:

Double Factorial of n:

Factorial of an integer n, denoted by $n!$ can be viewed as a function $f : \mathbb{N} \cup \{0\} \to \mathbb{N}$, where $\mathbb{N}$ is the set of all Natural numbers, defined as

$$f(n) = \begin{cases} 1 & \text{for } n = 0, \\ n \times (n-1) \times (n-2) \times .... \times 3 \times 2 \times 1 & \text{for } n \neq 0. \end{cases}$$

One can define $n!!$ (double factorial of n) as

$$g(n) = \begin{cases} 1 & \text{for } n = 0, \\ n \times (n-2) \times (n-4) \times .... \times 4 \times 2 & \text{for n is even} \\ n \times (n-2) \times (n-4) \times .... \times 3 \times 1 & \text{for n is odd} \end{cases}$$

Accordingly, $5!! = 5 \times 3 \times 1 = 15$ and $8!! = 8 \times 6 \times 4 \times 2 = 384$.
Note that $n!! \neq (n!)!$ as $4!! = 8$ where as $(4!)! = (24)!$

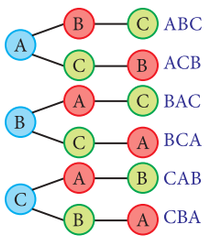
4.4 Permutations

2C4NZB

What is a permutation ?

Permuations come in various disguises.

Suppose three friends A, B and C have to stand in line for a photograph. In how many order can they stand? Some of the possible arrangements (from left to right) are



A, B, C: A, C, B: B, A, C
B, C, A: C, B, A: C, A, B.

Thus there are six possible ways in which they can arrange themselves for the photograph.

Thus if 3 objects have to be arranged in a row there are $3 \times 2 \times 1 = 3!$ possible permutations. The number of permutations of 4 objects taken all at a time is $4 \times 3 \times 2 \times 1 = 4!$ Thus if n objects have to be arranged in a line there are $n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 = n!$ possible arrangements or permutations.

Suppose you have 7 letters A,B,C,D,E,F and G. We want to make a 4 letter string. We have 7 choices for the 1st letter. Having chosen the first letter, we have 6 choices for the second letter. Proceeding this way, we have 4 choices for the 4th letter.
Hence, the number of permutations of 4 letters chosen from 7 letters is

$$7 \times 6 \times 5 \times 4 = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{7!}{3!} = \frac{7!}{(7-4)!}.$$

More generally, the number of distinct permutations of r objects which can be made from n distinct objects is $\frac{n!}{(n-r)!}$. It is denoted by ${}^nP_r$. The formal proof of this result will be proved in this section.

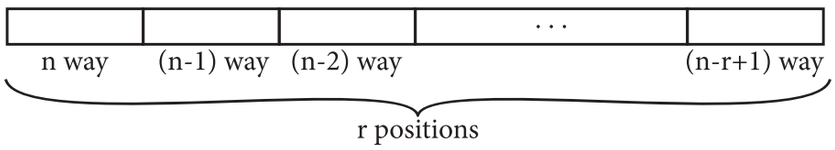
4.4.1 Permutations of distinct objects

In terms of function on any finite set say $S = \{x_1, x_2, ...x_n\}$, a permutation can be defined as a bijective mapping on the set S onto itself. The number of permutation on the set S is the same as the total number of bijective mappings on the set S.

We denote the number of permutations by ${}^nP_r$.

Theorem 4.1: If n, r are positive integers and $r \leq n$, then the number of permutations of n distinct objects taken r at a time is $n\,(n-1)\,(n-2)\cdots(n-r+1)$.

Proof. A permutation is an ordering. A permutation of n distinct objects taken r at a time is formed by filling of r positions, in a row with objects chosen from the given n distinct objects.



There are n objects that can be filled in the first position. For the second position there are remaining $n-1$ objects. There are $n-2$ objects for the third position. Continuing like this until finally we place one of the $(n-(r-1))$ possible objects in the rth position. By the rule of product we conclude ${}^nP_r = n\,(n-1)\,(n-2)\cdots(n-r+1)$.

■

Theorem 4.2: If $n \geq 1$, and $0 \leq r \leq n$, then ${}^nP_r = \dfrac{n!}{(n-r)!}$.

Proof. By Theorem 4.1, we have,

$$\begin{aligned}
{}^nP_r &= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) \\
&= \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) \times (n-r) \times (n-r-1) \cdots 2 \times 1}{(n-r) \times (n-r-1) \times \cdots \times 2 \times 1} \\
&= \frac{n!}{(n-r)!}.
\end{aligned}$$

■

To be specific, if n is a positive integer, and r is any non-negative integer, we can represent,

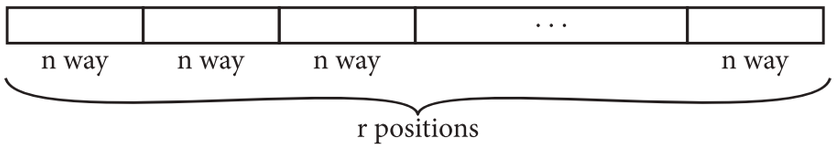
$${}^nP_r = \begin{cases} \dfrac{n!}{(n-r)!} & \text{for } r \leq n, \\ 0 & \text{for } r > n. \end{cases}$$

$${}^nP_r = \begin{cases} {}^nP_n = n! & \text{for } r = n, \\ {}^nP_0 = 1 & \text{for } r = 0. \end{cases}$$

The n different objects arranged in a row is ${}^nP_n = n!$ ways.

Theorem 4.3: The number of permutations of n different objects taken r at a time where repetition is allowed, is n^r.

Proof. As in Theorem 4.1.



We can fill the first position with n objects. For the second position (still we can use the object used in first position), there are n objects, and so on the rth position can be filled with n objects.By the rule of product, The number of permutations of n different objects taken r at a time when repetition allowed is $n \times n \times n \times \cdots n$ (r times) $= n^r$. ■

4.4.2 Properties of Permutations

1.
$$^nP_n =^n P_{n-1}.$$

Proof. We have,

$$^nP_{n-1} = \frac{n!}{(n-(n-1))!} = \frac{n!}{1!} = n! = \frac{n!}{(n-n)!} = {}^nP_n$$ ■

2.
$$^nP_r = n \times^{n-1} P_{r-1}$$

Proof. We have,

$$n \times {}^{n-1}P_{r-1} = n \times \frac{(n-1)!}{((n-1)-(r-1))!} = \frac{n!}{(n-r)!} = {}^nP_r.$$

Continuing this process, we arrive at

$$\begin{aligned} {}^nP_r &= n \times {}^{n-1}P_{r-1} = n \times (n-1) \times^{n-2} P_{r-2} \\ &= n \times (n-1) \times (n-2) \times^{n-3} P_{r-3} \times \cdots \times (n-(r-1))^{n-r} P_0 \\ &= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1). \end{aligned}$$

$$\boxed{{}^nP_r = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1).}$$ ■

3. $^nP_r = {}^{n-1}P_r + r \times {}^{n-1}P_{r-1}$

Proof. We have,

$$\begin{aligned} {}^{n-1}P_r + r \times {}^{n-1}P_{r-1} &= \frac{(n-1)!}{((n-1)-r)!} + r\frac{(n-1)!}{(n-r)!} \\ &= \frac{(n-1)!}{(n-1-r)!} + r\frac{(n-1)!}{(n-r)!} \\ &= \frac{(n-1)! \times (n-r)}{(n-1-r)! \times (n-r)} + r\frac{(n-1)!}{(n-r)!} \\ &= \frac{(n-1)! \times (n-r)}{(n-r)!} + r\frac{(n-1)!}{(n-r)!} \\ &= \frac{(n-1)!\,((n-r)+r)}{(n-r)!} = \frac{(n-1)!n}{(n-r)!} \\ &= \frac{n!}{(n-r)!} = {}^nP_r \end{aligned}$$ ■

Example 4.25 Evaluate:

$$\text{(i) } {}^4P_4 \quad \text{(ii) } {}^5P_3 \quad \text{(iii)} {}^8P_4 \quad \text{(iv) } {}^6P_5.$$

Solution:

(i) ${}^4P_4 = 4 \times 3 \times 2 \times 1 = 4! = 24.$

(ii) ${}^5P_3 = 5 \times 4 \times 3 = 60.$

(iii) ${}^8P_4 = 8 \times 7 \times 6 \times 5 = 1680.$

(iv) ${}^6P_5 = 6 \times 5 \times 4 \times 3 \times 2 = 6! = 720.$

Example 4.26 If ${}^{(n+2)}P_4 = 42 \times {}^nP_2$, find n.

Solution:

$$\begin{aligned} {}^{(n+2)}P_4 &= 42 \times {}^nP_2 \\ \Rightarrow \frac{{}^{(n+2)}P_4}{{}^nP_2} &= 42 \\ \Rightarrow \frac{(n+2)(n+1)(n)(n-1)}{n(n-1)} &= 42 \\ \Rightarrow (n+2)(n+1) &= 42 = 7 \times 6 \\ \Rightarrow n+2 &= 7 \Rightarrow n = 5. \end{aligned}$$

Example 4.27 If ${}^{10}P_r = {}^7P_{r+2}$ find r.

Solution:

$$\begin{aligned} {}^{10}P_r &= {}^7P_{r+2} \\ \frac{10!}{(10-r)!} &= \frac{7!}{(5-r)!} \end{aligned}$$

i.e.,

$$\frac{10 \times 9 \times 8 \times 7!}{(10-r) \times (9-r) \times (8-r) \times (7-r) \times (6-r) \times (5-r)!} = \frac{7!}{(5-r)!}$$

$$(10-r) \times (9-r) \times (8-r) \times (7-r) \times (6-r) = 10 \times 9 \times 8 = 6 \times 5 \times 4 \times 3 \times 2.$$

Therefore, $10 - r = 6 \Rightarrow r = 4$.

Example 4.28 How many 'letter strings' together can be formed with the letters of the word "VOWELS" so that

(i) the strings begin with E

(ii) the strings begin with E and end with W.

Solution:
The given strings contains 6 letters (V,O,W,E,L,S).

(i) Since all strings must begin with E, we have the remaining 5 letters which can be arranged in ${}^5P_5 = 5!$ ways
Therefore the total number of strings with E as the starting letter is $5! = 120$.

(ii) Since all strings must begin with E, and end with W, we need to fix E and W. The remaining 4 letters can be arranged in ${}^4P_4 = 4!$ ways.

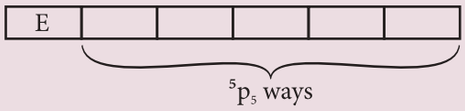


Figure 4.10

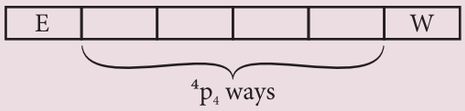


Figure 4.11

Therefore the total number of strings with E as the starting letter and W as the final letter is $4! = 24$.

Example 4.29 A number of four different digits is formed with the use of the digits 1,2,3,4 and 5 in all possible ways. Find the following

(i) How many such numbers can be formed?
(ii) How many of these are even?
(iii) How many of these are exactly divisible by 4?

Solution:

(i) The solution for this is the same as the number of permutations taking four digits out of 5 digits is ${}^5P_4 = 5\times4\times3\times2 = 120$.

(ii) For even number last digits must be 2 or 4 which is filled in ${}^2P_1$ ways and remaining 3 places filled from remaining 4 digits in ${}^4P_3$ ways. Therefore the required number of ways is ${}^2P_1 \times {}^4P_3 = 2 \times 24 = 48$.

(iii) Since the number divisible by 4, then last two digit must be divisible by 4. The Last two digits become 12,24,32,52 (4 ways). The remaining first two places filled from remaining 3 digits in ${}^3P_2$ ways. The required number of numbers which are divisible by 4 is ${}^4P_1 \times {}^3P_2 = 4 \times 6 = 24$.

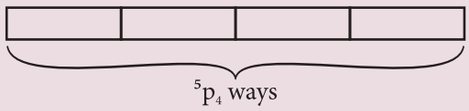


Figure 4.12

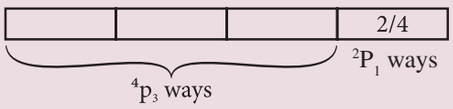


Figure 4.13

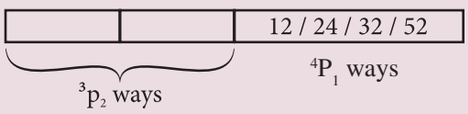


Figure 4.14

4.4.3 Objects always together (String method)

The number of permutations of n different objects, taken all at a time, when m specified objects are always together,

- Consider a string of m specified objects as a single unit
- Then we have $(n-m+1)$ objects. Permute this $(n-m+1)$ objects in $(n-m+1)!$ ways.
- Then permute the m specified objects between themselves in $m!$ ways.
- Finally, the answer is $m! \times (n-m+1)!$.

4.4.4 No two things are together (Gap method)

To obtain the number of permutations of n different objects when no two of k given objects occur together and there are no restrictions on the remaining $m = n-k$ objects, we follow the procedure as follows:

- First of all, arrange the m objects on which there is no restriction in a row. These m objects can be permuted in ${}^mP_m = m!$ ways.
- Then count the number of gaps between every two of m objects on which there is no restriction including the end positions. Number of such gaps will be one more than m that is $(m+1)$. In this $m+1$ gaps, we can permute the k objects in ${}^{m+1}P_k$ ways.
- Then the required number of ways are $m! \times^{(m+1)} P_k$.

Example 4.30 How many different strings can be formed together using the letters of the word "EQUATION" so that

(i) the vowels always come together?
(ii) the vowels never come together?

Solution:

(i) There are 8 letters in the word "EQUATION" which includes 5 vowels (E,U,A,I,O) and 3 consonants (Q,T,N). Considering 5 vowels as one letter, we have 4 letters which can be arranged in ${}^4P_4 = 4!$ ways. But corresponding each of these arrangements, the vowels E,U,A,I,O can be put in ${}^5P_5 = 5!$ ways.

Hence, by the rule of product required number of words is $4! \times 5! = 24 \times 120 = 2880$.

(ii) The total number of strings formed by using all the eight letters of the word "EQUATION" is

$$ {}^8P_8 = 8! = 40320. $$

So, the total number of strings in which vowels are never together is the same as the difference between the total number of strings and the number of strings in which vowels are together is $40320 - 2880 = 37440$.

Example 4.31 There are 15 candidates for an examination. 7 candidates are appearing for mathematics examination while the remaining 8 are appearing for different subjects. In how many ways can they be seated in a row so that no two mathematics candidates are together?

Solution:

Let us arrange the 8-non-mathematics candidates in ${}^8P_8 = 8!$ ways. Each of these arrangements create 9 gaps. Therefore, the 7 mathematics candidates can be placed in these 9 gaps in ${}^9P_7$ ways.

$$ _O_1_O_2_O_3_O_4_O_5_O_6_O_7_O_8_ $$

By the rule of product, the required number of arrangements is

$$ 8! \times {}^9P_7 = 8! \times \frac{9!}{2!} = \frac{8! \times 9!}{2!}. $$

Example 4.32 In how many ways 5 boys and 4 girls can be seated in a row so that no two girls are together.

Solution:

$$_B_1_B_2_B_3_B_4_B_5_$$

The 5 boys can be seated in the row in ${}^5P_5 = 5!$ ways. In each of these arrangements 6 gaps are created. Since no two girls are to sit together, we may arrange 4 girls in this 6 gaps. This can be done in ${}^6P_4$ ways. Hence, the total number of seating arrangements is

$$5! \times {}^6P_4 = 120 \times 360 = 43200.$$

Example 4.33 4 boys and 4 girls form a line with the boys and girls alternating. Find the number of ways of making this line.

Solution:
4 boys can be arranged in a line in ${}^4P_4 = 4!$ ways. By keeping boys as first in each of these arrangements, 4 gaps are created. In these 4 gaps, 4 girls can be arranged in ${}^4P_4 = 4!$ ways.

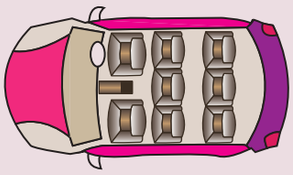
$$B_1_B_2_B_3_B_4_ \text{ or } G_1_G_2_G_3_G_4_$$

Therefore, keeping boys as first, the total number of arrangements are $4! \times 4!$. Similarly, keeping girls as first, by a similar argument, the total number of arrangements are $4! \times 4!$. Hence, by the rule of sum, keeping either a boy or a girl first, the total number of arrangements are

$$(4! \times 4!) + (4! \times 4!) = 2(4!)^2 = 1152.$$

Example 4.34 A van has 8 seats. It has two seats in the front with two rows of three seats behind. The van belongs to a family, consisting of seven members, $F, M, S_1, S_2, S_3,$ D_1, D_2. How many ways can the family sit in the van if

(i) There are no restriction?
(ii) Either F or M drives the van?
(iii) D_1, D_2 sits next to a window and F is driving?



Solution:

(i) As there 8 seats to be occupied out of which one seat is for the one who drives. Since there are no restrictions any one can drive the van. Hence the number of ways of occupying the driver seat is ${}^7P_1 = 7$ ways . The number of ways of occupying the remaining 7 seats by the remaining 6 people is ${}^7P_6 = 5040$. Hence the total number of ways the family can be seated in the car is $7 \times 5040 = 35280$.

(ii) As the driver seat can be occupied by only F or M, there are only two ways it can be occupied. Hence the total number of ways the family can be seated in the car is $2 \times 5040 = 10080$.

(iii) As there are only 5 window seats available for $D_1 \& D_2$ to occupy the number of ways of seated near the windows by the two family members is ${}^5P_2 = 20$. As the driver seat is occupied by F, the remaining 4 people can be seated in the available 5 seats in ${}^5P_4 = 120$. Hence the total number of ways the family can be seated in the car is
$20 \times 1 \times 120 = 2400$.

To understand the next problem we now define, **The Rank of a word in the dictionary.** It is the place at which the given word comes when writing all the strings formed by the letters of the given word in the dictionary order or lexicographic order.

Example 4.35 If the letters of the word TABLE are permuted in all possible ways and the words thus formed are arranged in the dictionary order (alphabetical order), find the ranks of the words (i) TABLE, (ii) BLEAT

Solution:
The dictionary order of the letters of given word is A, B, E, L, T. In the dictionary order of the words which begin with A come first. If we fill the first place with A, remaining 4 letters (B, E, L, T) can be arranged in $4!$ ways. On proceeding like this we get

(i) The rank of the word TABLE

$A - - - - = 4! = 24$ ways
$B - - - - = 4! = 24$ ways
$E - - - - = 4! = 24$ ways
$L - - - - = 4! = 24$ ways
$TABEL = 1$ way
$TABLE = 1$ way

The rank of the word TABLE is $4 \times 4! + 1 + 1 = 98$.

(ii) The rank of the word BLEAT

$A - - - - = 4! = 24$ ways
$BA - -- = 3! = 6$ ways
$BE - -- = 3! = 6$ ways
$BLA - - = 2! = 2$ ways
$BLEAT = 1$ way

The rank of the word BLEAT is $24 + 6 + 6 + 2 + 1 = 39$.

4.4.5 Permutations of not all distinct objects

Consider permuting the letters of the word JEE. In this case the letters of the word are not different. There are 2 E's, which are of same kind. Let us treat, temporarily, the 2 E's as different, say E_1 and E_2. The number of permutations of 3 different letters taken all at a time is $3!$.

Permutations when E_1, E_2 are different	Permutations when E_1, E_2 are the same.
JE_1E_2, JE_2E_1	JEE
E_1JE_2, E_2JE_1	EJE
E_1E_2J, E_2E_1J	EEJ

It is because of the two E_1, E_2 permuted internally will give rise to the same permutations. Since they are same, the required number of permutations is $\frac{3!}{2!} = 3$.

Theorem 4.4: The number of permutations of n objects, where p objects are of the same kind and rest are all different is $\frac{n!}{p!}$.

Generally, the number of permutations of n objects, where p_1 objects are of one kind, p_2 objects are of second kind, $\cdots$ p_k, are of kth kind and the rest of it are of different kind is $\frac{n!}{p_1! \times p_2! \times \cdots \times p_k!}$.

Example 4.36 Find the number of ways of arranging the letters of the word BANANA.

Solution:
This word has 6 letters in which there are 3 A'S, 2 N's and one B. The number of ways of arrangements is $\frac{6!}{3! \times 2!} = 60$.

Example 4.37 Find the number of ways of arranging the letters of the word RAMANUJAN so that the relative positions of vowels and consonants are not changed.

Solution:
In the word RAMANUJAN there are 4 vowels (A,A,U,A)in that 3 A's, 1 U and 5 consonants (R,M,N,J,N) in that two N's and rest are distinct. The 4 vowels (A,A,A,U) can be arranged themselves in $\frac{4!}{3!} = 4$ ways. The 5 consonants (R,M,N,J,N) can be arranged themselves in $\frac{5!}{2!} = 60$ ways. Therefore the number of required arrangements are $\frac{4!}{3!} \times \frac{5!}{2!} = 4 \times 60 = 240$.

Example 4.38 Three twins pose for a photograph standing in a line. How many arrangements are there (i). when there are no restrictions. (ii). when each person is standing next to his or her twin?

Solution:

(i) The six persons without any restriction may be arranged in ${}^6P_6 = 6! = 720$ ways.
(ii) Let us consider three twins as T_1, T_2, T_3. Each twin is considered as a single unit and these three can be permuted in $3!$ ways. Again each twin can be permuted between themselves in $2!$ ways. Hence, the total number of arrangements is $3! \times 2! \times 2! \times 2! = 48$ ways.

Example 4.39 How many numbers can be formed using the digits 1,2,3,4,2,1 such that, even digits occupies even place?

Solution:
There are 6 places in that there are 3 even places we have 2,4,2 as even numbers. The number of ways of permuting 2,4,2 in the 3 even places in $\frac{3!}{2!} = 3$ ways. The remaining numbers 1,3,1 can be permuted in the remaining 3 places in $\frac{3!}{2!} = 3$ ways. Hence, the required number of numbers is $3 \times 3 = 9$.

Example 4.40 How many paths are there from start to end on a 6×4 grid as shown in the picture?

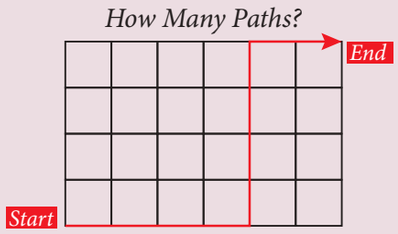


Figure 4.15

Solution:
Note that any such path comprises 6 horizontal unit lengths and 4 vertical unit lengths. This each path consists of 10 unit lengths where 6 are of one kind (horizontal) and 4 are of another kind (vertical).
Thus the total number of paths is $\dfrac{10!}{4! \times 6!} = 210$.

Example 4.41 If the different permutations of all letters of the word BHASKARA are listed as in a dictionary, how many strings are there in this list before the first word starting with B?

Solution:
The required number of strings is the total number of strings starting with A and using the letters A,A,B,H,K,R,S is $\dfrac{7!}{2!} = 2520$.

Example 4.42 If the letters of the word IITJEE are permuted in all possible ways and the strings thus formed are arranged in the lexicographic order, find the rank of the word IITJEE

Solution:
The lexicographic order of the letters of given word is E, E, I, I, J, T. In the lexicographic order, the strings which begin with E come first. If we fill the first place with E, remaining 5 letters (E, I, I, J, T) can be arranged in $\dfrac{5!}{2!}$ ways. On proceeding like this we get,

$E - - - - - = \dfrac{5!}{2!} = 60$ ways

$IE - - - = 4! = 24$ ways

$IIE - - - = 3! = 6$ ways

$IIJ - - - = \dfrac{3!}{2!} = 3$ ways

$IITE - - = 2! = 2$ ways

$IITJEE = 1$ way

The rank of the word IITJEE is 60 +24 + 6 + 3 + 2 + 1 = 96.

Example 4.43 Find the sum of all 4-digit numbers that can be formed using the digits 1, 2, 4, 6, 8.

Solution:
The number of 4-digit numbers that can be formed using the given 5 digits is ${}^5P_4 = 120$. We first find the sum of the digits in the unit place of all these 120 numbers. By filling the 1 in unit place,

the remaining three places can be filled with remaining 4 digits in ${}^4P_3 = 24$ ways. This means, the number of 4-digit numbers having 1 in units place is ${}^4P_3 = 24$. Similarly, each of the digits 2, 4, 6, 8 appear 24 times in units place. An addition of all these digits gives the sum of all the unit digits of all 120 numbers. Therefore,

$$\begin{aligned}
&({}^4P_3 \times 1) + ({}^4P_3 \times 2) + ({}^4P_3 \times 4) + ({}^4P_3 \times 6) + ({}^4P_3 \times 8) \\
&\qquad = {}^4P_3 \times (1+2+4+6+8) \\
&\qquad = {}^4P_3 \times \text{(sum of the digits)} \\
&\qquad = {}^4P_3 \times 21.
\end{aligned}$$

Similarly, we get the sum of the digits in 10th place as ${}^4P_3 \times 21$. Since it is in 10th place, its value is ${}^4P_3 \times 21 \times 10$. Similarly, the values of the sum of the digits in 100th place and 1000^{th}place are ${}^4P_3 \times 21 \times 100$ and ${}^4P_3 \times 21 \times 1000$ respectively. Hence the sum of all the 4 digit numbers formed by using the digits 1, 2, 4, 6, 8 is

$$\begin{aligned}
&({}^4P_3 \times 21) + ({}^4P_3 \times 21 \times 10) + ({}^4P_3 \times 21 \times 100) + ({}^4P_3 \times 21 \times 1000) \\
&\qquad = {}^4P_3\,(21 \times 1111) \\
&\qquad = 24 \times 21 \times 1111 \\
&\qquad = 559944.
\end{aligned}$$

Deduction 4.1: The sum of all r-digit numbers that can be formed using the given n non zero digits is ${}^{(n-1)}P_{(r-1)} \times$ (sum of the digits) $\times\, 111\cdots1$(r times)

Deduction 4.2: If 0 is one digit among the given n digits, then we get that the sum of the r-digits numbers that can be formed using the given n digits (including 0) is
$\{{}^{(n-1)}P_{(r-1)} \times$ (sum of the digits) $\times\, 111\cdots1$(r times)$\} -$
$\{{}^{(n-2)}P_{(r-2)} \times$ (sum of the digits) $\times\, 111\cdots1$((r-1) times)$\}$.

Permutation as Function

Permutation on any finite set $S_n = \{x_1, x_2, x_3, \cdots, x_n\}$ is a bijective function from $S_n \to S_n$. Therefore the set of all permutations on a finite set with n elements is the same as the total number of bijective functions on the set. This is precisely $n!$. Hence the study of permutation is the same as the study of the bijective mappings on the set. Few representations for a permutation on S_3 are given by

$$\begin{bmatrix} A & B & C \\ B & C & A \end{bmatrix}, \begin{bmatrix} A & B & C \\ C & A & B \end{bmatrix}, \begin{bmatrix} A & B & C \\ C & B & A \end{bmatrix}, \cdots$$

Exercise - 4.2

1. If ${}^{(n-1)}P_3 :^n P_4 = 1 : 10$, find n.
2. If ${}^{10}P_{r-1} = 2 \times^6 P_r$, find r.
3. (i) Suppose 8 people enter an event in a swimming meet. In how many ways could the gold, silver and bronze prizes be awarded?
   (ii) Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them?

4. Determine the number of permutations of the letters of the word SIMPLE if all are taken at a time?
5. A test consists of 10 multiple choice questions. In how many ways can the test be answered if
   (i) Each question has four choices?
   (ii) The first four questions have three choices and the remaining have five choices?
   (iii) Question number n has $n+1$ choices?
6. A student appears in an objective test which contain 5 multiple choice questions. Each question has four choices out of which one correct answer.
   (i) What is the maximum number of different answers can the students give?
   (ii) How will the answer change if each question may have more than one correct answers?
7. How many strings can be formed from the letters of the word ARTICLE, so that vowels occupy the even places?
8. 8 women and 6 men are standing in a line.
   (i) How many arrangements are possible if any individual can stand in any position?
   (ii) In how many arrangements will all 6 men be standing next to one another?
   (iii) In how many arrangements will no two men be standing next to one another?
9. Find the distinct permutations of the letters of the word MISSISSIPPI?
10. How many ways can the product $a^2b^3c^4$ be expressed without exponents?
11. In how many ways 4 mathematics books, 3 physics books, 2 chemistry books and 1 biology book can be arranged on a shelf so that all books of the same subjects are together.
12. In how many ways can the letters of the word SUCCESS be arranged so that all Ss are together?
13. A coin is tossed 8 times,
    (i) How many different sequences of heads and tails are possible?
    (ii) How many different sequences containing six heads and two tails are possible?
14. How many strings are there using the letters of the word INTERMEDIATE, if
    (i) The vowels and consonants are alternative (ii) All the vowels are together
    (iii) Vowels are never together (iv) No two vowels are together.
15. Each of the digits 1, 1, 2, 3, 3 and 4 is written on a separate card. The six cards are then laid out in a row to form a 6-digit number.
    (i) How many distinct 6-digit numbers are there?
    (ii) How many of these 6-digit numbers are even?
    (iii) How many of these 6-digit numbers are divisible by 4?
16. If the letters of the word GARDEN are permuted in all possible ways and the strings thus formed are arranged in the dictionary order, then find the ranks of the words (i) GARDEN (ii) DANGER.
17. Find the number of strings that can be made using all letters of the word THING. If these words are written as in a dictionary, what will be the 85th string?
18. If the letters of the word FUNNY are permuted in all possible ways and the strings thus formed are arranged in the dictionary order, find the rank of the word FUNNY.
19. Find the sum of all 4-digit numbers that can be formed using digits 1, 2, 3, 4, and 5 repetitions not allowed?
20. Find the sum of all 4-digit numbers that can be formed using digits 0, 2, 5, 7, 8 without repetition?

4.5 Combinations

Let us suppose there are four persons A, B, C and D (actual names may be used here) and we have to select three of them to be a part of a committee. In how many ways can we make this selection? For example, A, B, C is one possible choice. Here the order of selection is immaterial. Thus A, B, C is the same as B, A, C or C, A, B as long as the same three persons are selected. Thus the possible

distinct choices or selections are $A, B, C; A, B, D; A, C, D$ and B, C, D. We may thus conclude that there are 4 ways of selecting 3 people out of 4. Each choice or selection is referred to as a combination of 4 different objects taken 3 at a time.

Suppose two persons are to be selected from four persons. The possible choices are: $A, B : A, C$: $A, D : B, C : B, D : C, D$. Thus the number of combinations of 4 different objects taken 2 at a time is 6. The number of combinations of n different objects taken r at a time is represented by ${}^nC_r$. From the above we may conclude that ${}^4C_3 = 4$ and ${}^4C_2 = 6$. Now, ${}^4C_3$ is the number of combinations of 4 objects taken 3 at a time. Note that in each combination, the three objects may be arranged in $3!$ ways. Thus the total number of permutations of 4 objects taken 3 at a time is ${}^4C_3 \times 3!$. This is also equal to ${}^4P_3$. Hence ${}^4P_3 = {}^4C_3 \times 3!$.

In general, this leads to an important relationship between permutations and combinations as,

$$ {}^nP_r = {}^nC_r \times r!. $$

Normally for any reader there may be a confusion between permutation and combination. The following table with an example may be helpful in clearing the confusion.

S.No	Description	Permutation	Combination
1	What is a	Number of Arrangement or Listing of objects	Number of Selections or Grouping of objects
2	Where to use	If the ordering of objects matters	If the ordering of objects does not matter
3	Representation	${}^nP_r$	${}^nC_r$
		Examples	
4	In a game of cricket	The number of batting line up of 11 players out of the 15 players	The number of teams consisting of 11 players out of 15 players
5	In a process of prize distribution	The number of ways of distributing 3 distinct prizes	The number of ways of distributing 3 identical prizes
6	In a committee formation	The number of ways of choosing a President and a Vice-President for a committee of 13 members	The number of ways of forming a committee of 2 persons from 13 members
7	In a process of choosing objects	The number of ways of choosing 3 out of 15 distinct objects one after another	The number of ways of choosing 3 out of 15 distinct objects simutaneously

Theorem 4.5: The number of combinations of n distinct objects taken r at a time is given by

$$ {}^nC_r = \frac{n!}{r!(n-r)!}, 0 \le r \le n. $$

Proof. By the relation between permutations and combinations, we have

$$^nC_r = \frac{^nP_r}{r!} = \frac{n!}{r!(n-r)!} \text{ (by a result from permutation).}$$

■

4.5.1 Properties of Combinations

Property 1: (i) $^nC_0 = 1$, (ii) $^nC_n = 1$, (iii) $^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$.

Proof.

$$\begin{aligned}
(i)^nC_0 &= \frac{n!}{0!(n-0)!} = 1.\\
(ii)^nC_n &= \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = 1.\\
(iii)^nC_r &= \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.
\end{aligned}$$

■

In view of,

$$^nC_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} =^n C_r,$$

we have,

Property 2:

$$^nC_r =^n C_{n-r}.$$

Property 3: If $^nC_x =^n C_y$ then either $x = y$ or $x + y = n$.

Proof. By the property 2 we have, $^nC_y =^n C_{n-y}$

Therefore, $^nC_x =^n C_y =^n C_{n-y}$ gives us $x = y$ or $x = n - y$ equivalently, $x = y$ or $x + y = n$. ■

Property 4: $^nC_r +^n C_{r-1} =^{n+1} C_r$.

Proof. Using the expressions for the "combination" we have,

$$\begin{aligned}
^nC_r +^n C_{r-1} &= \frac{n!}{r!.(n-r)!} + \frac{n!}{(r-1)!.(n-(r-1))!}\\
&= \frac{n!}{r! \times (n-r)!} + \frac{n!}{(r-1)! \times (n-r+1)!}\\
&= \frac{n!}{r.(r-1)! \times (n-r)!} + \frac{n!}{(r-1)!.(n-r)!(n-r+1)}\\
&= \frac{n!}{(r-1)! \times (n-r)!} \times \left(\frac{1}{r} + \frac{1}{(n-r+1)}\right)\\
&= \frac{n!}{(r-1)! \times (n-r)!} \times \frac{(n-r+1+r)}{r(n-r+1)}\\
&= \frac{n!}{(r-1)! \times (n-r)!} \times \frac{(n+1)}{r(n-r+1)}\\
&= \frac{(n+1)!}{r! \times (n+1-r)!} =^{n+1} C_r.
\end{aligned}$$

■

Property 5:

$$^{n}C_r = \frac{n}{r} \times^{(n-1)} C_{(r-1)}.$$

Proof.

$$\begin{aligned}\frac{n}{r} \times^{(n-1)} C_{(r-1)} &= \frac{n}{r}\frac{(n-1)!}{(r-1)! \times ((n-1)-(r-1))!} \\ &= \frac{n(n-1)!}{r(r-1)!(n-r)!} =^{n} C_r\end{aligned}$$

■

Example 4.44 Evaluate the following: (i) $^{10}C_3$ (ii) $^{15}C_{13}$ (iii) $^{100}C_{99}$ (iv) $^{50}C_{50}$.

Solution:

(i)

$$\begin{aligned}^{10}C_3 &= \frac{10!}{7! \times 3!} = \frac{10 \times 9 \times 8 \times 7!}{7! \times 3 \times 2 \times 1} \\ &= \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120\end{aligned}$$

(ii)

$$\begin{aligned}^{15}C_{13} &= \frac{15!}{2! \times 13!} = \frac{15 \times 14 \times 13!}{2 \times 1 \times 13!} \\ &= \frac{15 \times 14}{2 \times 1} = 105\end{aligned}$$

(iii)

$$^{100}C_{99} = \frac{100 \times 99!}{99!} = 100$$

(iv)

$$^{50}C_{50} = \frac{50!}{50! \times 0!} = 1.$$

Example 4.45 Find the value of $^{5}C_2$ and $^{7}C_3$ using the property 5

Solution:

$$^{n}C_r = \frac{n}{r} \times {}^{n-1}C_{r-1}.$$

Substituting $n = 5$ and $r = 2$, we get

$$\begin{aligned}^{5}C_2 &= \frac{5}{2} \times {}^{4}C_1 = \frac{5}{2} \times \frac{4}{1} \\ &= \frac{5 \times 4}{2 \times 1} = 10.\end{aligned}$$

Substituting $n = 7$ and $r = 3$, we get

$$\begin{aligned}^{7}C_3 &= \frac{7}{3} \times {}^{6}C_2 = \frac{7}{3} \times \frac{6}{2} \times {}^{5}C_{2-1} \\ &= \frac{7}{3} \times \frac{6}{2} \times {}^{5}C_1 = \frac{7}{3} \times \frac{6}{2} \times \frac{5}{1} \\ &= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35\end{aligned}$$

Example 4.46 If ${}^nC_4 = 495$, What is n?.

Solution:
We know that,

$$ {}^nC_4 = 495. $$

Therefore,

$$ \frac{n \times (n-1) \times (n-2) \times (n-3)}{4 \times 3 \times 2 \times 1} = 495 $$

$$ \Longrightarrow n \times (n-1) \times (n-2) \times (n-3) = 495 \times 4 \times 3 \times 2 \times 1 $$

Factoring $495 = 3 \times 3 \times 5 \times 11$, and writing this product as a product of 4 consecutive numbers in the descending order we get, $n \times (n-1) \times (n-2) \times (n-3) = 12 \times 11 \times 10 \times 9$. Equating n with the maximum number, we obtain $n = 12$.

Example 4.47 If ${}^nP_r = 11880$ and ${}^nC_r = 495$, Find n and r.

Solution:
We know that,

$$ \frac{{}^nP_r}{{}^nC_r} = r!. $$

Therefore,

$$ r! = \frac{11880}{495} = 24 = 4!, $$

gives $r = 4$. Using this $r = 4$, in ${}^nc_4 = 495$, and applying the result of the Example (4.46) we get, $n = 12$.

Example 4.48 Prove that ${}^{24}C_4 + \sum_{r=0}^{4} {}^{(28-r)}C_3 = {}^{29}C_4$

Solution:

$$ \begin{aligned} {}^{24}C_4 + \sum_{r=0}^{4} {}^{(28-r)}C_3 &= {}^{24}C_4 + {}^{28}C_3 + {}^{27}C_3 + {}^{26}C_3 + {}^{25}C_3 + {}^{24}C_3 \\ &= {}^{24}C_4 + {}^{24}C_3 + {}^{25}C_3 + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 \\ &= {}^{25}C_4 + {}^{25}C_3 + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 \\ &= {}^{26}C_4 + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 \\ &= {}^{27}C_4 + {}^{27}C_3 + {}^{28}C_3 \\ &= {}^{28}C_4 + {}^{28}C_3 \\ &= {}^{29}C_4 \end{aligned} $$

Example 4.49 Prove that ${}^{10}C_2 + 2 \times {}^{10}C_3 + {}^{10}C_4 = {}^{12}C_4$

Solution:

$$\begin{aligned}
{}^{10}C_2 + 2 \times {}^{10}C_3 + {}^{10}C_4 &= {}^{10}C_2 + ({}^{10}C_3 + {}^{10}C_3) + {}^{10}C_4 \\
&= ({}^{10}C_2 + {}^{10}C_3) + ({}^{10}C_3 + {}^{10}C_4) \\
&= {}^{11}C_3 + {}^{11}C_4 \\
&= {}^{12}C_4
\end{aligned}$$

Example 4.50 If ${}^{(n+2)}C_7 : {}^{(n-1)}P_4 = 13 : 24$ find n.

Solution:

$$\begin{aligned}
{}^{(n+2)}C_7 : {}^{(n-1)}P_4 &= 13 : 24. \\
\frac{{}^{(n+2)}C_7}{{}^{(n-1)}P_4} &= \frac{13}{24}. \\
\frac{(n+2)!}{(n-5)! \times 7!} \times \frac{(n-5)!}{(n-1)!} &= \frac{13}{24}. \\
\frac{(n+2)(n+1)n(n-1)!}{(n-1)! \times 7!} &= \frac{13}{24}. \\
(n+2)(n+1)(n) &= \frac{13}{24} \times 7! = \frac{13}{24} \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\
(n+2)(n+1)(n) &= 13 \times 14 \times 15 \\
n+2 &= 15 \Longrightarrow n = 13.
\end{aligned}$$

Example 4.51 A salad at a certain restaurant consists of 4 of the following fruits: apple, banana, guava, pomegranate, grapes, papaya and pineapple. Find the total possible number of fruit salads.

Solution:
There are seven fruits and we have to select four fruits for the fruit salad. Hence, the total number of possible ways of making a fruit salad is ${}^{7}C_4 = {}^{7}C_3 = 35$.

Example 4.52 A Mathematics club has 15 members. In that 8 are girls. 6 of the members are to be selected for a competition and half of them should be girls. How many ways of these selections are possible?

Solution:
There are 8 girls and 7 boys in the mathematics club. The number of ways of selecting 6 members in that half of them girls (3 girls and 3 others) is ${}^{8}C_3 \times {}^{7}C_3 = 56 \times 35 = 1960$.

Example 4.53 In rating 20 brands of cars, a car magazine picks a first, second, third, fourth and fifth best brand and then 7 more as acceptable. In how many ways can it be done?

Solution:
The picking of 5 brands for a first, second, third, fourth and fifth best brand from 20 brands in ${}^{20}P_5$ ways. From the remaining 15 we need to select 7 acceptable in ${}^{15}C_7$ ways. By the rule of product it can be done in ${}^{20}P_5 \times {}^{15}C_7$ ways.

Example 4.54 From a class of 25 students, 10 students are to be chosen for an excursion party. There are 4 students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen?

Solution:
There are two possibilities (i) All the 4 students will got to the excursion party then, we need to select 6 students out of 21 students. It can be done in ${}^{21}C_6 = \dfrac{21!}{6! \times 15!}$ ways.

(ii) All the 4 students will not go to the excursion party then, we need to select 10 students out of 21 students. It can be done in ${}^{21}C_{10} = \dfrac{21!}{10! \times 11!}$. Hence, the total number of ways is
${}^{21}C_6 + {}^{21}C_{10} = \dfrac{21!}{6! \times 15!} + \dfrac{21!}{10! \times 11!}$.

Example 4.55 A box of one dozen apple contains a rotten apple. If we are choosing 3 apples simultaneously, in how many ways, one can get only good apples.

Solution:
The total number of ways of selecting 3 apples from 12 apples is ${}^{12}C_3 = 220$.

The total number of ways of getting a rotten apple when selecting 3 apples from 12 apples is equal to selecting 1 rotten apple and remaining 2 apples can be selected from 11 apples is ${}^{11}C_2 = 55$.

Therefore, the total number of ways of getting only good apples is

$$ {}^{12}C_3 - {}^{11}C_2 = 220 - 55 = 165 $$

.

Example 4.56 An exam paper contains 8 questions, 4 in Part A and 4 in Part B. Examiners are required to answer 5 questions. In how many ways can this be done if
(i) There are no restrictions of choosing a number of questions in either parts.
(ii) At least two questions from Part A must be answered.

Solution:

(i) **There are no restrictions.** Totally there are 8 questions in both Part A and Part B. The total number of ways of attempting 5 questions from 8 questions is ${}^8C_5 = {}^8C_3 = 56$.

(ii) **At least two questions from Part A needs to be answered.**Accordingly, various choices are tabulated as follows.

Part A	Part B	Number of selections
2	3	${}^4C_2 \times {}^4C_3$
3	2	${}^4C_3 \times {}^4C_2$
4	1	${}^4C_4 \times {}^4C_1$

Therefore, the required number of ways of answering is

$$ {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_2 + {}^4C_4 \times {}^4C_1 = 24 + 24 + 4 = 52. $$

Example 4.57 Out of 7 consonants and 4 vowels, how many strings of 3 consonants and 2 vowels can be formed?

Solution:
Number of ways of selecting (3 consonants out of 7) and (2 vowels out of 4) is

$$^7C_3 \times^4 C_2$$

Each string contains 5 letters. Number of ways of arranging 5 letters among themselves is $5! = 120$
Hence required number of ways is,

$$^7C_3 \times^4 C_2 \times 5! = 35 \times 6 \times 120 = 25200$$

.

Example 4.58 Find the number of strings of 5 letters that can be formed with the letters of the word PROPOSITION.

Solution:
There are 11 letters in the word, with respect to number of repetitions of letters there are 4 distinct letters (R, S, T, N), 2 sets of two alike letters (PP,II), 1 set of three alike letters (OOO). The following table will illustrate the combination of these sets and the number of words

S.No	Letter Options	Selections	Arrangements
1	5 distinct (R, S, T, N, P, I, O)	$^7C_5$	$^7C_5 \times 5! = 2520$
2	1 set of 3 alike (OOO), 1 set of 2 alike (PP, II)	$^1C_1 \times^2 C_1$	$^1C_1 \times^2 C_1 \times \frac{5!}{3! \times 2!} = 20$
3	1 set of 3 alike (OOO), 2 distinct (R, S, T, N, P, I)	$^1C_1 \times^6 C_2$	$^1C_1 \times^6 C_2 \times \frac{5!}{3!} = 300$
4	2 sets of 2 alike (PP, II, OO), 1 distinct (R, S, T, N and remaining one in 2 alike)	$^3C_2 \times^5 C_1$	$^3C_2 \times^5 C_1 \times \frac{5!}{2! \times 2!} = 450$
5	1 set of 2 alike (PP, II, OO), 3 distinct (R, S, T, N and remaining two in 2 alike)	$^3C_1 \times^6 C_3$	$^3C_1 \times^6 C_3 \times \frac{5!}{2!} = 3600$

Hence, the total number of strings are $2520 + 20 + 300 + 450 + 3600 = 6890$.

Example 4.59 If a set of m parallel lines intersect another set of n parallel lines (not parallel to the lines in the first set), then find the number of parallelograms formed in this lattice structure.

Solution:
Whenever we select 2 lines from the first set of m lines and 2 lines from the second set of n lines, one parallelogram is formed. Thus the number of parallelograms formed is $^mC_2 \times^n C_2$.

Example 4.60 How many diagonals are there in a polygon with n sides?

Solution:
A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon. Number of line segments obtained by joining the vertices of a n sided polygon taken two at a time is ${}^nC_2 = \dfrac{n(n-1)}{2}$. Out of these lines, there are n sides of polygon. Therefore, number of diagonals of the polygon is

$$\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}.$$

In particular for a pentagon and heptagon (Septagon), number of diagonals respectively are $\frac{5(5-3)}{2} = 5$ and $\frac{7(7-3)}{2} = 14$.

Exercise - 4.3

1. If ${}^nC_{12} = {}^nC_9$ find ${}^{21}C_n$.
2. If ${}^{15}C_{2r-1} = {}^{15}C_{2r+4}$, find r.
3. If ${}^nP_r = 720$, and ${}^nC_r = 120$, find n, r.
4. Prove that ${}^{15}C_3 + 2 \times {}^{15}C_4 + {}^{15}C_5 = {}^{17}C_5$.
5. Prove that ${}^{35}C_5 + \sum_{r=0}^{4} {}^{(39-r)}C_4 = {}^{40}C_5$.
6. If ${}^{(n+1)}C_8 : {}^{(n-3)}P_4 = 57 : 16$, find the value of n.
7. Prove that ${}^{2n}C_n = \dfrac{2^n \times 1 \times 3 \times \cdots (2n-1)}{n!}$.
8. Prove that if $1 \le r \le n$ then $n \times {}^{(n-1)}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$.
9. (i) A Kabaddi coach has 14 players ready to play. How many different teams of 7 players could the coach put on the court?
   (ii) There are 15 persons in a party and if each 2 of them shakes hands with each other, how many handshakes happen in the party?
   (iii) How many chords can be drawn through 20 points on a circle?
   (iv) In a parking lot one hundred , one year old cars, are parked. Out of them five are to be chosen at random for to check its pollution devices. How many different set of five cars can be chosen?
   (v) How many ways can a team of 3 boys,2 girls and 1 transgender be selected from 5 boys, 4 girls and 2 transgenders?
10. Find the total number of subsets of a set with
    [Hint: ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$]
    (i) 4 elements (ii) 5 elements (iii) n elements.
11. A trust has 25 members.
    (i) How many ways 3 officers can be selected?
    (ii) In how many ways can a President, Vice President and a Secretary be selected?

12. How many ways a committee of six persons from 10 persons can be chosen along with a chair person and a secretary?
13. How many different selections of 5 books can be made from 12 different books if,
    (i) Two particular books are always selected?
    (ii) Two particular books are never selected?
14. There are 5 teachers and 20 students. Out of them a committee of 2 teachers and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees
    (i) a particular teacher is included?
    (ii) a particular student is excluded?
15. In an examination a student has to answer 5 questions, out of 9 questions in which 2 are compulsory. In how many ways a student can answer the questions?
16. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly three aces in each combination.
17. Find the number of ways of forming a committee of 5 members out of 7 Indians and 5 Americans, so that always Indians will be the majority in the committee.
18. A committee of 7 peoples has to be formed from 8 men and 4 women. In how many ways can this be done when the committee consists of
    (i) exactly 3 women?
    (ii) at least 3 women?
    (iii) at most 3 women?
19. 7 relatives of a man comprises 4 ladies and 3 gentlemen, his wife also has 7 relatives; 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relative and 3 of the wife' s relatives?
20. A box contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box, if at least one black ball is to be included in the draw?
21. Find the number of strings of 4 letters that can be formed with the letters of the word EXAMINATION?.
22. How many triangles can be formed by joining 15 points on the plane, in which no line joining any three points?
23. How many triangles can be formed by 15 points, in which 7 of them lie on one line and the remaining 8 on another parallel line?
24. There are 11 points in a plane. No three of these lies in the same straight line except 4 points, which are collinear. Find,
    (i) the number of straight lines that can be obtained from the pairs of these points?
    (ii) the number of triangles that can be formed for which the points are their vertices?
25. A polygon has 90 diagonals. Find the number of its sides?

4.6 Mathematical induction

Let us consider the sum of the first n positive odd numbers. These are $1, 3, 5, 7, \cdots, 2n-1$. The first odd number 1 which is equal to 1. The first two odd numbers are 1 and 3 and their sum is 4. Writing these as follows helps us to see a pattern.

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \\ 1+3+5+7+9 &= 25 \end{aligned}$$

and so on. We note that the right hand side of the expressions are the perfect squares $1, 4, 9, 16, 25 \cdots$. This pattern compels us to make the conjecture that the sum of the first n odd numbers is equal to n^2. Symbolically, we express this as,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

However we have only made a conjecture. In order to prove the conjecture we shall use the Principle of Mathematical Induction. Mathematical Induction is a method or technique of proving mathematical results or theorems of the above kind. This technique relies upon making conjectures by observing all possible cases of a specific result. It is well suited for proving results in algebra or in other disciplines of mathematics where results or theorems are stated in terms of n, n being a positive integer. The process of Mathematical Induction may be compared to that of climbing an infinite staircase.

In order to ensure that we complete the climb, it is sufficient to ensure the following.

(a) We can climb the first step.
(b) Once we have reached a particular step of the staircase, we can climb to the next step.

Being sure of (a) and (b) will enable us to climb all the steps in the staircase. Similarly, when we apply this method to prove a mathematical statement $P(n)$, the process of induction involves the following steps.

Figure 4.16

Step 1: Verify that the statement is true for n = 1, that is, verify that P(1) is true. This is akin to climbing the first step of the staircase and is referred to as the **initial step**.

Step 2: Verify that the statement is true for $n = k + 1$ whenever it is true for $n = k$, where k is a positive integer. This means that we need to prove that $P(k + 1)$ is true whenever $P(k)$ is true. This is referred to as the **inductive step**.

Step 3: If steps 1 and 2 have been established then the statement $P(n)$ is true for all positive integers n.

One of the interesting method of proof in Mathematics is by the Mathematical induction. We shall illustrate the method through problems. As an illustration of the process let us revisit a well known result through an example below:

Example 4.61 By the principle of mathematical induction, prove that, for all integers $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution:
Let,

$$P(n) := 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Substituting the value of $n = 1$, in the statement we get, $P(1) = \dfrac{1(1+1)}{2} = 1.$ Hence, $P(1)$ is true.

Let us assume that the statement is true for $n = k$. Then

$$P(k) = 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}.$$

We need to show that $P(k+1)$ is true. Consider,

$$P(k+1) = \underbrace{1 + 2 + 3 + \cdots + k} + (k+1) = \frac{k(k+1)}{2} + (k+1).$$

That is,

$$P(k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

This implies, $P(k+1)$ is true. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction, for all integers $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Example 4.62 Prove that the sum of first n positive odd numbers is n^2.

Solution:
Let $P(n) = 1 + 3 + 5 + \cdots + (2n-1)$. Therefore $P(1) = 1 = 1^2$ is true.

We assume that $P(k) = 1 + 3 + 5 + ...(2k-1)$ is true for $n = k$. That is $P(k) = k^2$

We need to prove $P(k+1) = (k+1)^2$.

$$\begin{aligned} P(k+1) &= 1 + 3 + 5 + ...(2(k+1) - 1) \\ &= \underbrace{1 + 3 + 5 + 7 + ... + (2k-1)} + 2k + 1 \\ &= P(k) + 2k + 1 \\ &= k^2 + 2k + 1 = (k+1)^2 \end{aligned}$$

This implies, $P(k+1)$ is true. Hence, by the principle of mathematical induction, $P(n)$ is true for all natural numbers.

Example 4.63 By the principle of mathematical induction, prove that, for all integers $n \geq 1$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution:
Let,

$$P(n) := 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Substituting $n = 1$ in the statement we get, $P(1) = \dfrac{1(1+1)(2(1)+1)}{6} = 1$. Hence, $P(1)$ is true.
Let us assume that the statement is true for $n = k$. Then

$$P(k) = 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

We need to show that $P(k+1)$ is true. Consider

$$\begin{aligned}
P(k+1) &= \underbrace{1^2 + 2^2 + 3^2 + \cdots + k^2} + (k+1)^2 \\
&= P(k) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)\left(k(2k+1) + 6(k+1)\right)}{6} \\
&= \frac{(k+1)\left(2k^2 + 7k + 6\right)}{6} \\
&= \frac{(k+1)\left[(k+2)(2k+3)\right]}{6} \\
&= \frac{(k+1)\left[((k+1)+1)(2(k+1)+1)\right]}{6}.
\end{aligned}$$

That is,

$$P(k+1) = \frac{(k+1)((k+1)+1)\left(2(k+1)+1\right)}{6}.$$

This implies, $P(k+1)$ is true. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, \text{ for all } n \geq 1.$$

Example 4.64 Using the Mathematical induction, show that for any natural number n,

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Solution:
Let $P(n) := \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$. Substituting the value of $n = 1$, in the statement we get, $P(1) = \frac{1}{1.2} = \frac{1}{2}$. Hence, $P(1)$ is true. Let us assume that the statement is true for $n = k$. Then

$$P(k) = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

We need to show that $P(k+1)$ is true. Consider,

$$\begin{aligned}
P(k+1) &= \underbrace{\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{k(k+1)}} + \frac{1}{(k+1)(k+2)} \\
&= P(k) + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{1}{k+1}\left(\frac{k}{1} + \frac{1}{k+2}\right) \\
&= \frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right) \\
&= \frac{1}{k+1}\left(\frac{(k+1)^2}{k+2}\right) = \frac{(k+1)}{(k+2)}.
\end{aligned}$$

This implies, $P(k+1)$ is true. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore, by the principle of mathematical induction, for any natural number n,

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Example 4.65 Prove that for any natural number n, $a^n - b^n$ is divisible by $a - b$, where $a > b$

Solution:
Let

$$P(n) := a^n - b^n, \text{ is divisible by } a - b.$$

Substituting the value of $n = 1$, in the statement we get,

$$P(1) = a - b,$$

which is divisible by $a - b$. Hence, $P(1)$ is true. Let us assume that the statement is true for $n = k$. Then $P(k) = a^k - b^k$, is divisible by $a - b$. We can write

$$P(k) = a^k - b^k = \lambda(a - b),\ \lambda \in \mathbb{N}.$$

We need to show that $P(k+1) = a^{k+1} - b^{k+1}$, is divisible by $a - b$.

$$\begin{aligned}
P(k+1) &= a^{k+1} - b^{k+1} \\
&= a^{k+1} - ab^k + ab^k - b^{k+1} \\
&= a(a^k - b^k) + b^k(a - b) \\
&= a(\lambda(a - b)) + b^k(a - b) \\
&= (a - b)\left(a\lambda + b^k\right) \\
&= (a - b)\lambda_1,\ \lambda_1 = a\lambda + b^k,\ \lambda_1 \in \mathbb{N}.
\end{aligned}$$

which is divisible by $a-b$. This implies that $P(k+1)$ is true. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction, $a^n - b^n$ is divisible by $a - b$, where $a > b$, for all natural numbers n.

Example 4.66 Prove that $3^{2n+2} - 8n - 9$ is divisible by 8 for all $n \geq 1$.

Solution:
Let

$$P(n) := 3^{2n+2} - 8n - 9, \text{ is divisible by 8.}$$

Substituting the value of $n = 1$, in the statement we get,

$$P(1) = 3^{2+2} - 8(1) - 9 = 64,$$

which is divisible by 8. Hence, $P(1)$ is true. Let us assume that the statement is true for $n = k$. Then $P(k) = 3^{2k+2} - 8k - 9$, is divisible by 8. We can write

$$\begin{aligned} P(k) &= 3^{2k+2} - 8k - 9 = 8k_1, \quad k_1 \in \mathbb{N} \text{ and therefore,} \\ 3^{2k+2} &= 8k_1 + 8k + 9. \end{aligned}$$

We need to show that $P(k+1) = 3^{2(k+1)+2} - 8(k+1) - 9$, is divisible by 8.
Consider,

$$\begin{aligned} P(k+1) &= 3^{2(k+1)+2} - 8(k+1) - 9 \\ &= 3^2 3^{2k+2} - 8k - 8 - 9 \\ &= 3^2\,(8k_1 + 8k + 9) - 8k - 17 \\ &= 72k_1 + 64k + 64 \\ &= 8\,(9k_1 + 8k + 1) \\ &= 8k_2,\ k_2 = 9k_1 + 8k + 1 \in \mathbb{N} \end{aligned}$$

which is divisible by 8. This implies that $P(k+1)$ is true. This means that the validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction, $3^{2n+2} - 8n - 9$ is divisible by 8 for all $n \geq 1$.

Example 4.67 Using the Mathematical induction, show that for any integer

$$n \geq 2,\ 3n^2 > (n+1)^2$$

Solution:
Let $P(n)$ be the statement that $3n^2 > (n+1)^2$ with $n \geq 2$. Therefore the first stage is $n = 2$.
Now, $P(2) = 3 \times 2^2 = 12$ and $3^2 = 9$. As $12 > 9$ we get $P(2)$ is true.
We assume that $P(n)$ is true for $n = k$.

Now,

$$\begin{aligned} P(k+1) &= 3(k+1)^2 = 3k^2+6k+3 \\ &= P(k)+6k+3 \\ &> (k+1)^2+6k+3 \\ &= k^2+8k+4 \\ &= k^2+4k+4+4k \\ &= (k+2)^2+4k \\ &> (k+2)^2 \text{ since } k>0. \end{aligned}$$

This is the statement $P(k+1)$. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction, for all $n \geq 2$, $3n^2 > (n+1)^2$.

Example 4.68 Using the Mathematical induction, show that for any integer

$$n \geq 2, \quad 3^n > n^2$$

Solution:
Let $P(n)$ be the statement that $3^n > n^2$ with $n \geq 2$. Therefore the first stage is $n = 2$. Now, $P(2) = 3^2 = 9$ and $2^2 = 4$. As $9 > 4$, we get $P(2)$ is true
We assume that $P(n)$ is true for $n = k$. That is $P(k) > k^2$. Now,

$$\begin{aligned} P(k+1) &= 3^{k+1} = 3 \times 3^k = 3 \times P(k) \\ &> 3k^2 \\ &> (k+1)^2, \text{ by Example 4.67.} \end{aligned}$$

Hence, for any integer $n \geq 2$, $3^n > n^2$.

Example 4.69 By the principle of mathematical induction, prove that, for $n \in \mathbb{N}$,

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(n-1)\beta) = \cos\left(\alpha + \frac{(n-1)\beta}{2}\right) \times \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Solution:
Let $P(n) := \cos(\alpha) + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ... + \cos(\alpha+(n-1)\beta)$. Then,

$$P(1) = \cos(\alpha) = \frac{\cos(\alpha).\sin(\frac{\beta}{2})}{\sin(\frac{\beta}{2})},$$

which shows $P(1)$ is true. We now assume that $P(n)$ is true for $n = k$. That is,

$$\cos(\alpha) + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(k-1)\beta) = \cos\left(\alpha + \frac{(k-1)\beta}{2}\right) \times \frac{\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}.$$

We need to prove $P(k+1)$ is true. Now,

$$P(k+1) = \underbrace{\cos(\alpha) + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(k-1)\beta)} + \cos(\alpha+k\beta)$$

Then,

$P(k+1) = P(k) + \cos(\alpha + k\beta)$

$$
\begin{aligned}
&= \frac{\cos\left(\alpha + \frac{(k-1)\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)}{\sin(\frac{\beta}{2})} + \cos(\alpha + k\beta) \\
&= \frac{1}{\sin(\frac{\beta}{2})}\left[\cos\left(\alpha + \frac{(k-1)\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + \cos(\alpha + k\beta)\sin\left(\frac{\beta}{2}\right)\right] \\
&= \frac{1}{\sin(\frac{\beta}{2})}\left[\cos\left(\left(\alpha + \frac{k\beta}{2}\right) - \frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + cos(\alpha + k\beta)\sin\left(\frac{\beta}{2}\right)\right] \\
&= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\left(\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\right.\right. \\
&\quad \left.\left. + \sin\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{\beta}{2}\right)\right)\sin\left(\frac{k\beta}{2}\right) + \cos(\alpha + k\beta)\sin\left(\frac{\beta}{2}\right)\right] \\
&= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)\right. \\
&\quad \left. + \sin\frac{\beta}{2}\left(\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + \cos(\alpha + k\beta)\right)\right] \\
&= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)\right. \\
&\quad \left. + \frac{\sin\frac{\beta}{2}}{2}\left(2\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + 2\cos(\alpha + k\beta)\right)\right] \\
&= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)\right. \\
&\quad \left. + \frac{\sin\frac{\beta}{2}}{2}\left(\cos\alpha - \cos\left(\alpha + k\beta\right) + 2\cos(\alpha + k\beta)\right)\right] \\
&= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + \frac{\sin\frac{\beta}{2}}{2}\left(\cos\alpha + \cos(\alpha + k\beta)\right)\right] \\
&= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + \frac{\sin\frac{\beta}{2}}{2}\left(2\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{-k\beta}{2}\right)\right)\right] \\
&= \frac{\cos\left(\alpha + \frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\left[\sin\left(\frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right) + \sin\frac{\beta}{2}\cos\left(\frac{k\beta}{2}\right)\right] \\
&= \frac{\cos\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{(k+1)\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \cdots + \cos(\alpha + (k-1)\beta) + \cos(\alpha + k\beta) \\
&\qquad = \cos\left(\alpha + \frac{k\beta}{2}\right) \times \frac{\sin\left(\frac{(k+1)\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}
\end{aligned}
$$

This implies that $P(k+1)$ is true. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction,

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(n-1)\beta)$$

$$= \cos\left(\alpha + \frac{(n-1)\beta}{2}\right) \times \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Example 4.70 Using the Mathematical induction, show that for any natural number n, with the assumption $i^2 = -1$,

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta),$$

Solution:
Let,

$$P(n) = (r(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta).$$

Substituting the value of $n = 1$, in the statement we get,

$$P(1) = (r(\cos\theta + i\sin\theta))^1 = r(\cos\theta + i\sin\theta).$$

Hence, $P(1)$ is true. Let us assume that the statement is true for $n = k$. Then

$$(r(\cos\theta + i\sin\theta))^k = r^k(\cos k\theta + i\sin k\theta),$$

We need to show that $P(k+1)$ is true. Consider,

$$\begin{aligned}
P(k+1) &= (r(\cos\theta + i\sin\theta))^{k+1} \\
&= (r(\cos\theta + i\sin\theta))^k \times r(\cos\theta + i\sin\theta) \\
&= r^k(\cos k\theta + i\sin k\theta) \times r(\cos\theta + i\sin\theta) \\
&= r^{k+1} \times \big((\cos k\theta\cos\theta + i^2\sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)\big) \\
&= r^{k+1} \times ((\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)) \\
&= r^{k+1} \times (\cos(k+1)\theta + i\sin(k+1)\theta).
\end{aligned}$$

This implies that $P(k+1)$ is true. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction, for any natural number n,

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

.

What we have proved in Example 4.70 is called the **Demoivre's theorem** for natural numbers, which will be studied in detail in the second year of Higher Secondary course.

Exercise - 4.4

1. By the principle of mathematical induction, prove that, for $n \geq 1$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

2. By the principle of mathematical induction, prove that, for $n \geq 1$

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$

3. Prove that the sum of the first n non-zero even numbers is $n^2 + n$.
4. By the principle of Mathematical induction, prove that, for $n \geq 1$

$$1.2 + 2.3 + 3.4 + \cdots + n.(n+1) = \frac{n(n+1)(n+2)}{3}.$$

5. Using the Mathematical induction, show that for any natural number $n \geq 2$,

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

6. Using the Mathematical induction, show that for any natural number $n \geq 2$,

$$\frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots + \frac{1}{1+2+3+\cdots+n} = \frac{n-1}{n+1}.$$

7. Using the Mathematical induction, show that for any natural number n,

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \cdots + \frac{1}{n.(n+1).(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}.$$

8. Using the Mathematical induction, show that for any natural number n,

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}.$$

9. Prove by Mathematical Induction that

$$1! + (2 \times 2!) + (3 \times 3!) + ... + (n \times n!) = (n+1)! - 1.$$

10. Using the Mathematical induction, show that for any natural number n, $x^{2n} - y^{2n}$ is divisible by $x + y$.
11. By the principle of Mathematical induction, prove that, for $n \geq 1$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 > \frac{n^3}{3}.$$

12. Use induction to prove that $n^3 - 7n + 3$, is divisible by 3, for all natural numbers n.
13. Use induction to prove that $5^{n+1} + 4 \times 6^n$ when divided by 20 leaves a remainder 9, for all natural numbers n.

14. Use induction to prove that $10^n + 3 \times 4^{n+2} + 5$, is divisible by 9, for all natural numbers n.
15. Prove that using the Mathematical induction

$$\sin(\alpha) + \sin\left(\alpha + \frac{\pi}{6}\right) + \sin\left(\alpha + \frac{2\pi}{6}\right) + \cdots + \sin\left(\alpha + \frac{(n-1)\pi}{6}\right) = \frac{\sin\left(\alpha + \frac{(n-1)\pi}{12}\right) \times \sin\left(\frac{n\pi}{12}\right)}{\sin\left(\frac{\pi}{12}\right)}.$$

2CDK1Y

Exercise - 4.5

Choose the correct or the most suitable answer

1. The sum of the digits at the 10th place of all numbers formed with the help of 2, 4, 5, 7 taken all at a time is

   (1) 432 (2) 108 (3) 36 (4) 18

2. In an examination there are three multiple choice questions and each question has 5 choices . Number of ways in which a student can fail to get all answer correct is

   (1) 125 (2) 124 (3) 64 (4) 63

3. The number of ways in which the following prize be given to a class of 30 boys first and second in mathematics, first and second in physics, first in chemistry and first in English is

   (1) $30^4 \times 29^2$ (2) $30^3 \times 29^3$ (3) $30^2 \times 29^4$ (4) 30×29^5.

4. The number of 5 digit numbers all digits of which are odd is

   (1) 25 (2) 5^5 (3) 5^6 (4) 625.

5. In 3 fingers, the number of ways four rings can be worn is $\cdots\cdots\cdots$ ways.

   (1) $4^3 - 1$ (2) 3^4 (3) 68 (4) 64

6. If ${}^{(n+5)}P_{(n+1)} = (\frac{11(n-1)}{2})^{(n+3)}P_n$, then the value of n are

   (1) 7 and 11 (2) 6 and 7 (3) 2 and 11 (4) 2 and 6.

7. The product of r consecutive positive integers is divisible by

   (1) $r!$ (2) $(r-1)!$ (3) $(r+1)!$ (4) r^r.

8. The number of five digit telephone numbers having at least one of their digits repeated is

   (1) 90000 (2) 10000 (3) 30240 (4) 69760.

9. If ${}^{a^2-a}C_2 = {}^{a^2-a}C_4$ then the value of 'a' is

   (1) 2 (2) 3 (3) 4 (4) 5

10. There are 10 points in a plane and 4 of them are collinear. The number of straight lines joining any two points is

    (1) 45 (2) 40 (3) 39 (4) 38.

11. The number of ways in which a host lady invite 8 people for a party of 8 out of 12 people of whom two do not want to attend the party together is

    (1) $2 \times {}^{11}C_7 + {}^{10}C_8$ (2) ${}^{11}C_7 + {}^{10}C_8$ (3) ${}^{12}C_8 - {}^{10}C_6$ (4) ${}^{10}C_6 + 2!$.

12. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines.

(1) 6 (2) 9 (3) 12 (4) 18

13. Everybody in a room shakes hands with everybody else. The total number of shake hands is 66. The number of persons in the room is $\cdots\cdots\cdots\cdots$

(1) 11 (2) 12 (3) 10 (4) 6

14. Number of sides of a polygon having 44 diagonals is $\cdots\cdots$

(1) 4 (2) $4!$ (3) 11 (4) 22

15. If 10 lines are drawn in a plane such that no two of them are parallel and no three are concurrent, then the total number of points of intersection are

(1) 45 (2) 40 (3)$10!$ (4) 2^{10}

16. In a plane there are 10 points are there out of which 4 points are collinear, then the number of triangles formed is

(1) 110 (2) ${}^{10}C_3$ (3) 120 (4) 116

17. In ${}^{2n}C_3 :^n C_3 = 11 : 1$ then n is

(1) 5 (2) 6 (3)11 (4)7

18. ${}^{(n-1)}C_r +^{(n-1)} C_{(r-1)}$ is

(1) ${}^{(n+1)}C_r$ (2) ${}^{(n-1)}C_r$ (3) ${}^{n}C_r$ (4)${}^{n}C_{r-1}$.

19. The number of ways of choosing 5 cards out of a deck of 52 cards which include at least one king is

(1) ${}^{52}C_5$ (2) ${}^{48}C_5$ (3)${}^{52}C_5 +^{48} C_5$ (4)${}^{52}C_5 -^{48} C_5$.

20. The number of rectangles that a chessboard has $\cdots$

(1) 81 (2) 9^9 (3)1296 (4) 6561

21. The number of 10 digit number that can be written by using the digits 2 and 3 is

(1) ${}^{10}C_2 +^9 C_2$ (2) 2^{10} (3)$2^{10} - 2$ (4) $10!$

22. If P_r stands for ${}^rP_r$ then the sum of the series $1 + P_1 + 2P_2 + 3P_3 + \cdots + nP_n$ is

(1) P_{n+1} (2) $P_{n+1} - 1$ (3) $P_{n-1} + 1$ (4)${}^{(n+1)}P_{(n-1)}$

23. The product of first n odd natural numbers equals

(1) ${}^{2n}C_n \times^n P_n$ (2) $(\frac{1}{2})^n \times {}^{2n}C_n \times^n P_n$ (3)$(\frac{1}{4})^n \times {}^{2n}C_n \times^{2n} P_n$ (4)${}^{n}C_n \times^n P_n$

24. If ${}^nC_4$, ${}^nC_5$, ${}^nC_6$ are in AP the value of n can be

(1) 14 (2) 11 (3)9 (4)5

25. $1 + 3 + 5 + 7 + \cdots + 17$ is equal to

(1) 101 (2) 81 (3) 71 (4) 61

Summary

In this chapter, we acquired the knowledge of

- Factorial of a natural number n is the product of the first n natural numbers.
- $n! = n(n-1)!$, for any integer $n \geq 1$.
- The number of ways of arranging n unlike objects is $n!$.
- The number of distinct permutations of r objects which can be made from n distinct objects is
$$^{n}P_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1).$$
- The number of permutations of n objects taken all at a time where P_1 objects one of first kind, P_2 objects one of second kind, $\cdots$ P_k objects one of the kth kind and the rest, if any are all different and is given by
$$\frac{n!}{P_1!P_2!\cdots P_k!}.$$
- Order matters for a permutation where as order does not matter for a combination.
- The number of combinations of n different objects taken r at a time denoted by $^{n}C_r$ is given by
$$^{n}C_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

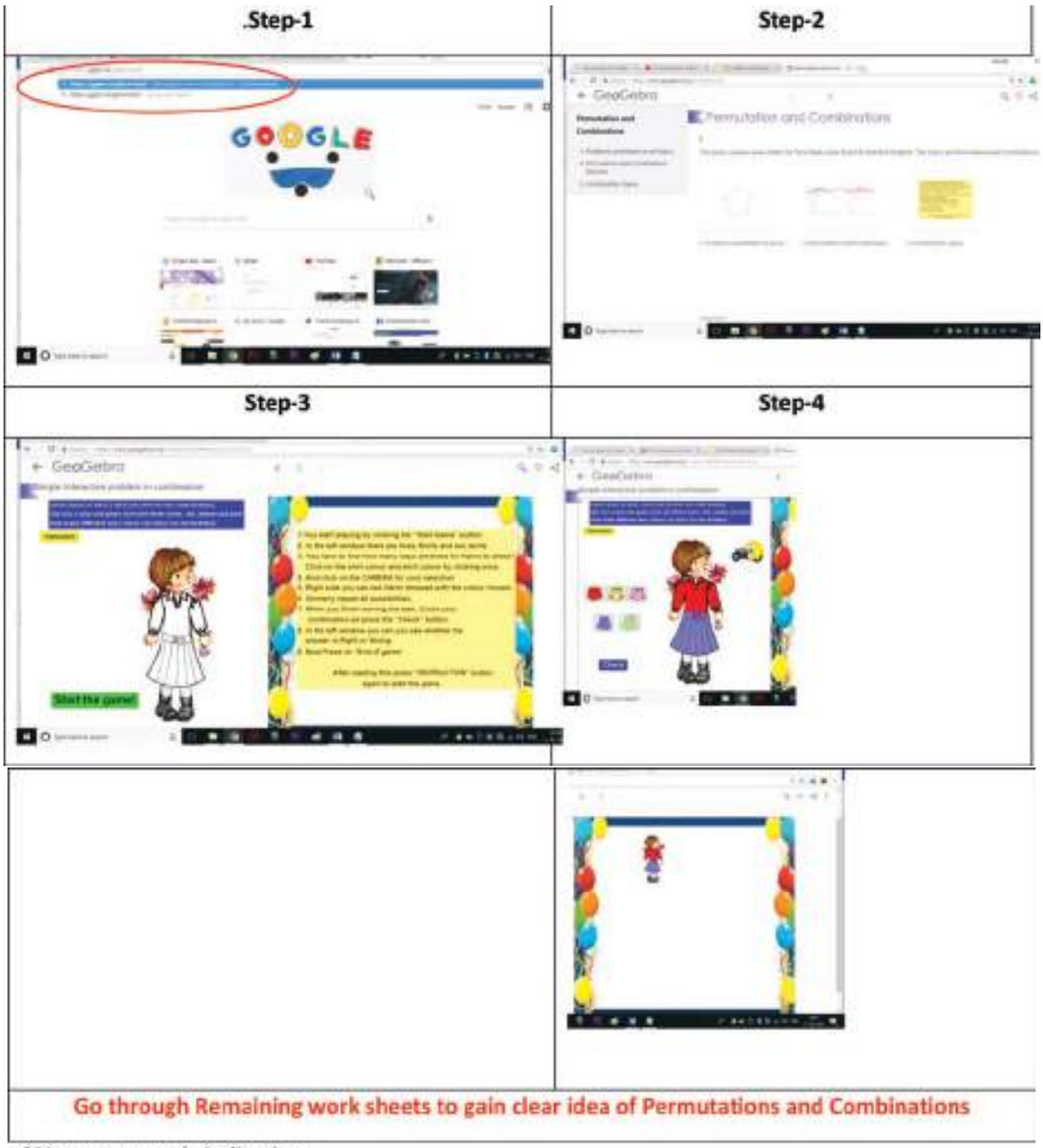
ICT CORNER-4(a)

Expected Outcome $\Rightarrow$

Step–1: Open the Browser and type the URL Link given below (or) Scan the QR Code. GeoGebra worksheet Permutations and Combination will appear. You can select the worksheet you want to open, for example open Combination Game

Step–2: Press Instruction button and read how to play. After reading once again press Instruction button and press Start the game.

Step–3: Find the number of combination to dress up the girl with 3 Shirts and 2 Skirts. After selecting each pair, click on Camera picture, so that the selected will appear in right space. After completing the selection press check button to see the result.



*Pictures are only indicatives.

Browse in the link

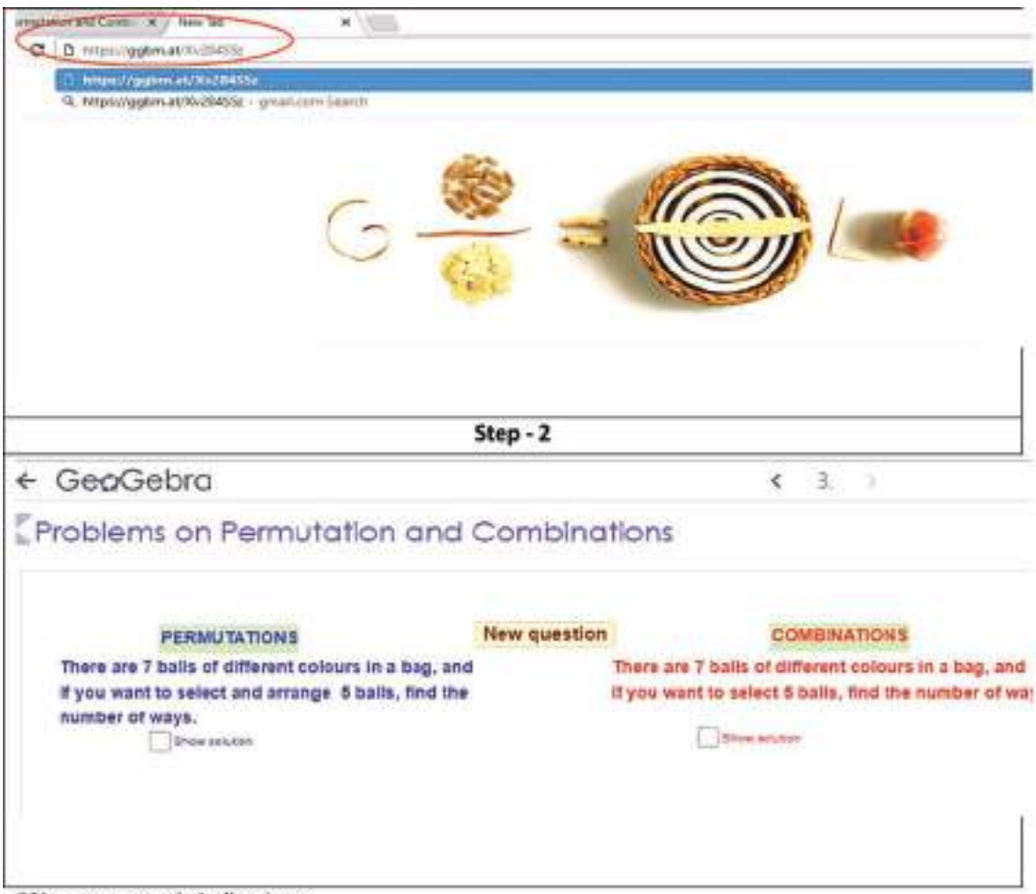
Permutations and Combinations: https://ggbm.at/gVnmcKp9

ICT CORNER-4(b)

Expected Outcome ⇒

Step–1: Open the Browser and type the URL Link given below (or) Scan the QR Code.

Step–2: GeoGebra worksheet "Problems on Permutations and Combination" will appear. By clicking on "New Problem" new question will be generated as many times you want to do. You have to work out the problem yourself and find the answer. Now click on "Show Solution" on each Permutation and Combination to get the respective solution and check with your answer.



*Pictures are only indicatives.

Browse in the link

Permutations and Combinations: https://ggbm.at/Xv284SSz

Chapter 5 Binomial Theorem, Sequences and Series

"Life stands before me like an eternal spring with new and brilliant clothes"

Johann Carl Friedrich Gauss

2CMG3L

5.1 Introduction

Binomial theorem facilitates the algebraic expansion of the binomial $(a+b)$ for a positive integral exponent n. Binomial theorem is used in all branches of Mathematics and also in other Sciences. For example using the Binomial theorem one can easily find the coefficient of x^{20} in the expansion of $(2x-7)^{23}$. If one wants to know the maturity amount after 10 years on a sum of money deposited in a nationalised bank at the rate of 8% compound interest per year or to know the size of population of our country after 15 years if the annual growth rate and present population size are known, Binomial theorem helps us in finding the above quantities. The coefficients in the binomial expansion of $(a+b)^n$, $n \in \mathbb{N}$, are called binomial coefficients. Binomial theorem plays a vital role in determining the probabilities of events when the random experiment involves finite sample space and each outcome is either success or failure. In this chapter we learn binomial theorem and some of its applications.

Greek Mathematician Euclid mentioned the special case of binomial theorem for exponent 2. Binomial theorem for exponent 3 was known by 6th century in India. In 1544, Michael Stifel (German Mathematician) introduced the term **binomial coefficient** and expressed $(1+x)^n$ in terms of $(1+x)^{n-1}$.

The German Mathematician Johann Carl Friedrich Gauss is one of the most renowned Mathematicians in history. Many have referred to him as the "Prince of Mathematics ". He has contributed in the areas of Number theory, Physics, Astronomy etc., Number Theory was Gauss's favourite field and he referred to Number theory as the "Queen of Mathematics". Anecdote involves, his school teacher who wanted to test the students asked them to sum the integers from 1 to 100. Within a few seconds Gauss shown the answer has 5050. Nobody is sure which method of summing an arithmetic sequence Gauss used as a child.

Johann Carl Friedrich Gauss (1777–1855)

Over the period of thousand years, legends have developed mathematical problems involving sequences and series. One of the famous legends about series concerns the invention of chess, where the cells of chess board were related to $1, 2, 4, 8, \ldots$ (imagine the number related to 64th cell). There are many applications of arithmetic and geometric progressions to real life situations.

In the earlier classes we have learnt about sequences, series. Roughly speaking a sequence is an arrangement of objects in some order and a series is the sum of the terms of a sequence of numbers. The concept of infinite series helps us to compute many values, like $\sin \frac{9}{44}\pi$, $\log 43$ and e^{20} to a desired level of approximation. Sequences are important in differential equations and analysis. We learn more about sequences and series.

Learning Objectives

On completion of this chapter, the students are expected to know

- the concept of Binomial Theorem, to compute binomial coefficients and their applications
- the concepts of sequences and series
- how to compute arithmetic, geometric and harmonic means
- how to find the sum of finite and infinite series of real numbers
- how to add series using telescopic summation
- how to apply binomial, exponential and logarithmic series

5.2 Binomial Theorem

The prefix **bi** in the words bicycle, binocular, binary and in many more words means **two**. The word ***binomial*** stands for expressions having two terms. For examples $(1+x)$, $(x+y)$, (x^2+xy) and $(2a+3b)$ are some binomial expressions.

5.2.1 Binomial Coefficients

In Chapter 4 we have learnt and used the symbol ${}^n\mathrm{C}_r$ which is defined as

$$ {}^n\mathrm{C}_r = \frac{n(n-1)(n-2)\ldots(n-(r-1))}{r(r-1)(r-2)\ldots 1} = \frac{n!}{(n-r)!\,r!}. $$

Since ${}^n\mathrm{C}_r$ occurs as the coefficients of x^r in $(1+x)^n$ $n \in \mathbb{N}$ and as the coefficients of $a^r b^{n-r}$ in $(a+b)^n$, they are called binomial coefficients. Though the values of ${}^n\mathrm{C}_r$ can be computed by formula, there is an interesting simple way to find ${}^n\mathrm{C}_r$ without doing cumbersome multiplications.

Pascal Triangle

The Pascal triangle is an arrangement of the values of ${}^n\mathrm{C}_r$ in a triangular form. The $(k+1)^{st}$ row consists values of

$$ {}^k\mathrm{C}_0\,,\ {}^k\mathrm{C}_1\,,\ {}^k\mathrm{C}_2\,,\ {}^k\mathrm{C}_3\,,\ldots,\ {}^k\mathrm{C}_k\,. $$

In fact, the Pascal triangle is

$$ \begin{array}{ccccccccc}
 & & & & {}^0\mathrm{C}_0 & & & & \\
 & & & {}^1\mathrm{C}_0 & & {}^1\mathrm{C}_1 & & & \\
 & & {}^2\mathrm{C}_0 & & {}^2\mathrm{C}_1 & & {}^2\mathrm{C}_2 & & \\
 & {}^3\mathrm{C}_0 & & {}^3\mathrm{C}_1 & & {}^3\mathrm{C}_2 & & {}^3\mathrm{C}_3 & \\
{}^4\mathrm{C}_0 & & {}^4\mathrm{C}_1 & & {}^4\mathrm{C}_2 & & {}^4\mathrm{C}_3 & & {}^4\mathrm{C}_4 \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots
\end{array}
$$

Recall the expansion and observe the coefficients of each term of the identities $(a+b)^0$, $(a+b)^1$, $(a+b)^2$, $(a+b)^3$. There is a pattern in the arrangements of coefficents

$$\begin{array}{lc} (a+b)^0 = 1 & 1 \\ (a+b)^1 = a+b & 1 \quad 1 \\ (a+b)^2 = a^2+2ab+b^2 & 1 \quad 2 \quad 1 \\ (a+b)^3 = a^3+3a^2b+3ab^2+b^3 & 1 \quad 3 \quad 3 \quad 1 \end{array}$$

If we observe carefully the Pascal triangle, we may notice that each row starts and ends with 1 and other entries are the sum of the two numbers just above it. For example '3' is the sum of 1 and 2 above it; '10' is the sum of 4 and 6 above it. We will prove in a short while that

$$(a+b)^n = {}^n\mathrm{C}_0\, a^n b^0 + {}^n\mathrm{C}_1\, a^{n-1}b^1 + \cdots + {}^n\mathrm{C}_r\, a^{n-r}b^r + \cdots + {}^n\mathrm{C}_n\, a^0 b^n.$$

which is the ***binomial expansion*** of $(a+b)^n$. The binomial expansion of $(a+b)^n$ for any $n \in \mathbb{N}$ can be written using Pascal triangle. For example, from the fifth row we can write down the expansion of $(a+b)^4$ and from the sixth row we can write down the expansion of $(a+b)^5$ and so on. We know the terms (without coefficients) of $(a+b)^5$ are

$$a^5,\ a^4b,\ a^3b^2,\ a^2b^3,\ ab^4,\ b^5$$

and the sixth row of the Pascal triangle is

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1.$$

Using these two we can write

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

The Pascal triangle can be constructed using addition alone, without using any multiplication or division. So without multiplication we can write down the binomial expansion for $(a+b)^n$ for any $n \in \mathbb{N}$.

The above pattern resembling a triangle, is credited in the name of the seventeenth century French Mathematician Blaise Pascal, who studied mathematical properties of this structure and used this concept effectively in Probability Theory.

5.2.2 Binomial theorem for positive integral index

Now we prove the most celebrated theorem called Binomial Theorem.

Theorem 5.1 (Binomial theorem for positive integral index): If n is any positive integer, then

$$(a+b)^n = {}^n\mathrm{C}_0\, a^n b^0 + {}^n\mathrm{C}_1\, a^{n-1}b^1 + \cdots + {}^n\mathrm{C}_r\, a^{n-r}b^r + \cdots + {}^n\mathrm{C}_n\, a^0 b^n.$$

Proof. We prove the theorem by using mathematical induction. For any positive integer n, let $P(n)$ be the statement

$$(a+b)^n = {}^n\mathrm{C}_0\, a^n b^0 + {}^n\mathrm{C}_1\, a^{n-1}b^1 + \cdots + {}^n\mathrm{C}_r\, a^{n-r}b^r + \cdots + {}^n\mathrm{C}_n\, a^0 b^n.$$

Since

$${}^1\mathrm{C}_0 = 1 \text{ and } {}^1\mathrm{C}_1 = 1,$$

the expression in the right hand side of $P(1)$ is $a^1b^0 + a^0b^1$ which is same as $a+b$; the left hand side is $(a+b)^1$. Hence $P(1)$ is true.

We assume that for a positive integer k, $P(k)$ is true. That is,

$$(a+b)^k = {}^k\mathrm{C}_0\, a^k b^0 + {}^k\mathrm{C}_1\, a^{k-1}b^1 + \cdots + {}^k\mathrm{C}_r\, a^{k-r}b^r + \cdots + {}^k\mathrm{C}_k\, a^0 b^k.$$

Let us use the identity

$${}^n\mathrm{C}_r + {}^n\mathrm{C}_{r-1} = {}^{n+1}\mathrm{C}_r$$

in the proof. Now

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k \\
&= (a+b)\left[{}^k\mathrm{C}_0\, a^k b^0 + {}^k\mathrm{C}_1\, a^{k-1}b^1 + \cdots + {}^k\mathrm{C}_r\, a^{k-r}b^r + \cdots + {}^k\mathrm{C}_k\, a^0 b^k\right] \\
&= \left[{}^k\mathrm{C}_0\, a^{k+1}b^0 + {}^k\mathrm{C}_1\, a^k b^1 + \cdots + {}^k\mathrm{C}_r\, a^{k-r+1}b^r + \cdots + {}^k\mathrm{C}_k\, a^1 b^k\right] \\
&\quad + \left[{}^k\mathrm{C}_0\, a^k b^1 + {}^k\mathrm{C}_1\, a^{k-1}b^2 + \cdots + {}^k\mathrm{C}_r\, a^{k-r}b^{r+1} + \cdots + {}^k\mathrm{C}_k\, a^0 b^{k+1}\right] \\
&= {}^k\mathrm{C}_0\, a^{k+1}b^0 + \left[{}^k\mathrm{C}_1 + {}^k\mathrm{C}_0\right] a^k b^1 + \cdots + \left[{}^k\mathrm{C}_r + {}^k\mathrm{C}_{r-1}\right] a^{k-r+1}b^r \\
&\quad + \cdots + \left[{}^k\mathrm{C}_k + {}^k\mathrm{C}_{k-1}\right] a^1 b^k + {}^k\mathrm{C}_k\, a^0 b^{k+1} \\
&= {}^{k+1}\mathrm{C}_0\, a^{k+1}b^0 + {}^{k+1}\mathrm{C}_1\, a^k b^1 + {}^{k+1}\mathrm{C}_2\, a^{k-1}b^2 + \cdots + {}^{k+1}\mathrm{C}_r\, a^{k-r+1}b^r \\
&\quad + \cdots + {}^{k+1}\mathrm{C}_k\, a^1 b^k + {}^{k+1}\mathrm{C}_{k+1}\, a^0 b^{k+1} \\
(a+b)^{k+1} &= {}^{k+1}\mathrm{C}_0\, a^{(k+1)}b^0 + {}^{k+1}\mathrm{C}_1\, a^{(k+1)-1}b^1 + {}^{k+1}\mathrm{C}_2\, a^{(k+1)-2}b^2 + \cdots \\
&\quad + {}^{k+1}\mathrm{C}_r\, a^{(k+1)-r}b^r + \cdots + {}^{k+1}\mathrm{C}_k\, a^1 b^{(k+1)-1} + {}^{k+1}\mathrm{C}_{k+1}\, a^0 b^{k+1}.
\end{aligned}$$

This shows that $P(k+1)$ is true whenever $P(k)$ is true. Thus, by the principle of mathematical induction, $P(n)$ is true for all natural numbers n. Hence,

$$(a+b)^n = {}^n\mathrm{C}_0\, a^n b^0 + {}^n\mathrm{C}_1\, a^{n-1}b^1 + \cdots + {}^n\mathrm{C}_r\, a^{n-r}b^r + \cdots + {}^n\mathrm{C}_n\, a^0 b^n, n \in \mathbb{N}.$$

(i) The expansion of $(a+b)^n, n \in \mathbb{N}$ can also be written as

$$(a+b)^n = \Sigma_{k=0}^{n}\, {}^n\mathrm{C}_k\, a^{n-k}b^k \text{ or } \Sigma_{k=0}^{n}\, {}^n\mathrm{C}_k\, a^k b^{n-k}.$$

(ii) The expansion of $(a+b)^n, n \in \mathbb{N}$, contains exactly $(n+1)$ terms.

(iii) In $(a+b)^n = \Sigma_{k=0}^{n}\, {}^n\mathrm{C}_k\, a^{n-k}b^k$, the powers of a decreases by 1 in each term, whereas the powers of b increases by 1 in each term. However, the sum of powers of a and b in each term is always n.

(iv) The $(r+1)^{th}$ term in the expansion of $(a+b)^n, n \in \mathbb{N}$, is

$$T_{r+1} = {}^n\mathrm{C}_r\, a^{n-r}b^r, \quad r = 0, 1, 2, \cdots, n.$$

(v) In the product $(a+b)(a+b)\cdots(a+b), n$ times, to get b^r, we need any r factors out of these n factors. This can be done in ${}^n\mathrm{C}_r$ ways. That is why, we have ${}^n\mathrm{C}_r$ as the coefficient of $a^{n-r}b^r$.

(vi) In the expansion of $(a+b)^n, n \in \mathbb{N}$, the coefficients at equidistant from the beginning and from the end are equal due to the fact that ${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}$.

(vii) In the expansion of $(a+b)^n, n \in \mathbb{N}$, the greatest coefficient is ${}^n\mathrm{C}_{\frac{n}{2}}$ if n is even and the greatest coefficients are ${}^n\mathrm{C}_{\frac{n-1}{2}}$ or ${}^n\mathrm{C}_{\frac{n+1}{2}}$, if n is odd.

(viii) In the expansion of $(a+b)^n, n \in \mathbb{N}$,
if n is even, the middle term is $T_{\frac{n}{2}+1} = {}^n\mathrm{C}_{\frac{n}{2}}\, a^{n-\frac{n}{2}} b^{\frac{n}{2}}$. If n is odd, then the two middle terms are $T_{\frac{n-1}{2}+1}$ and $T_{\frac{n+1}{2}+1}$.

5.3 Particular cases of Binomial Theorem

(i) Replacing b by $(-b)$, in the binomial expansion of $(a+b)^n, n \in \mathbb{N}$, we get

$$\begin{aligned}(a-b)^n &= {}^n\mathrm{C}_0\, a^n b^0 - {}^n\mathrm{C}_1\, a^{n-1} b^1 + {}^n\mathrm{C}_2\, a^{n-2} b^2 - \cdots \\ &\quad + (-1)^r\, {}^n\mathrm{C}_r\, a^{n-r} b^r + \cdots + (-1)^n\, {}^n\mathrm{C}_n\, a^0 b^n.\end{aligned}$$

Observe that the sign '$+$' and '$-$' appear alternately in the binomial expansion of $(a-b)^n$.

(ii) Replacing a by 1 and b by x, in the binomial expansion of $(a+b)^n$, we get

$$(1+x)^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1\, x + {}^n\mathrm{C}_2\, x^2 + \cdots + {}^n\mathrm{C}_r\, x^r + \cdots + {}^n\mathrm{C}_n\, x^n.$$

In particular, when $x = 1$, ${}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + \cdots + {}^n\mathrm{C}_n = 2^n$.

If X is a set containing n elements, then we know that ${}^n\mathrm{C}_r$ is the number of subsets of X having exactly r elements. So by adding ${}^n\mathrm{C}_r$ for $r = 0, 1, 2, \ldots, n$ we get the number of subsets of X. So by using the above identity we see that **a set of n elements has 2^n subsets.**

(iii) $(1-x)^n = {}^n\mathrm{C}_0 - {}^n\mathrm{C}_1\, x + {}^n\mathrm{C}_2\, x^2 - \cdots + (-1)^r\, {}^n\mathrm{C}_r\, x^r + \cdots + (-1)^n x^n$. In particular, when $x = 1$, ${}^n\mathrm{C}_0 + {}^n\mathrm{C}_2 + {}^n\mathrm{C}_4 + \cdots = {}^n\mathrm{C}_1 + {}^n\mathrm{C}_3 + {}^n\mathrm{C}_5 + \cdots = 2^{n-1}$

Example 5.1 Find the expansion of $(2x+3)^5$.

Solution:
By taking $a = 2x$, $b = 3$ and $n = 5$ in the binomial expansion of $(a+b)^n$ we get

$$\begin{aligned}(2x+3)^5 &= (2x)^5 + 5(2x)^4 3 + 10(2x)^3 3^2 + 10(2x)^2 3^3 + 5(2x)3^4 + 3^5 \\ &= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243.\end{aligned}$$

Example 5.2 Evaluate 98^4.

Solution:
By taking $a = 100$, $b = 2$ and $n = 4$ in the binomial expansion of $(a-b)^n$ we get

$$\begin{aligned}98^4 &= (100-2)^4 \\ &= {}^4\mathrm{C}_0\, 100^4 - {}^4\mathrm{C}_1\, 100^3 2 + {}^4\mathrm{C}_2\, 100^2 2^2 - {}^4\mathrm{C}_3\, 100^1 2^3 + {}^4\mathrm{C}_4\, 100^0 2^4 \\ &= 100000000 - 8000000 + 240000 - 3200 + 16 \\ &= 92236816.\end{aligned}$$

Example 5.3 Find the middle term in the expansion of $(x+y)^6$.

Solution:
Here $n = 6$; which is even. Thus the middle term in the expansion of $(x+y)^6$ is the term containing x^3y^3, that is the term ${}^6\mathrm{C}_3\, x^3y^3$ which is equal to $20x^3y^3$.

Example 5.4 Find the middle terms in the expansion of $(x+y)^7$.

Solution:
As $n = 7$ which is odd, the terms containing x^4y^3 and x^3y^4 are the two middle terms. They are ${}^7\mathrm{C}_3\, x^4y^3$ and ${}^7\mathrm{C}_4\, x^3y^4$ which are equal to $35x^4y^3$ and $35x^3y^4$.

Example 5.5 Find the coefficient of x^6 in the expansion of $(3+2x)^{10}$.

Solution:
Let us take $a = 3$ and $b = 2x$ in the binomial expansion of $(a+b)^{10}$. Then, x^6 will appear in the term containing $(2x)^6$ and nowhere else. So the term containing x^6 is

$$ {}^{10}\mathrm{C}_6\, a^4b^6 = {}^{10}\mathrm{C}_4\, a^4b^6 = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}\, 3^4(2x)^6 = 210\times 3^4\times 2^6x^6. \quad [\because {}^{10}\mathrm{C}_6 = {}^{10}\mathrm{C}_4] $$

So coefficient of x^6 in the expansion of $(3+2x)^{10}$ is $210\times 3^4 2^6$.

Example 5.6 Find the coefficient of x^3 in the expansion of $(2-3x)^7$.

Solution:
Let us take $a = 2$ and $b = -3x$ in the binomial expansion of $(a+b)^7$. Then, x^3 will appear in the term containing $(-3x)^3$ and nowhere else. So the term containing x^3 is

$$ {}^7\mathrm{C}_3\, a^4b^3 = \frac{7\times 6\times 5}{3\times 2\times 1}\, 2^4(-3x)^3 = 35\times 2^4\times(-3)^3x^3. $$

So coefficient of x^3 in the expansion of $(2-3x)^7$ is $35\times 16\times(-27) = -15120$.

Example 5.7 The 2nd, 3rd and 4th terms in the binomial expansion of $(x+a)^n$ are 240, 720 and 1080 for a suitable value of x. Find x, a and n.

Solution:
It is given that $T_2 = 240$, $T_3 = 720$ and $T_4 = 1080$.

$$ \begin{aligned} T_2 &= {}^n\mathrm{C}_1\, x^{n-1}a &= 240 \qquad (1)\\ T_3 &= {}^n\mathrm{C}_2\, x^{n-2}a^2 &= 720 \qquad (2)\\ T_4 &= {}^n\mathrm{C}_3\, x^{n-3}a^3 &= 1080 \qquad (3) \end{aligned} $$

Dividing (2) by (1) and (3) by (2) we get

$$\frac{a}{x} = \tfrac{6}{n-1} \qquad (4)$$
$$\frac{a}{x} = \tfrac{9}{2(n-2)} \qquad (5)$$

From (4) and (5)

$$\frac{6}{n-1} = \frac{9}{2(n-2)}$$

Thus $n = 5$. Substituting $n = 5$ in (1) and (4) also dividing (1) by (4), we get

$$\frac{5x^4 a}{\frac{a}{x}} = \frac{240}{\frac{6}{4}}$$

Thus $5x^5 = 160$ and hence $x = 2$. Substituting in (4) we get $a = 3$.

Example 5.8 Expand $\left(2x - \frac{1}{2x}\right)^4$.

Solution:
We have

$$\begin{aligned}\left(2x - \frac{1}{2x}\right)^4 &= {}^4\mathrm{C}_0\,(2x)^4\left(-\frac{1}{2x}\right)^0 + {}^4\mathrm{C}_1\,(2x)^3\left(-\frac{1}{2x}\right)^1 + {}^4\mathrm{C}_2\,(2x)^2\left(-\frac{1}{2x}\right)^2 \\ &\quad + {}^4\mathrm{C}_3\,(2x)^1\left(-\frac{1}{2x}\right)^3 + {}^4\mathrm{C}_4\,(2x)^0\left(-\frac{1}{2x}\right)^4 \\ &= (2x)^4 - 4(2x)^3\left(\frac{1}{2x}\right) + 6(2x)^2\left(\frac{1}{2x}\right)^2 - 4(2x)\left(\frac{1}{2x}\right)^3 + \left(\frac{1}{2x}\right)^4 \\ &= 16x^4 - 16x^2 + 6 - \frac{1}{x^2} + \frac{1}{16x^4}\end{aligned}$$

Example 5.9 Expand $\left(x^2 + \sqrt{1-x^2}\right)^5 + \left(x^2 - \sqrt{1-x^2}\right)^5$.

Solution:
We have

$$\begin{aligned}\left(x^2 + \sqrt{1-x^2}\right)^5 &= {}^5\mathrm{C}_0\,(x^2)^5\left(\sqrt{1-x^2}\right)^0 + {}^5\mathrm{C}_1\,(x^2)^4\left(\sqrt{1-x^2}\right)^1 \\ &\quad + {}^5\mathrm{C}_2\,(x^2)^3\left(\sqrt{1-x^2}\right)^2 + {}^5\mathrm{C}_3\,(x^2)^2\left(\sqrt{1-x^2}\right)^3 \\ &\quad + {}^5\mathrm{C}_4\,(x^2)^1\left(\sqrt{1-x^2}\right)^4 + {}^5\mathrm{C}_5\,(x^2)^0\left(\sqrt{1-x^2}\right)^5 \\ &= x^{10} + 5x^8\sqrt{1-x^2} + 10x^6(1-x^2) + 10x^4(1-x^2)\sqrt{1-x^2} \\ &\quad + 5x^2(1-x^2)^2 + (1-x^2)^2(\sqrt{1-x^2})\end{aligned}$$

$$\begin{aligned}\left(x^2-\sqrt{1-x^2}\right)^5 &= {}^5\mathrm{C}_0\,(x^2)^5\left(\sqrt{1-x^2}\right)^0 - {}^5\mathrm{C}_1\,(x^2)^4\left(\sqrt{1-x^2}\right)^1\\ &\quad + {}^5\mathrm{C}_2\,(x^2)^3\left(\sqrt{1-x^2}\right)^2 - {}^5\mathrm{C}_3\,(x^2)^2\left(\sqrt{1-x^2}\right)^3\\ &\quad + {}^5\mathrm{C}_4\,(x^2)^1\left(\sqrt{1-x^2}\right)^4 - {}^5\mathrm{C}_5\,(x^2)^0\left(\sqrt{1-x^2}\right)^5\\ &= x^{10}-5x^8\sqrt{1-x^2}+10x^6(1-x^2)-10x^4(1-x^2)\sqrt{1-x^2}\\ &\quad +5x^2(1-x^2)^2-(1-x^2)^2(\sqrt{1-x^2})\end{aligned}$$

Thus

$$\begin{aligned}\left(x^2+\sqrt{1-x^2}\right)^5+\left(x^2-\sqrt{1-x^2}\right)^5 &= 2[x^{10}+10x^6(1-x^2)+5x^2(1-x^2)^2]\\ &= 2[x^{10}+10x^6-10x^8+5x^2(1-2x^2+x^4)]\\ &= 2[x^{10}-10x^8+15x^6-10x^4+5x^2]\end{aligned}$$

Example 5.10 Using Binomial theorem, prove that 6^n-5n always leaves remainder 1 when divided by 25 for all positive integer n.

Solution:
To prove this it is enough to prove, $6^n-5n=25k+1$ for some integer k. We first consider the expansion

$$(1+x)^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1\,x + {}^n\mathrm{C}_2\,x^2+\cdots+{}^n\mathrm{C}_{n-1}\,x^{n-1}+{}^n\mathrm{C}_n\,x^n, n\in\mathbb{N}$$

Taking $x=5$ we get $(1+5)^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1\,5 + {}^n\mathrm{C}_2\,5^2+\cdots+{}^n\mathrm{C}_{n-1}\,5^{n-1}+{}^n\mathrm{C}_n\,5^n$. The above equality reduces to $6^n=1+5n+25({}^n\mathrm{C}_2+5\,{}^n\mathrm{C}_3+\cdots+{}^n\mathrm{C}_n\,5^{n-2})$.
That is,

$$6^n-5n=1+25({}^n\mathrm{C}_2+5\,{}^n\mathrm{C}_3+\cdots+{}^n\mathrm{C}_n\,5^{n-2})=1+25k, k\in\mathbb{N}.$$

Thus 6^n-5n always leaves remainder 1 when divided by 25 for all positive integer n.

Example 5.11 Find the last two digits of the number 7^{400}.

Solution:
We have

$$\begin{aligned}7^{400} &= (7^2)^{200}=(50-1)^{200}\\ &= {}^{200}\mathrm{C}_0\,50^{200} - {}^{200}\mathrm{C}_1\,50^{199}+\cdots\\ &\qquad + {}^{200}\mathrm{C}_{198}\,50^2(-1)^{198}+{}^{200}\mathrm{C}_{199}\,50(-1)^{199}+{}^{200}\mathrm{C}_{200}\,(-1)^{200}\\ &= 50^2\left({}^{200}\mathrm{C}_0\,50^{198}-{}^{200}\mathrm{C}_1\,50^{197}+\cdots+{}^{200}\mathrm{C}_{198}\,(-1)^{198}\right)-200\times 50+1.\end{aligned}$$

As 50^2 and 200 are divisible by 100, the last two digits of 7^{400} is 0 1.

Exercise - 5.1

1. Expand (i) $\left(2x^2 - \frac{3}{x}\right)^3$ (ii) $\left(2x^2 - 3\sqrt{1-x^2}\right)^4 + \left(2x^2 + 3\sqrt{1-x^2}\right)^4$.
2. Compute (i) 102^4 (ii) 99^4 (iii) 9^7.
3. Using binomial theorem, indicate which of the following two number is larger: $(1.01)^{1000000}$, 10000.
4. Find the coefficient of x^{15} in $\left(x^2 + \dfrac{1}{x^3}\right)^{10}$.
5. Find the coefficient of x^2 and the coefficient of x^6 in $\left(x^2 - \dfrac{1}{x^3}\right)^6$.
6. Find the coefficient of x^4 in the expansion of $(1+x^3)^{50}(x^2 + \dfrac{1}{x})^5$.
7. Find the constant term of $\left(2x^3 - \dfrac{1}{3x^2}\right)^5$.
8. Find the last two digits of the number 3^{600}.
9. If n is a positive integer, using Binomial theorem, show that, $9^{n+1} - 8n - 9$ is always divisible by 64.
10. If n is an odd positive integer, prove that the coefficients of the middle terms in the expansion of $(x+y)^n$ are equal.
11. If n is a positive integer and r is a nonnegative integer, prove that the coefficients of x^r and x^{n-r} in the expansion of $(1+x)^n$ are equal.
12. If a and b are distinct integers, prove that $a-b$ is a factor of $a^n - b^n$, whenever n is a positive integer. [Hint: write $a^n = (a-b+b)^n$ and expand]
13. In the binomial expansion of $(a+b)^n$, if the coefficients of the 4th and 13th terms are equal then, find n.
14. If the binomial coefficients of three consecutive terms in the expansion of $(a+x)^n$ are in the ratio $1:7:42$, then find n.
15. In the binomial expansion of $(1+x)^n$, the coefficients of the 5th, 6th and 7th terms are in AP. Find all values of n.
16. Prove that $C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = \dfrac{(2n)!}{(n!)^2}$.

5.4 Finite Sequences

A sequence is a list of elements with a particular order. While the idea of a sequence of numbers, $a_1, a_2, \cdots$, is straight forward, it is useful to think of a sequence as a function whose domain is either the set of first n natural numbers or $\mathbb{N}$. Throughout this chapter, we consider only sequences of real numbers and we will refer to them as sequences. The arithmetic sequences and geometric sequences are also known as arithmetic progressions(AP) and geometric progressions (GP). Let us recall, the basic definitions of sequences and series.

- If X is any set and $n \in \mathbb{N}$, then any function $f : \{1, 2, 3, \ldots, n\} \to X$ is called a ***finite sequence*** on X and any function $g : \mathbb{N} \to X$ is called an ***infinite sequence*** on X. The value $f(n)$ of the function f at n is denoted by a_n and the sequence itself is denoted by (a_n).
- If the set X happens to be a set of real numbers, the sequence is called a numerical sequence or a sequence of real numbers.
- Though every sequence is a function, a function is not necessarily a sequence.
- Unlike sets, where elements are not repeated, the terms in a sequence may be repeated. In particular, a sequence in which all terms are same is called a constant sequence.
- A useful way to visualise a sequence (a_n) is to plot the graph of $\{(n, a_n) : n \in \mathbb{N}\}$ which gives some details about the sequence.

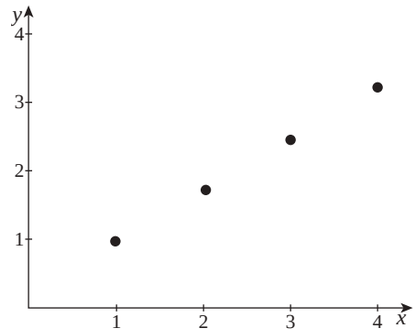


Figure 5.1

5.4.1 Arithmetic and Geometric Progressions

Progressions are some special cases of sequences where the terms of the sequences are either in increasing form or decreasing form.

We recall some definitions and results which has been discussed in earlier classes on arithmetic and geometric progressions.

Arithmetic Progression (AP)

- A sequence of the form

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots,\ a+(n-1)d,\ a+nd,\ \ldots$$

  is called an ***arithmetic progression*** or an ***arithmetic sequence***. In other words, each term (other than the first term) of the sequence is obtained by adding a constant to its previous term; the constant d is called ***common difference*** and the term a is called the ***initial term*** or ***first term***.
- The nth term of an arithmetic progression is given by $T_n = a + (n-1)d$.
- The sequences $\sqrt{2},\ \sqrt{2}+\sqrt{3},\ \sqrt{2}+2\sqrt{3},\ \sqrt{2}+3\sqrt{3},\ \ldots$ and $12,\ 9,\ 6,\ 3,\ \ldots$ are arithmetic sequences with common differences $\sqrt{3}$ and -3 respectively.
- It is interesting to observe that $3, 7, 11$ are three prime numbers which form an AP.
- For $n \in \mathbb{N}, T_n = an + b$ where a and b are relatively prime, form an AP which contains infinitely many prime numbers along with infinitely many composite numbers.

Geometric Progression (GP)

- A sequence of the form

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^{n-1},\ ar^n,\ \ldots$$

  with $a \neq 0$, and $r \neq 0$ is called a ***geometric progression*** or a ***geometric sequence***. In other words, each term (other than the first term) of the sequence is obtained by multiplying its previous term by a constant; the constant r is called ***common ratio*** and the term a is called the ***initial term*** or ***first term***.
- The nth term of a geometric progression is given by $T_n = ar^{n-1}$.
- The sequences $1,\ 2,\ 4,\ 8,\ 16,\ \ldots$ and $\sqrt{2},\ 2,\ 2\sqrt{2},\ 4,\ 4\sqrt{2},\ 16,\ \ldots$ are geometric sequences with common ratios 2 and $\sqrt{2}$ respectively.
- Taking logarithm of each term in a geometric progression with positive common ratio yields an arithmetic progression. $i.e.$, If $a, ar, ar^2, \ldots$ is a GP with $r > 0$, then $\log a, \log(ar), \log(ar^2), \ldots$ is an AP with common difference $\log r$.

It is interesting to note that the constant sequence $c,\ c,\ c, \ldots$ is an arithmetic sequence and is also a geometric sequence if $c \neq 0$.

Let us consider the special constant sequence $0, 0, 0, 0, \ldots$. We have no problem in seeing this as an arithmetic sequence. But when we try to see this as a geometric sequence clearly the initial term

a must be 0. What can we say about the common ratio r? If we take r as 1, 2 or any other number we get the same sequence $0, 0, 0, 0, \ldots$. We are left with the situation where a geometric sequence has infinitely many common ratios. To overcome these confusions some mathematicians exclude this sequence from the class of geometric sequences by assuming $a \neq 0$ in the definition. (We made this assumption)

5.4.2 Arithmetico-Geometric Progression (AGP)

Combining arithmetic and geometric progressions, a new progression called arithmetico geometric progression is formed. As we use the abbreviations AP and GP for arithmetic progressions and geometric progressions, we use the abbreviation AGP for arithmetico geometric progression. AGP's arise in various applications, such as the computation of expected value in probability theory.

Definition 5.1

A sequence of the form

$$a,\ (a+d)r,\ (a+2d)r^2,\ (a+3d)r^3,\ \ldots,\ (a+(n-1)d)r^{n-1},\ (a+nd)r^n,\ \ldots$$

is called an ***arithmetico-geometric progression*** or an ***arithmetico-geometric sequence***.

Consider an AP: $a, a+d, a+2d, \ldots$
GP: $1, r, r^2, \ldots$

Then the AGP is $a, (a+d)r, (a+2d)r^2, \ldots$

Here, a is the ***initial term***, d is the ***common difference*** and r is the ***common ratio*** of the AGP.

If we take $r = 1$, then the AGP will become an AP and if we take $d = 0$, then it will become a GP. So the arithmetic and Geometric progressions become particular cases of AGP. This is a nice situation to know the concept of generalization in mathematics.

We note that the nth term of an AGP is given by $T_n = (a+(n-1)d)r^{n-1}$. All APs and all GPs are AGPs.

For example, the AP $0, 1, 2, 3, 4, \ldots$ and the GP $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ give the AGP $\frac{0}{1}, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \ldots$

The sequence $4, 14, 40, 104, 256, 608, \ldots$ is also an example of an arithmetico-geometric progression. For this sequence $a = 4$, $d = 3$ and $r = 2$.

5.4.3 Harmonic Progression (HP)

Harmonic progression is one of many important sequences and is closely related to the arithmetic progression. Harmonic progression is widely used.

Definition 5.2

A sequence $h_1, h_2, h_3, \ldots$ is said to a ***harmonic sequence*** or a ***harmonic progression*** if $\frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \frac{1}{h_4}, \ldots$ is an arithmetic sequence.

Note that a sequence is in harmonic progression if reciprocal of its term are in arithmetic progression. But we should not say that harmonic progressions are reciprocals of arithmetic progressions; in fact, if an arithmetic sequence contains a zero term, then its reciprocal is not meaningful. Of course, if an arithmetic progression contains no zero term, then its reciprocal is a harmonic progression. So a general harmonic progression will be of the form

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots$$

As the denominator of a fraction cannot be 0, $a+kd \neq 0$ for any integer $k \geq 0$. So the condition $\frac{-a}{d}$ is not a whole number is essential. Problems in harmonic progression can be converted into problems in arithmetic progression and be solved using arithmetic progression techniques and formulae.

(i) The sequence $\left(\frac{1}{n}\right) = 1, \frac{1}{2}, \frac{1}{3}, \cdots$ is a harmonic sequence. One can draw the graph of
$\{(n, \frac{1}{n}) : n \in \mathbb{N}\}$ and visualize the harmonic sequence $\left(\frac{1}{n}\right)$.

(ii) If a, b, c are in HP, then $b = \dfrac{2ac}{a+c}$.

(iii) In a triangle, if the altitudes are in AP, then the sides are in HP

Example 5.12 Prove that if a, b, c are in HP, if and only if $\frac{a}{c} = \frac{a-b}{b-c}$.

Solution:
If a, b, c are in HP, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP. Thus, we have $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$, which gives $ab - ac = ac - bc$. So, $a(b-c) = c(a-b)$, which gives $\frac{a}{c} = \frac{a-b}{b-c}$. On the other hand, if $\frac{a}{c} = \frac{a-b}{b-c}$, then $a(b-c) = c(a-b)$. Dividing each terms of both sides by abc, we get $\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$. Thus, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP and hence a, b, c are in HP.

Example 5.13 If the 5th and 9th terms of a harmonic progression are $\frac{1}{19}$ and $\frac{1}{35}$, find the 12th term of the sequence.

Solution:
Let h_n be the harmonic progression and let $a_n = \dfrac{1}{h_n}$. Then $a_5 = 19$ and $a_9 = 35$. As a_n's form an arithmetic progression, we have $a + 4d = 19$ and $a + 8d = 35$. Solving these two equations, we get $a = 3$ and $d = 4$. Thus $a_{12} = a + 11d = 47$. Thus the 12th term of the harmonic progression is $\dfrac{1}{47}$.

What can we say about constant sequences? All constant sequences other than the zero sequence are harmonic progressions also.

5.4.4 Arithmetic, Geometric and Harmonic Mean

We know the concept of "*average*". There are many "*averages*". Arithmetic mean (AM), Geometric mean(GM) and Harmonic mean(HM) are also some averages. Let us now recall the definitions of arithmetic mean and geometric mean, where the terms need not in AP or GP.

Arithmetic Mean and Geometric Mean

Definition 5.3

Let n be any positive integer. Let $a_1, a_2, a_3, \ldots, a_n$ be n numbers . Then the number

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$$

is called the ***arithmetic mean*** of the numbers $a_1, a_2, a_3, \ldots, a_n$.

The numbers $a_1, a_2, a_3, \ldots, a_n$ need not be distinct and it is not necessary that the numbers are positive. It easily follows from the definition that 16 is the arithmetic mean of the numbers $14, 14, 17, 20, 15$.

Taking the multiplication in place of addition and n^{th} root in place of division by n in the definition of arithmetic mean we get the definition of geometric mean.

Definition 5.4

Let n be any positive integer. Let $a_1, a_2, a_3, \ldots, a_n$ be n non-negative numbers. Then the number

$$\sqrt[n]{a_1 a_2 a_3 \ldots a_n}$$

is called the ***geometric mean*** of the numbers $a_1, a_2, a_3, \ldots, a_n$.

Here also the numbers $a_1, a_2, a_3, \ldots, a_n$ need not be distinct but it is necessary that the numbers are non-negative. The geometric mean of the numbers $4, 6, 9$ is $\sqrt[3]{216} = 6$. The arithmetic mean of these three numbers is $\frac{19}{3} = 6\frac{1}{3}$. Observe that the arithmetic mean is greater than the geometric mean in this case. Is this true always?

It can be proved that "*For any set of n non-negative numbers, the arithmetic mean is greater than or equal to the geometric mean*". That is, if AM denotes the arithmetic mean and GM denotes the geometric mean, then $AM \geq GM$.

Let us prove this inequality $AM \geq GM$ for two non-negative numbers.

Theorem 5.2: If AM and GM denote the arithmetic mean and the geometric mean of two nonnegative numbers, then $AM \geq GM$. The equality holds if and only if the two numbers are equal.

Proof. Let a and b be any two nonnegative numbers. Then

$$AM = \tfrac{a+b}{2} \quad \text{and} \quad GM = \sqrt{ab}.$$

We have, $(a+b)^2 - 4ab = (a-b)^2 \geq 0$ Thus, $(a+b)^2 - 4ab \geq 0$ which gives $(a+b) \geq 2\sqrt{ab}$. Hence $\frac{a+b}{2} \geq \sqrt{ab}$.
In other words, $AM \geq GM$.

Moreover, the equality holds if and only if $(a+b)^2 - 4ab = 0$. This holds if and only if $(a-b)^2 = 0$ which holds if and only if $a = b$. Thus $AM = GM$ if and only if $a = b$. ■

Geometrical Proof for $AM \geq GM$

Let a and b be any two nonnegative real numbers. If at least one of them is zero, then GM is 0 and hence we have nothing to prove. So let us assume that $a > 0$ and $b > 0$. We draw a straight line segment AB of length $a + b$ and a semi-circle having AB as diameter. Let M be the midpoint of AB. Then M is the center of the semi-circle drawn. Since M is the midpoint of AB, we have $AM = MB = \frac{a+b}{2}$. So the radius of the circle is $\frac{a+b}{2}$. Let D be the point on AB so that $AD = a$; then $DB = b$.

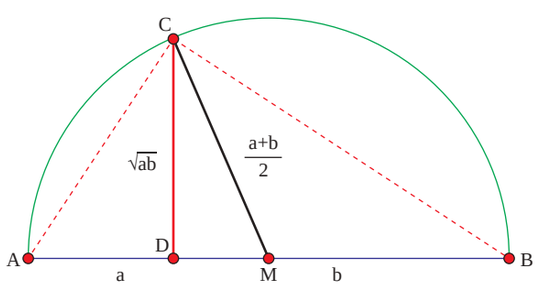


Figure 5.2

Through D we draw the perpendicular to AB and let it to meet the semi-circle at C. We draw straight lines CA, CB and CM. Since M is the center $CM =$ radius $= \frac{a+b}{2}$. Clearly $MD = \frac{a+b}{2} - a$. Using the similar triangles ΔACD and ΔCBD we have $\frac{CD}{AD} = \frac{BD}{CD}$ and hence $CD^2 = AD \times BD = ab$. So $CD = \sqrt{ab}$. (Using Pythagoras theorem also we can prove that $CD = \sqrt{ab}$.) Since the length of any half chord is less than or equal to the radius, we have $CD \leq CM$. In other words, $\sqrt{ab} \leq \frac{a+b}{2}$. This means $GM \leq AM$.

The length of the half chord DC is equal to the radius if and only if $D = M$. Thus equality $AM = GM$ holds if and only if $a = b$.

Result 5.1: If $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression, every term a_k $(k > 1)$ is the arithmetic mean of its immediate predecessor a_{k-1} and immediate successor a_{k+1}.

Proof. Let $a_1, a_2, a_3, \ldots, a_n$ be an arithmetic progression with initial term a and common difference d. Then

$$a_k = a + (k-1)d, \quad a_{k-1} = a + (k-2)d \quad \text{and} \quad a_{k+1} = a + kd.$$

Thus

$$\frac{a_{k-1} + a_{k+1}}{2} = \frac{a + (k-2)d + a + kd}{2} = \frac{2a + (2k-2)d}{2} = a + (k-1)d = a_k.$$

Therefore, a_k is the arithmetic mean of a_{k-1} and a_{k+1}. ■

Result 5.2: If $a_1, a_2, a_3, \ldots, a_n$ is a geometric progression, every term a_k $(k > 1)$ is the geometric mean of its immediate predecessor a_{k-1} and immediate successor a_{k+1}.

Proof. Let $a_1, a_2, a_3, \ldots, a_n$ be a geometric progression with initial term a and common ratio r. Then

$$a_k = ar^{k-1}, \quad a_{k-1} = ar^{k-2} \quad \text{and} \quad a_{k+1} = ar^k.$$

Thus

$$\sqrt{a_{k-1}\, a_{k+1}} = \sqrt{ar^{k-2}ar^k} = \sqrt{a^2r^{2k-2}} = ar^{k-1} = a_k.$$

Therefore, a_k is the geometric mean of a_{k-1} and a_{k+1}. ■

Example 5.14 Find seven numbers $A_1, A_2, \ldots, A_7$ so that the sequence $4, A_1, A_2, \ldots, A_7, 7$ is in arithmetic progression and also 4 numbers G_1, G_2, G_3, G_4 so that the sequence $12, G_1, G_2, G_3, G_4, \frac{3}{8}$ is in geometric progression.

Solution:
Since $a = 4$ and $4 + 8d = 7$ we get $d = \frac{3}{8}$. So the required 7 numbers are $4\frac{3}{8}$, $4\frac{6}{8}$, $5\frac{1}{8}$, $5\frac{4}{8}$, $5\frac{7}{8}$, $6\frac{2}{8}$, $6\frac{5}{8}$.

Since $a = 12$ and $ar^5 = \frac{3}{8}$ we get $r^5 = \frac{1}{32}$ and hence $r = \frac{1}{2}$. Thus the required 4 numbers are $6, 3, 1\frac{1}{2}, \frac{3}{4}$.

Example 5.15 If the product of the 4th, 5th and 6th terms of a geometric progression is 4096 and if the product of the 5th, 6th and 7th-terms of it is 32768, find the sum of first 8 terms of the geometric progression.

Solution:
Let $a, ar, ar^2, \ldots$ be the geometric series having the given properties. Since the 4th, 5th and 6th terms are ar^3, ar^4 and ar^5, their product is a^3r^{12}. Thus $a^3r^{12} = 4096$. Similarly $a^3r^{15} = 32768$.

Therefore $\dfrac{a^3r^{15}}{a^3r^{12}} = \dfrac{32768}{4096}$. Hence $r^3 = 8$. This implies that $r = 2$.
$a^3r^{12} = 4096$ we have $a^3 = 1$. Therefore $a = 1$.
The sum of the first 8 terms is $\dfrac{a(1-r^8)}{1-r} = \dfrac{1-2^8}{1-2} = 255$.

Harmonic Mean

The harmonic mean of a set of positive numbers is the reciprocal of the arithmetic mean of the reciprocals of the set of numbers. That is, if $h_1, h_2, \ldots, h_n$ are positive numbers, then their reciprocals are $\frac{1}{h_1}, \frac{1}{h_2}, \ldots, \frac{1}{h_n}$; the arithmetic mean of the reciprocals is

$$\frac{\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_n}}{n}$$

and the reciprocal of this arithmetic mean, that is the harmonic mean of the numbers $h_1, h_2, \ldots, h_n$ is

$$\frac{n}{\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_n}}.$$

Definition 5.5

The harmonic mean of a set $\{h_1, h_2, \ldots, h_n\}$ of positive numbers is defined as

$$\frac{n}{\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_n}}.$$

In particular, the harmonic mean of two positive numbers a and b is $\dfrac{2}{\frac{1}{a} + \frac{1}{b}}$ which is equal to $\frac{2ab}{a+b}$.

It can be proved that "*For any set of n positive numbers, the geometric mean is greater than or equal to the harmonic mean*". That is, $GM \geq HM$.

Let us prove this inequality $GM \geq HM$ for two non-negative numbers.

Theorem 5.3: If GM and HM denote the geometric mean and the harmonic mean of two non-negative numbers, then $GM \geq HM$. The equality holds if and only if the two numbers are equal.

Proof. Let a and b be any two positive numbers. Then

$$\begin{aligned}
GM &= \sqrt{ab} \quad \text{and} \quad HM = \frac{2ab}{a+b}. \\
GM - HM &= \sqrt{ab} - \frac{2ab}{a+b} \\
&= \frac{\sqrt{ab}(a+b) - 2ab}{a+b} \\
&= \frac{\sqrt{ab}((a+b) - 2\sqrt{ab})}{a+b} \\
&= \frac{\sqrt{ab}(\sqrt{a} - \sqrt{b})^2}{a+b} \\
&\geq 0
\end{aligned}$$

Thus $GM - HM \geq 0$ and hence $GM \geq HM$. ■

We have already proved that $AM \geq GM$ in theorem 5.2 and now we have $GM \geq HM$. Combining these two, we have an important inequality $AM \geq GM \geq HM$.

Result 5.3: For any two positive numbers, the three means AM, GM and HM are in geometric progression.

Proof. Let a and b be any two positive real numbers. Then

$$AM = \frac{a+b}{2}, \quad GM = \sqrt{ab}, \quad \text{and } HM = \frac{2ab}{a+b}.$$

Now

$$AM \times HM = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right) = ab = (\sqrt{ab})^2 = GM^2.$$

That is, $AM \times HM = GM^2$ and hence AM, GM and HM are in geometric progression. ■

We note the following interesting results.

- If b is the arithmetic mean of a and c, then a, b, c is an arithmetic progression.
- If b is the geometric mean of a and c, then a, b, c is a geometric progression.
- If b is the harmonic mean of a and c, then a, b, c is a harmonic progression.

If a vehicle travels at a speed of x kmph. covering certain distance and it returns the same distance with a speed of y kmph., then the average speed of the vehicle in the whole travel is the harmonic mean of the upward and downward speeds. Indeed, if d is the distance, then time taken for upward journey is $\frac{d}{x}$ and the time taken for downward journey is $\frac{d}{y}$.

Thus average speed is $\dfrac{2d}{\dfrac{d}{x}+\dfrac{d}{y}} = \dfrac{2xy}{x+y}$.

For example if a vehicle travels at a speed of 60 kmph. covering certain distance and it returns the same distance with a speed of 40 kmph., then the average speed of the vehicle in the whole travel is the harmonic mean of 60 and 40. That is $\frac{2\times60\times40}{60+40} = 48$ kmph speed.

Exercise - 5.2

1. Write the first 6 terms of the sequences whose n^{th} terms are given below and classify them as arithmetic progression, geometric progression, arithmetico-geometric progression, harmonic progression and none of them.

(i) $\dfrac{1}{2^{n+1}}$ (ii) $\dfrac{(n+1)(n+2)}{n+3(n+4)}$ (iii) $4\left(\frac{1}{2}\right)^n$ (iv) $\dfrac{(-1)^n}{n}$ (v) $\dfrac{2n+3}{3n+4}$ (vi) 2018 (vii) $\dfrac{3n-2}{3^{n-1}}$

2. Write the first 6 terms of the sequences whose n^{th} term a_n is given below.

(i) $a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$

(ii) $a_n = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ a_{n-1} + a_{n-2} & \text{if } n > 2 \end{cases}$

(iii) $a_n = \begin{cases} n & \text{if } n \text{ is } 1, 2 \text{ or } 3 \\ a_{n-1} + a_{n-2} + a_{n-3} & \text{if } n > 3 \end{cases}$

3. Write the nth term of the following sequences.

(i) $2,\ 2,\ 4,\ 4,\ 6,\ 6,\ \ldots$ (ii) $\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \frac{5}{6},\ \ldots$
(iii) $\frac{1}{2},\ \frac{3}{4},\ \frac{5}{6},\ \frac{7}{8},\ \frac{9}{10},\ \ldots$ (iv) $6,\ 10,\ 4,\ 12,\ 2,\ 14,\ 0, 16,\ -2, \ldots$

4. The product of three increasing numbers in GP is 5832. If we add 6 to the second number and 9 to the third number, then resulting numbers form an AP. Find the numbers in GP.
5. Write the nth term of the sequence $\dfrac{3}{1^2 2^2}, \dfrac{5}{2^2 3^2}, \dfrac{7}{3^2 4^2}, \ldots$ as a difference of two terms.
6. If t_k is the kth term of a GP, then show that t_{n-k}, t_n, t_{n+k} also form a GP for any positive integer k.
7. If a, b, c are in geometric progression, and if $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$, then prove that x, y, z are in arithmetic progression.
8. The AM of two numbers exceeds their GM by 10 and HM by 16. Find the numbers.
9. If the roots of the equation $(q-r)x^2 + (r-p)x + p - q = 0$ are equal, then show that p, q and r are in AP.
10. If a, b, c are respectively the pth, q^{th} and r^{th} terms of a GP, show that $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$.

5.5 Finite Series

Roughly speaking a series is the sum of terms of a sequence of numbers; a finite series is the sum of terms of a finite sequence of numbers. If (a_n) is a sequence of numbers, then the expression $a_1 + a_2 + \cdots + a_n$ is called a ***finite series***. The expression $a_1 + a_2 + \cdots + a_n$ is denoted as $\sum\limits_{k=1}^{n} a_k$. Sometimes, depending upon the problem under consideration and for simplicity a series may be given as $a_0 + a_1 + a_2 + \cdots$ with first term as a_0.

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5.5.1 Sum of Arithmetic, Geometric and Arithmetico-Geometric Progressions

In the earlier classes we studied about the sum of a few terms, like sum of first n terms, of arithmetic and geometric progressions. We now recall them.

Sum of Arithmetic and Geometric Progressions

- A series is said to be an ***arithmetic series*** if the terms of the series form an arithmetic sequence. A series is said to be a ***geometric series*** if the terms of the series form a geometric sequence.
- The sum S_n of the first n terms of the arithmetic sequence $(a + (n-1)d)$ is given by $S_n = na + \frac{(n-1)n}{2}d = \frac{n}{2}[2a + (n-1)d]$.
- The sum S_n of the first n terms of the geometric sequence (ar^{n-1}) is given by $S_n = \frac{a(1-r^n)}{1-r}$ provided $r \neq 1$. If $r = 1$, then the sequence is nothing but the constant sequence $a, a, a, \ldots$ and the sum of the first n terms is clearly na. Thus, if $r \neq 1$, then $1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$.

Sum of Arithmetico-Geometric Progressions

- A series is said to be an ***arithmetico-Geometric series*** if the terms of the series form an arithmetico-Geometric sequence.
- The sum S_n of the first n terms of the arithmetico-Geometric sequence $((a + (n-1)d)r^{n-1})$ is given by

$$S_n = \frac{a - (a + (n-1)d)r^n}{1-r} + dr\left(\frac{1 - r^{n-1}}{(1-r)^2}\right)$$

for $r \neq 1$.

Example 5.16 Find the sum up to n terms of the series: $1 + \frac{6}{7} + \frac{11}{49} + \frac{16}{343} + \cdots$.

Solution:
Here $a = 1$, $d = 5$ and $r = \frac{1}{7}$.

$$\begin{aligned} S_n &= \frac{a - (a + (n-1)d)r^n}{1 - r} + dr\left(\frac{1 - r^{n-1}}{(1-r)^2}\right) \\ &= \frac{1 - (1 + 5(n-1))(\frac{1}{7})^n}{1 - \frac{1}{7}} + 5 \times \frac{1}{7}\left(\frac{(1 - \left(\frac{1}{7}\right)^{n-1})}{(1 - \frac{1}{7})^2}\right) \\ &= \frac{1 - \frac{5n-4}{7^n}}{\frac{6}{7}} + \frac{\frac{5}{7}(7^{n-1} - 1)}{7^{n-1}(\frac{6}{7})^2} \\ &= \frac{7^n - 5n + 4}{7^{n-1}6} + \frac{5(7^{n-1} - 1)}{7^{n-2}36} \end{aligned}$$

5.5.2 Telescopic Summation for Finite Series

Telescopic summation is a more general method used for summing a series either for finite or infinite terms. This technique expresses sum of n terms of a given series just in two terms, usually first and last term, by making the intermediate terms cancel each other. After canceling intermediate terms, we bring the last term which is far away from the first term very close to the first term. So this process is called "Telescopic Summation".

Example 5.17 Find the sum of the first n terms of the series $\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots$

Solution:
Let t_k denote the kth term of the given series. Then $t_k = \frac{1}{\sqrt{k}+\sqrt{k+1}}$. If we are successful in writing the kth term as a difference of two expressions, then we can solve using this technique. We have

$$t_k = \frac{1}{\sqrt{k} + \sqrt{k+1}} = \frac{\sqrt{k} - \sqrt{k+1}}{(\sqrt{k} + \sqrt{k+1})(\sqrt{k} - \sqrt{k+1})} = \frac{\sqrt{k} - \sqrt{k+1}}{k - (k+1)} = \sqrt{k+1} - \sqrt{k}$$

Thus

$$t_1 + t_2 + \cdots + t_n = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{n+1} - \sqrt{n}) = \sqrt{n+1} - 1$$

Example 5.18 Find $\sum\limits_{k=1}^{n} \frac{1}{k(k+1)}$.

Solution:
Let t_k denote the kth term of the given series. Then $t_k = \frac{1}{k(k+1)}$. By using partial fraction we get

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Thus

$$t_1 + t_2 + \cdots + t_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Exercise - 5.3

1. Find the sum of the first 20-terms of the arithmetic progression having the sum of first 10 terms as 52 and the sum of the first 15 terms as 77.
2. Find the sum up to the 17th term of the series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots$
3. Compute the sum of first n terms of the following series:

   i. $8+88+888+8888+\cdots$ ii. $6+66+666+6666+\cdots$

4. Compute the sum of first n terms of $1+(1+4)+(1+4+4^2)+(1+4+4^2+4^3)+\cdots$
5. Find the general term and sum to n terms of the sequence $1, \frac{4}{3}, \frac{7}{9}, \frac{10}{27}, \ldots$.
6. Find the value of n, if the sum to n terms of the series $\sqrt{3}+\sqrt{75}+\sqrt{243}+\cdots$ is $435\sqrt{3}$.
7. Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ term of an AP. is equal to twice the mth term.
8. A man repays an amount of Rs.3250 by paying Rs.20 in the first month and then increases the payment by Rs.15 per month. How long will it take him to clear the amount?
9. In a race, 20 balls are placed in a line at intervals of 4 meters, with the first ball 24 meters away from the starting point. A contestant is required to bring the balls back to the starting place one at a time. How far would the contestant run to bring back all balls?
10. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and n^{th} hour?
11. What will Rs.500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?
12. In a certain town, a viral disease caused severe health hazards upon its people disturbing their normal life. It was found that on each day, the virus which caused the disease spread in Geometric Progression. The amount of infectious virus particle gets doubled each day, being 5 particles on the first day. Find the day when the infectious virus particles just grow over 1,50,000 units?

5.5.3 Some Special Finite Series

In this section we give some of the important formulas of summing up finitely many terms which follows either an AP, GP, or any specific series.

1. Summation of first n natural numbers:

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$

   [Treating this as an AP, one can find the sum.]

2. Summation of the squares of first n natural numbers:

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

   [Use the identity $a^3-b^3=(a-b)(a^2+ab+b^2)$ and try to prove this result.]

3. Summation of the cubes of first n natural numbers:

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

   [Use the identity $k^4-(k-1)^4=(4k^3-6k^2+4k-1)$ and try to prove this result.]

Note that the above three results were proved in the earlier classes.

5.6 Infinite Sequences and Series

A finite sum of real numbers is well defined by the properties of real numbers, but in order to make sense of an infinite series, we need to consider the concept of convergence. Consider the infinite sum: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ with each term positive. Can we assign a numerical value to the sum? While at first it may seem difficult or impossible. We can certainly do something similar where we have one quantity getting closer and closer to a fixed quantity.

Let us discuss an interesting problem. Let there be two plates A and B. Let a full cake be placed on the plate A and let B as empty. Let us cut the cake in A into exactly two equal parts and place one part on B leaving the other part in A. Let us cut the remaining part of the cake in A into exactly two equal parts and place one part on B leaving the other part in A. Let us again cut the remaining part of the cake in A into exactly two equal parts and place one part on B leaving the other part in A. If we go on doing this what will happen? What will be the amount of cake "*finally*" in A and in B? Let us list the stage by stage status:

Stage	Plate A	Plate B
0	1	0
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{4}$	$\frac{1}{2}+\frac{1}{4}$
3	$\frac{1}{8}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$
4	$\frac{1}{16}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$
5	$\frac{1}{32}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$
$\cdots$	$\cdots$	$\cdots$
n	$\frac{1}{2^n}$	$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots+\frac{1}{2^n}$
$\cdots$	$\cdots$	$\cdots$

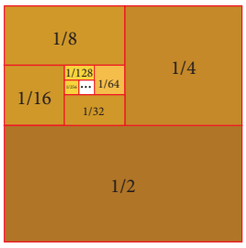

Figure 5.3

Intuitively we feel that "*finally*" nothing will remain in plate A and the full cake will be in plate B. In other words, the cake available in A is 0 and the cake available in B is 1. That is, intuitively we feel that

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \ldots$$

"*goes*" to 0 and

$$\frac{1}{2},\ \frac{1}{2}+\frac{1}{4},\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8},\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16},\ \ldots$$

"*goes*" to 1 or equivalently

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots \qquad \text{is 1.}$$

In this section let us learn the sense in which the words "*finally*" and "*goes*" are used and also let us learn the addition of infinitely many numbers.

We intuitively feel that $1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \ldots$ "*goes*" to 0. Similarly we feel that the sequence 1, $\frac{1}{10},\ \frac{1}{100},\ \frac{1}{1000},\ \frac{1}{10000},\ \frac{1}{100000},\ \ldots$ also "*goes*" to 0.

If (a_n) is a sequence and a is a number so that for any given small positive number, there is a stage after which the distance between a_n and a is smaller than that positive number, then we may say that a_n goes to a as n goes to infinity. In technical terms we may say that a_n *tends to* a *as* n *tends to infinity.* In other words, in the limiting case a_n becomes a or the limit of a_n is a as n tends to ∞. We also say that the sequence (a_n) converges to a. If (a_n) converges to a, then we write $\lim\limits_{n\to\infty} a_n = a$.

At the same time we cannot say that the sequence

$$1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ \ldots$$

goes to some number. In other words, this sequence do not converge to any limit. So a sequence may not have a limit. But we can prove that a sequence cannot converge to more than one limit; that is, **if a sequence converges to a limit, then it is unique**.

5.6.1 Fibonacci Sequence

The Fibonacci sequence is a sequence of numbers where a number other than first two terms, is found by adding up the two numbers before it. Starting with 1, the sequence goes 1, 1, 2, 3, 5, 8, 13, 21, 34, and so forth. Written as a rule, the expression is $x_n = x_{n-1} + x_{n-2}, n \geq 3$ with $x_1 = 1, x_2 = 1$

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Named after **Fibonacci**, also known as Leonardo of Pisa or Leonardo Pisano, Fibonacci numbers were first introduced in the book Liber abaci in 1202. The son of a Pisan merchant, Fibonacci traveled widely and traded extensively. Mathematics was incredibly important to those in the trading industry, and his passion for numbers was cultivated in his youth.

Knowledge of numbers is said to have first originated in the Hindu-Arabic arithmetic system, which Fibonacci studied while growing up in North Africa. Prior to the publication of Liber abaci, the Latin-speaking world had yet to be introduced to the decimal number system. He wrote many books about geometry, commercial arithmetic and irrational numbers. He also helped in the development of the concept of zero.

$$\begin{array}{lcccccccccccccccc} n & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \cdots \\ x_n & = & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 & 144 & 233 & 377 & \cdots \end{array}$$

For example, The 8th term is sum of 6th term and 7th term. Thus, $x_8 = 8 + 13 = 21$.

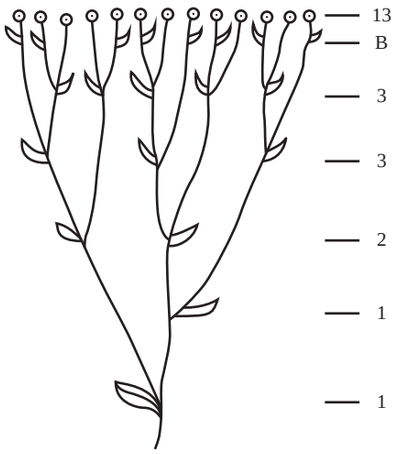


Figure 5.4

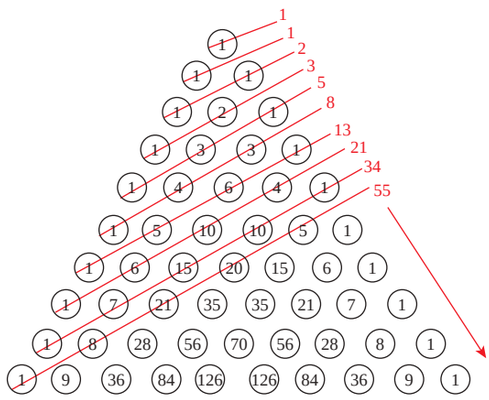


Figure 5.5

There is an interesting pattern in the Fibonacci sequence.

Observe that

(i) every third number is a multiple of 3rd term ($t_3 = 2$).
(ii) every fourth number is a multiple of 4th term ($t_4 = 3$).
(iii) every fifth number is a multiple of 5th term ($t_5 = 5$).
(iv) So, every n^{th} number is a multiple of nth term.

Infinite Series

If (a_n) is an infinite sequence of numbers, then the formal expression $a_1+a_2+\cdots$ is called an ***infinite series*** and is denoted as $\sum\limits_{k=1}^{\infty} a_k$.

In the beginning of the section we have seen the infinite series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

and intuitively felt that "*its sum*" is 1 (in the cake problem). Actually in the cake problem, the stage by stage availability of the cake in the plate B is given in the following sequence.

$$\frac{1}{2}, \frac{1}{2}+\frac{1}{4}, \frac{1}{2}+\frac{1}{4}+\frac{1}{8}, \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} \cdots$$

The nth term of this sequence is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}$$

which is equal to $\frac{2^n-1}{2^n}$. If s_n denote this sum, then $\lim\limits_{n\to\infty} s_n = 1$. This is one of the reasons for us to feel that the sum

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots \qquad \text{is } 1.$$

Motivated by this we may define the sum of infinitely many numbers. Let (a_n) be a sequence of real numbers and let $s_n = a_1+a_2+a_3+\cdots+a_n$. If the sequence (s_n) converges to a limit s, then it is meaningful to say that the sequence (a_n) is "*summable*" and the sum is s. In this case, we write

$$a_1+a_2+a_3+\cdots = s$$

It is customary to say that the series converges to s.

Definition 5.6

Let $\sum\limits_{n=1}^{\infty} a_n$ be a series of real numbers and let

$$s_n = a_1+a_2+a_3+\cdots+a_n, n\in\mathbb{N}$$

The sequence (s_n) is called the ***partial sum sequence*** of $\sum\limits_{n=1}^{\infty} a_n$. If (s_n) converges and if $\lim\limits_{n\to\infty} s_n = s$, then the series is said to be a ***convergent series*** and s is called the ***sum*** of the series.

We write $\sum\limits_{n=1}^{\infty} a_n = s$. Let us see some examples. The series $\sum\limits_{n=1}^{\infty} (-1)^{n+1}$ does not converge because the partial sum sequence $1, 0, 1, 0, 1, 0, \ldots$ does not converge.

We cannot apply algebraic rules meant for finite series to an infinite series blindly. Consider $\sum\limits_{n=1}^{\infty} (-1)^{n+1} = 1-1+1-1+\cdots$ If $S = 1-1+1-1+\cdots$ then one may argue that $S = 0$ or 1 or $\frac{1}{2}$ according to $S = (1-1)+(1-1)+\cdots$, $S = 1+(-1+1)+(-1+1)+\cdots$ or $1-S = 1-(+1-1+1-1+\cdots = S)$ respectively.

The series $\sum\limits_{n=0}^{\infty} x^n$ converges if $x = \frac{1}{2}$ whereas it does not converge if $x = 2$. This shows that series like $\sum\limits_{n=0}^{\infty} a_n x^n$ converges for some values of x and does not converge for some values of x. The problem of finding the values of x for which sequences of this form converges is beyond the scope of this book. However in the rest of the chapter we list some series with the appropriate values of x for which the series converges and the sum of the series whenever it converges.

5.6.2 Infinite Geometric Series

The series Σx^n is called a geometric series or geometric progression. Let us start with the series: $\sum\limits_{n=0}^{\infty} x^n, x \neq 1$. If $s_n = x_0 + x_1 + x_2 + \cdots + x_n$, then $s_n = \frac{1-x^n}{1-x}$. As x^n tends to 0 if $|x| < 1$, we say that s_n tends to $\frac{1}{1-x}$ if $|x| < 1$.

- $\sum\limits_{n=0}^{\infty} x^n$ converges for all real number x with $|x| < 1$ and the sum is $\frac{1}{1-x}$. That is, for all real numbers x satisfying $|x| < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

- $\sum\limits_{n=0}^{\infty} (-1)^n x^n$ converges for all real number x with $|x| < 1$ and the sum is $\frac{1}{1+x}$. That is, for all real numbers x satisfying $|x| < 1$,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

- $\sum\limits_{n=0}^{\infty} (2x)^n$ converges for all real number x with $|x| < \frac{1}{2}$ and the sum is $\frac{1}{1-2x}$. That is, for real numbers x satisfying $|x| < \frac{1}{2}$,

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

- $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all real number x and the sum is e^x. That is,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

  for all real numbers x.
- $\sum\limits_{n=0}^{\infty} (-1)^n x$ converges only for $x = 0$.

Let us discuss some special series. By assuming the convergence of those series let us solve some problems.

5.6.3 Infinite Arithmetico-Geometric Series

- The sum of the arithmetico-Geometric series $\Sigma((a + (n-1)d))r^{n-1}$ is given by

$$S = \lim_{n\to\infty} S_n = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

  for $-1 < r < 1$.

Example 5.19 Find the sum: $1 + \frac{4}{5} + \frac{7}{25} + \frac{10}{125} + \cdots$.

Solution:
Here $a = 1$, $d = 3$ and $r = \frac{1}{5}$.

$$\begin{aligned} S_\infty &= \frac{a}{1-r} + \frac{dr}{(1-r)^2} \\ &= \frac{1}{1-\frac{1}{5}} + \frac{3 \times \frac{1}{5}}{(1-\frac{1}{5})^2} \\ &= \frac{5}{4} + \left(\frac{3}{5}\right)\left(\frac{25}{16}\right) \\ &= \frac{35}{16} \end{aligned}$$

5.6.4 Telescopic Summation for Infinite Series

We discussed about summing terms of a finite sequence using telescopic summation technique in Section 5.5.2. The same applies for infinite series also.

Example 5.20 Find $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+5n+6}$.

Solution:
Let a_n denote the nth term of the given series. Then $a_n = \frac{1}{n^2+5n+6}$. By using partial fraction, we get

$$a_n = \frac{1}{n+2} - \frac{1}{n+3}.$$

Let s_n denote the sum of first n terms of the given series. Then

$$s_n = a_1 + a_2 + \cdots + a_n = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{1}{3} - \frac{1}{n+3}.$$

But as n tends to infinity, $\frac{1}{n+3}$ tends to zero and hence $\frac{1}{3} - \frac{1}{n+3}$ tends to $\frac{1}{3}$. In other words s_n tends to $\frac{1}{3}$. Thus $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+5n+6} = \frac{1}{3}$.

5.6.5 Binomial Series

In the discussion on geometric series we have seen that

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \quad \frac{1}{1+x} = 1 - x + x^2 - \cdots \text{ and } \frac{1}{1-2x} = 1 + 2x + 4x^2 + \cdots$$

for some suitable values of x. But the expressions $\frac{1}{1-x}$, $\frac{1}{1+x}$ and $\frac{1}{1-2x}$ can be written as $(1-x)^{-1}$, $(1+x)^{-1}$ and $(1-2x)^{-1}$. This suggests us a possibility of having negative exponents, that is negative powers, for $(1+x)$, $(1-x)$ and so on. Yes. This is possible. We can have any power, positive or negative, integer or rational. We can even have irrational exponent for $(1+x)$. We already proved

Binomial theorem (Theorem 5.1) for positive integral exponent (integral exponent means integer power). Now let us state the Binomial theorem for rational power.

Binomial Theorem for Rational Exponent

Theorem 5.4 (Binomial Theorem for Rational Exponent): For any rational number n,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

for all real numbers x satisfying $|x| < 1$.

As the proof involves higher mathematical concepts, let us assume the theorem without proof and see some particular cases and solve some problems. In the theorem

1. By taking $-x$ in the place of x, we get

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \qquad (|x| < 1)$$

2. By taking $-n$ in the place of n, we get

$$(1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{2!}x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}x^3 + \cdots$$

Hence

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \qquad (|x| < 1)$$

3. by taking $-x$ and $-n$ in the places of x and n, we get

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \qquad (|x| < 1)$$

Even though we have explicitly mentioned that n is a rational number in the theorem, some of us may hesitate to use n in place of a general rational number. So we give the theorem using the representation $\frac{p}{q}(q \neq 0)$ for a rational number.

$$\begin{aligned}
(1+x)^{\frac{p}{q}} &= 1 + \frac{p}{q}x + \frac{\frac{p}{q}(\frac{p}{q}-1)}{2!}x^2 + \frac{\frac{p}{q}(\frac{p}{q}-1)(\frac{p}{q}-2)}{3!}x^3 + \cdots \\
&= 1 + \frac{p}{q}x + \frac{p(p-q)}{q^2 2!}x^2 + \frac{p(p-q)(p-2q)}{q^3 3!}x^3 + \cdots \qquad (|x| < 1) \\
(1-x)^{\frac{p}{q}} &= 1 - \frac{p}{q}x + \frac{p(p-q)}{q^2 2!}x^2 - \frac{p(p-q)(p-2q)}{q^3 3!}x^3 + \cdots \qquad (|x| < 1)
\end{aligned}$$

Though the theorem gives a formula to compute $(1+x)^n$, to solve numerical problems quickly we must remember and able to write certain expansions directly. Observation of the coefficient in each of such expansions will be very helpful in solving problems. Let us list some of them: (Try yourself!).

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
3. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots$
4. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots$

All the above expansions are valid only when $|x| < 1$.

Example 5.21 Expand $(1+x)^{\frac{2}{3}}$ up to four terms for $|x|<1$.

Solution:
Here $n=\frac{2}{3}$.

$$\begin{aligned}
\frac{n(n-1)}{2!} &= \frac{\frac{2}{3}(\frac{2}{3}-1)}{2!} \\
&= \frac{\frac{2}{3}(\frac{-1}{3})}{2} \\
&= \frac{-1}{9} \\
\frac{n(n-1)(n-2)}{3!} &= \frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)}{3!} \\
&= \frac{\frac{2}{3}(\frac{-1}{3})(\frac{-4}{3})}{6} \\
&= \frac{4}{81}
\end{aligned}$$

Thus

$$(1+x)^{\frac{2}{3}} = 1+\frac{2}{3}x-\frac{1}{9}x^2+\frac{4}{81}x^3+\cdots$$

Example 5.22 Expand $\dfrac{1}{(1+3x)^2}$ in powers of x. Find a condition on x for which the expansion is valid.

Solution:
If we take $3x=y$, then

$$\frac{1}{(1+3x)^2}=\frac{1}{(1+y)^2}.$$

Now $\dfrac{1}{(1+y)^2}$ can be expanded using binomial theorem in powers of y. The expansion is valid only for values of y satisfying $|y|<1$. Replacing y by $3x$ we can get an expansion of $\dfrac{1}{(1+3x)^2}$. The expansion is valid only for values of x satisfying $|3x|<1$; that is the expansion is valid only for values of x satisfying $|x|<\frac{1}{3}$.

$$\begin{aligned}
\frac{1}{(1+3x)^2} &= (1+3x)^{-2} \\
&= 1-2(3x)+\frac{2(2+1)}{2!}(3x)^2-\frac{2(2+1)(2+2)}{3!}(3x)^3 \\
&\quad +\frac{2(2+1)(2+2)(2+3)}{4!}(3x)^4-\cdots
\end{aligned}$$

Hence, $\frac{1}{(1+3x)^2}=1-6x+27x^2-108x^3+405x^4-\cdots$, $|x|<\frac{1}{3}$.

Example 5.23 Expand $\dfrac{1}{(3+2x)^2}$ in powers of x. Find a condition on x for which the expansion is valid.

Solution:
(Clearly we have to use the expansion of $(1+x)^{-2}$. So, we have to write $(3+2x)$ as $3\left(1+\dfrac{2x}{3}\right)$ and proceed.)

$$\begin{aligned}
\frac{1}{(3+2x)^2} &= \frac{1}{3^2\left(1+\frac{2x}{3}\right)^2} \\
&= \frac{1}{9}\left(1+\frac{2x}{3}\right)^{-2} \\
&= \frac{1}{9}(1+y)^{-2} \qquad \left(\text{where } \frac{2x}{3}=y\right) \\
&= \frac{1}{9}(1-2y+3y^2-4y^3+5y^4-\cdots), if|y|<1 \\
&= \frac{1}{9}\left(1-2\left(\frac{2x}{3}\right)+3\left(\frac{2x}{3}\right)^2-4\left(\frac{2x}{3}\right)^3+5\left(\frac{2x}{3}\right)^4-\cdots\right), |\frac{2x}{3}|<1 \\
&= \frac{1}{9}\left(1-\frac{4}{3}x+\frac{4}{3}x^2-\frac{32}{27}x^3+\frac{80}{81}x^4-\cdots\right)
\end{aligned}$$

Thus, $$\frac{1}{(3+2x)^2} = \frac{1}{9}-\frac{4}{27}x+\frac{4}{27}x^2-\frac{32}{243}x^3+\frac{80}{729}x^4-\cdots, |x|<\frac{3}{2}$$

The expansion is valid if $|y|<1$. So, the expansion is valid if $|x|<\frac{3}{2}$.

We can find square root, cube root and other roots of any positive number by using binomial theorem. Let us see one such problem.

Example 5.24 Find $\sqrt[3]{65}$.

Solution:
We know that for $|x|<1$,

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

$$\begin{aligned}
\sqrt[3]{65} &= 65^{\frac{1}{3}} \\
&= (64+1)^{\frac{1}{3}} \\
&= 64^{\frac{1}{3}}\left(1+\frac{1}{64}\right)^{\frac{1}{3}} \\
&= 4\left(1+\frac{1}{64}\right)^{\frac{1}{3}}
\end{aligned}$$

$$
\begin{aligned}
&= 4\left(1+\frac{1}{3}\times\frac{1}{64}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\times\left(\frac{1}{64}\right)^2+\cdots\right)\\
&= 4+\frac{1}{48}-4\times\frac{1}{9}\times\frac{1}{64}\times\frac{1}{64}+\cdots\\
&= 4+\frac{1}{48}-\frac{1}{9216}+\cdots\\
&\approx 4+0.02 \qquad \left(\text{since } \frac{1}{9216}+\cdots \text{ is very small}\right)\\
\sqrt[3]{65} &= 4.02 \text{ (approximately).}
\end{aligned}
$$

Example 5.25 Prove that $\sqrt[3]{x^3+7}-\sqrt[3]{x^3+4}$ is approximately equal to $\frac{1}{x^2}$ when x is large.

Solution:

$$
\begin{aligned}
\sqrt[3]{x^3+7} &= (x^3+7)^{\frac{1}{3}}\\
&= \left[x^3\left(1+\frac{7}{x^3}\right)\right]^{\frac{1}{3}}, \qquad (|\frac{7}{x^3}|<1 \text{ as } x \text{ is large})\\
&= x\left(1+\frac{7}{x^3}\right)^{\frac{1}{3}}\\
&= x\left(1+\frac{1}{3}\times\frac{7}{x^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{7}{x^3}\right)^2+\cdots\right)\\
&= x\left(1+\frac{7}{3}\times\frac{1}{x^3}-\frac{49}{9}\times\frac{1}{x^6}+\cdots\right)\\
&= x+\frac{7}{3}\times\frac{1}{x^2}-\frac{49}{9}\times\frac{1}{x^5}+\cdots\\
\sqrt[3]{x^3+4} &= (x^3+4)^{\frac{1}{3}}\\
&= \left[x^3\left(1+\frac{4}{x^3}\right)\right]^{\frac{1}{3}}\\
&= x\left(1+\frac{4}{x^3}\right)^{\frac{1}{3}} \qquad (|\frac{4}{x^3}|<1)\\
&= x\left(1+\frac{1}{3}\times\frac{4}{x^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{4}{x^3}\right)^2+\cdots\right)\\
&= x+\frac{4}{3}\times\frac{1}{x^2}-\frac{16}{9}\times\frac{1}{x^5}+\cdots
\end{aligned}
$$

Since x is large, $\frac{1}{x}$ is very small and hence higher powers of $\frac{1}{x}$ are negligible. Thus $\sqrt[3]{x^3+7}=x+\frac{7}{3}\times\frac{1}{x^2}$ and $\sqrt[3]{x^3+4}=x+\frac{4}{3}\times\frac{1}{x^2}$. Therefore

$$
\sqrt[3]{x^3+7}-\sqrt[3]{x^3+4} = \left(x+\frac{7}{3}\times\frac{1}{x^2}\right)-\left(x+\frac{4}{3}\times\frac{1}{x^2}\right)=\frac{1}{x^2}
$$

The binomial theorem is true for all real numbers n. For example, when $n = \sqrt{2}$, we have

$$(1+x)^{\sqrt{2}} = 1 + \sqrt{2}x + \frac{\sqrt{2}(\sqrt{2}-1)}{2!}x^2 + \frac{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}-2)}{3!}x^3 + \cdots, \quad |x| < 1.$$

5.6.6 Exponential Series

The series $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ is called an ***exponential series***. It can be proved that this series converges for all values of x.

For any real number x, $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ where

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$$

We have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{5.1}$$

for all values of x. By taking $-x$ in place of x in (5.1) we get

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots. \tag{5.2}$$

In particular,

$$\frac{1}{e} = e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots.$$

From (5.1) and (5.2) we get

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \text{ and } \frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

In particular we have

$$\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots \text{ and } \frac{e - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

By taking $2x$ in place of x in (5.1) we get

$$e^{2x} = 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots.$$

On simplification we see that

$$e^{2x} = 1 + \frac{2x}{1!} + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots.$$

5.6.7 Logarithmic Series

The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ is called a ***logarithmic series***. This series converges for all values of x satisfying $|x|<1$. This series converges when $x=1$ also.

For all values of x satisfying $|x|<1$, the sum of the series is $\log(1+x)$. Thus

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \text{ for all values of } x \text{ satisfying } |x|<1.$$

By taking $-x$ in place of x we get

$$\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots \text{ for all values of } x \text{ satisfying } |x|<1.$$

Now $\log\left(\frac{1+x}{1-x}\right)=\log(1+x)-\log(1-x)$. Using this we get

$$\log\left(\frac{1+x}{1-x}\right)=2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right].$$

Suppose we want to write $\log(1+2x)$ in a series, then we can replace $2x$ by y and use the expansion

$$\log(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\cdots$$

for all values of y satisfying $|y|<1$. But if $|y|<1$, then $|2x|<1$ and hence $|x|<\frac{1}{2}$. So if $|x|<\frac{1}{2}$, then

$$\log(1+2x)=2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}-\frac{(2x)^4}{4}+\cdots.$$

Thus $\log(1+2x)=2x-\frac{4x^2}{2}+\frac{8x^3}{3}-\frac{16x^4}{4}+\cdots$ for all values of x satisfying $|x|<\frac{1}{2}$.

Exercise - 5.4

1. Expand the following in ascending powers of x and find the condition on x for which the binomial expansion is valid.
   (i) $\frac{1}{5+x}$ (ii) $\frac{2}{(3+4x)^2}$ (iii) $(5+x^2)^{\frac{2}{3}}$ (iv) $(x+2)^{-\frac{2}{3}}$
2. Find $\sqrt[3]{1001}$ approximately (two decimal places).
3. Prove that $\sqrt[3]{x^3+6}-\sqrt[3]{x^3+3}$ is approximately equal to $\frac{1}{x^2}$ when x is sufficiently large.
4. Prove that $\sqrt{\frac{1-x}{1+x}}$ is approximately equal to $1-x+\frac{x^2}{2}$ when x is very small.
5. Write the first 6 terms of the exponential series (i) e^{5x} (ii) e^{-2x} (iii) $e^{\frac{1}{2}x}$.
6. Write the first 4 terms of the logarithmic series (i) $\log(1+4x)$ (ii) $\log(1-2x)$ (iii) $\log\left(\frac{1+3x}{1-3x}\right)$ (iv) $\log\left(\frac{1-2x}{1+2x}\right)$. Find the intervals on which the expansions are valid.
7. If $y=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$, then show that $x=y-\frac{y^2}{2!}+\frac{y^3}{3!}-\frac{y^4}{4!}+\cdots$.
8. If $p-q$ is small compared to either p or q, then show that $\sqrt[n]{\frac{p}{q}}\simeq\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}$. Hence find $\sqrt[8]{\frac{15}{16}}$.
9. Find the coefficient of x^4 in the expansion of $\frac{3-4x+x^2}{e^{2x}}$.
10. Find the value of $\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{1}{9^{n-1}}+\frac{1}{9^{2n-1}}\right)$.

Exercise - 5.5

2DF48J

Choose the correct or the most suitable answer.

1. The value of $2+4+6+\cdots+2n$ is

   (1) $\frac{n(n-1)}{2}$ (2) $\frac{n(n+1)}{2}$ (3) $\frac{2n(2n+1)}{2}$ (4) $n(n+1)$

2. The coefficient of x^6 in $(2+2x)^{10}$ is

   (1) ${}^{10}C_6$ (2) 2^6 (3) ${}^{10}C_6\, 2^6$ (4) ${}^{10}C_6\, 2^{10}$.

3. The coefficient of x^8y^{12} in the expansion of $(2x+3y)^{20}$ is

   (1) 0 (2) $2^8 3^{12}$ (3) $2^8 3^{12} + 2^{12} 3^8$ (4) ${}^{20}C_8\, 2^8 3^{12}$.

4. If ${}^{n}C_{10} > {}^{n}C_r$ for all possible r, then a value of n is

   (1) 10 (2) 21 (3) 19 (4) 20.

5. If a is the arithmetic mean and g is the geometric mean of two numbers, then

   (1) $a \le g$ (2) $a \ge g$ (3) $a = g$ (4) $a > g$.

6. If $(1+x^2)^2(1+x)^n = a_0 + a_1x + a_2x^2 + \cdots + x^{n+4}$ and if a_0, a_1, a_2 are in AP, then n is

   (1) 1 (2) 5 (3) 2 (4) 4.

7. If $a, 8, b$ are in AP, $a, 4, b$ are in GP, and if a, x, b are in HP then x is

   (1) 2 (2) 1 (3) 4 (4) 16.

8. The sequence $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}+\sqrt{2}}, \frac{1}{\sqrt{3}+2\sqrt{2}}, \cdots$ form an

   (1) AP (2) GP (3) HP (4) AGP.

9. The HM of two positive numbers whose AM and GM are 16, 8 respectively is

   (1) 10 (2) 6 (3) 5 (4) 4.

10. If S_n denotes the sum of n terms of an AP whose common difference is d, the value of $S_n - 2S_{n-1} + S_{n-2}$ is

    (1) d (2) 2d (3) 4d (4) d^2.

11. The remainder when 38^{15} is divided by 13 is

    (1) 12 (2) 1 (3) 11 (4) 5.

12. The nth term of the sequence $1, 2, 4, 7, 11, \cdots$ is

    (1) n^3+3n^2+2n (2) n^3-3n^2+3n (3) $\frac{n(n+1)(n+2)}{3}$ (4) $\frac{n^2-n+2}{2}$.

13. The sum up to n terms of the series $\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots$ is

    (1) $\sqrt{2n+1}$ (2) $\frac{\sqrt{2n+1}}{2}$ (3) $\sqrt{2n+1}-1$ (4) $\frac{\sqrt{2n+1}-1}{2}$.

14. The nth term of the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \cdots$ is

    (1) $2^n - n - 1$ (2) $1 - 2^{-n}$ (3) $2^{-n} + n - 1$ (4) 2^{n-1}.

15. The sum up to n terms of the series $\sqrt{2} + \sqrt{8} + \sqrt{18} + \sqrt{32} + \cdots$ is

    (1) $\frac{n(n+1)}{2}$ (2) $2n(n+1)$ (3) $\frac{n(n+1)}{\sqrt{2}}$ (4) 1.

16. The value of the series $\frac{1}{2}+\frac{7}{4}+\frac{13}{8}+\frac{19}{16}+\cdots$ is

(1) 14 (2) 7 (3) 4 (4) 6.

17. The sum of an infinite GP is 18. If the first term is 6, the common ratio is

(1) $\frac{1}{3}$ (2) $\frac{2}{3}$ (3) $\frac{1}{6}$ (4) $\frac{3}{4}$.

18. The coefficient of x^5 in the series e^{-2x} is

(1) $\frac{2}{3}$ (2) $\frac{3}{2}$ (3) $\frac{-4}{15}$ (4) $\frac{4}{15}$.

19. The value of $\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots$ is

(1) $\frac{e^2+1}{2e}$ (2) $\frac{(e+1)^2}{2e}$ (3) $\frac{(e-1)^2}{2e}$ (4) $\frac{e^2-1}{2e}$.

20. The value of $1-\frac{1}{2}\left(\frac{2}{3}\right)+\frac{1}{3}\left(\frac{2}{3}\right)^2-\frac{1}{4}\left(\frac{2}{3}\right)^3+\cdots$ is

(1) $\log\left(\frac{5}{3}\right)$ (2) $\frac{3}{2}\log\left(\frac{5}{3}\right)$ (3) $\frac{5}{3}\log\left(\frac{5}{3}\right)$ (4) $\frac{2}{3}\log\left(\frac{2}{3}\right)$.

Summary

In this chapter we have acquired the knowledge of

- Binomial theorem for any $n \in \mathbb{N}$,

$$(a+b)^n = {}^n\mathrm{C}_0\, a^n b^0 + {}^n\mathrm{C}_1\, a^{n-1}b^1 + \cdots + {}^n\mathrm{C}_n\, a^0 b^n.$$

- ${}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + \cdots + {}^n\mathrm{C}_n = 2^n$.
- ${}^n\mathrm{C}_1 + {}^n\mathrm{C}_3 + + {}^n\mathrm{C}_5 \cdots + = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_2 + + {}^n\mathrm{C}_4 \cdots + = 2^{n-1}$
- $AM \geq GM \geq HM$
- The nth term of an AP is given by $T_n = a+(n-1)d$.
- The nth term of an GP is given by $T_n = ar^{n-1}$.
- The nth term of an AGP is given by $T_n = (a+(n-1)d)r^{n-1}$.
- For any positive numbers a and b, we have

$$\mathrm{AM} = \frac{a+b}{2},\ \ \mathrm{GM} = \sqrt{ab},\ \ \mathrm{HM} = \frac{2ab}{a+b}.$$

- The sum of first n terms of an AP is given by $S_n = \dfrac{n}{2}(2a+(n-1)d)$.
- The sum of first n terms of an GP is given by $S_n = \dfrac{a(1-r^n)}{1-r}$ for $r \neq 1$.
- The sum of first n terms of an AGP is given by $S_n = \dfrac{a-(a+(n-1)d)r^n}{1-r} + dr\left(\frac{1-r^{n-1}}{(1-r)^2}\right)$ for $r \neq 1$.
- $\sum\limits_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$.
- $\sum\limits_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

- $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.
- Fibonacci sequence $1, 1, 2, 3, 5, \cdots$
- Binomial theorem for rational exponent

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

  for all real numbers x satisfying $|x| < 1$.
- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots$
- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.
- $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$.
- $\frac{e^x+e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$ and $\frac{e^x-e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$.
- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ for all values of x satisfying $|x| < 1$.
- $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$ for all values of x satisfying $|x| < 1$.
- $\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right]$ for all values of x satisfying $|x| < 1$.

ICT CORNER-5(a)

Expected Outcome ⇒

Step–1

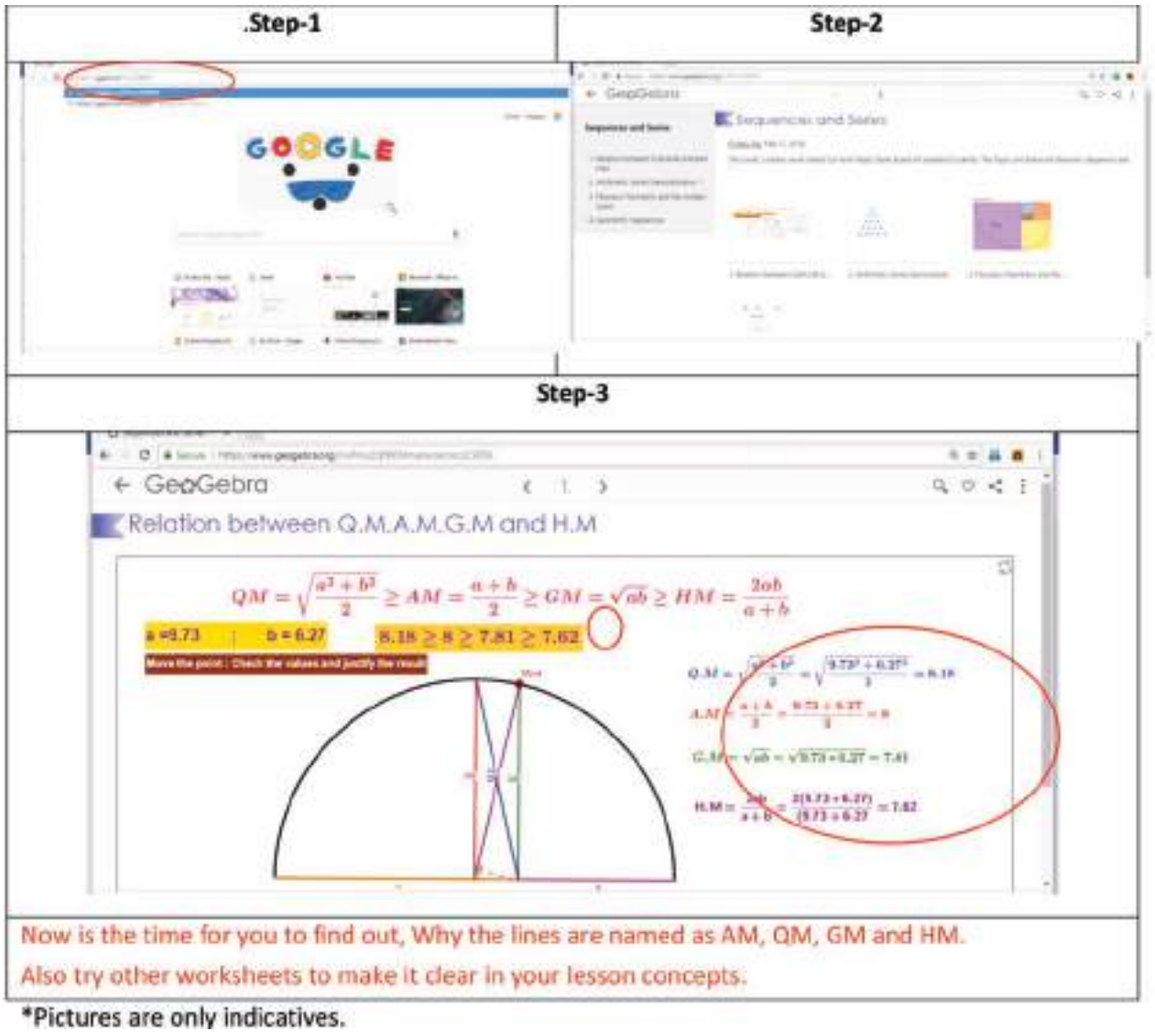
Open the Browser and type the URL Link given below (or) Scan the QR Code..

Step–2

Sequences and Series workbook will open. Several worksheets are given in this workbook. Select a worksheet “Relation between Q.M,A.M,G.M and H.M”

Step–3

You have All the Mean formula working with varying a and b. Move the point named “Move” and see that Mean values changes corresponding to a and b.

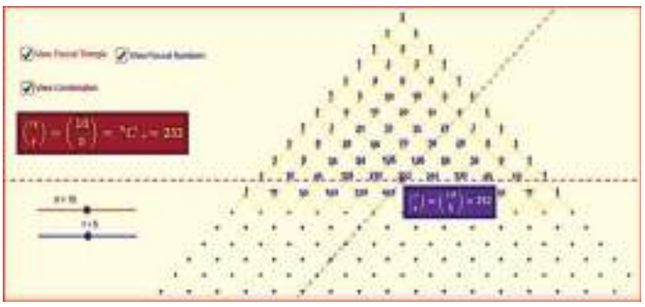


*Pictures are only indicatives.

Browse in the link Sequence and Series:
https://ggbm.at/Pmz2QfWDor Scan the QR Code.

ICT CORNER-5(b)

Expected Outcome ⇒



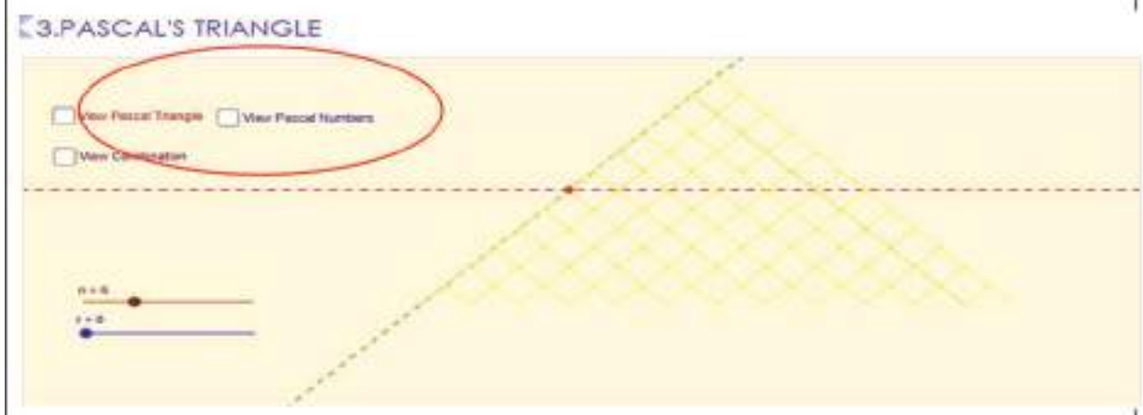
Step–1

Open the Browser and type the URL Link given below (or) Scan the QR Code. GeoGebra worksheet “PASCALS TRIANGLE” will appear. You can see pascal triangle arrangements. There are check boxes “View Pascal Triangle”, “View Pascal’s Numbers” and “View Combination”

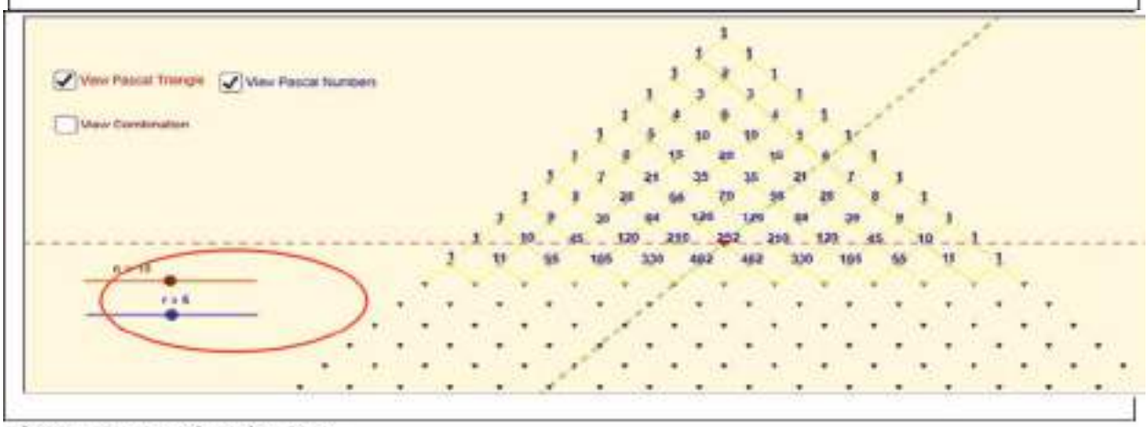
Step–2

Click on “View Pascal’s Numbers” and move the sliders “n” and “r”. The red colour point moves over the numbers. Leave on any number and click “View Calculation” to see combinatorial working. Now compare the pascal numbers with the calculation.

Step-1



Step-2



*Pictures are only indicatives.

Browse in the link Sequence and Series:
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Chapter 6 Two Dimensional Analytical Geometry

"My powers are ordinary. Only my application brings me success ".

Sir Isaac Newton

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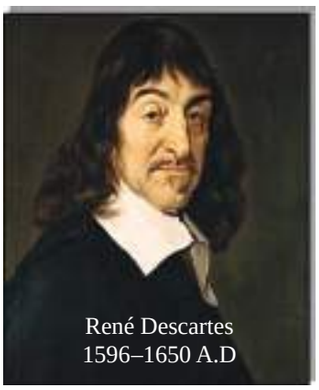
6.1 Introduction


René Descartes
1596–1650 A.D

Francois viete(1540-1603) introduced the first systematic *algebraic* notation and contributed to the theory of equations.Two French mathematicians-philosophers **René Descartes** and **Pierre de Fermat** independently founded analytical geometry in the-1630s by adapting Francois viete's algebra to the study of geometric loci. Descartes established analytical geometry as"a way of visualizing algebraic formulas" and developed the coordinate system as "a device to locate points on a plane". His main achievement was to bridge the gap between algebra and geometry. With regard to algebra, he explained in detail that how algebraic equations can be expressed and explained through the use of geometrical shapes. Analytical geometry is a great invention of Descartes and Fermat. Cartesian geometry, the alternative term used for analytical geometry is named after him.

From the17th century onwards, mathematics is being developed in two directions: pure and applied mathematics. One of the first areas of applied mathematic studied in the 17th century was the motion of an object in a straight line. The straight line graphs can be used in the fields of study as diverse as business, economics, social sciences, physics, and medicine. The problem of the shortest line plays a chief and historically important role in the foundations of geometry.

Given a real-world problem, our first task is to formulate the problem using the language of mathematics. Many techniques are used in the construction of mathematical models. Let us see how linear equations (models) can be constructed from a given set of information and solved using appropriate mathematical techniques. Consider some of the real-world, simple problems as illustrated below:

Real life situation 6.1: When a student walks from his house, at an average speed of 6 kmph, reaches his school by ten minutes before school starts. When his average speed is 4 kmph, he reaches his school five minutes late. If he starts to walk to school every day at 8.00 A.M, then how to find (i) the distance between house and the school (ii) the minimum average speed to reach the school on time and time taken to reach the school (iii) the time at which the school starts (iv) the pair of straight lines of his path of walk (Combined equation of two straight lines).

Real life situation 6.2: Suppose the Government has decided to erect a new Electrical Power Transmission Substation to provide better power supply to two villages namely A and B. The substation has

to be on the line l. The distances of villages A and B from the foot of the perpendiculars P and Q on the line l are 3 km and 5 km respectively and the distance between P and Q is 6 km. How to calculate **the smallest length** of cable required to connect the two villages (or the roads that connect the villages as well as the power station) from the power station and to find the equations of the cable lines (or roads) that connect the power station to two villages.

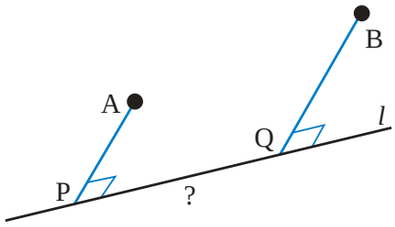


Figure 6.1

Real life situation 6.3:

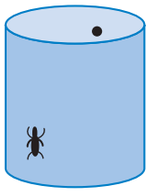


Figure 6.2

Consider a hollow cylindrical vessel, with circumference 24 cm and height 10 cm. An ant is located on the outside of vessel 4 cm from the bottom. There is a drop of honey at the diagrammatically opposite inside of the vessel, 3 cm from the top. What is the shortest distance the ant would need to crawl to get the honey? What is the equation of the path traced out by the ant. Here is a picture that illustrates the position of the ant and the honey.

Real life situation 6.4: The quantity demanded of a certain type of Compact Disk is 22,000 units when a unit price is ₹ 8. The customer will not buy the disk, at a unit price of ₹ 30 or higher. On the other side the manufacturer will not market any disk if the price is ₹ 6 or lower. However, if the price is ₹ 14 the manufacturer can supply 24,000 units. Assume that the quantity demanded and quantity supplied are linearly proportional to the price. How to find (i) the demand equation (ii) supply equation (iii) the market equilibrium quantity and price. (iv) The quantity of demand and supply when the price is ₹ 10.

The equation of the straight line for each of the problems stated above, not only solves the specific case of solutions but also helps us get many information through it. Later, in this chapter, let us try to solve these types of problems by using the concepts of straight lines. In order to understand the straight line, we need to get acquainted with some of its basic concepts. Let us discuss those in detail

Learning Objective

On completion of this chapter, the students expected to know

- the equation of a line in different forms
- whether two given lines are parallel or perpendicular;
- the distance of a given point from a given line and between two parallel lines,
- the family of straight lines for a given condition
- the equation of pair of straight lines, angle between them and angle bisectors

6.2 Locus of a point

Definition 6.1

A **point** is an exact position or location on a plane surface.

2DXVBU

It is important to understand that a point is not a thing, but a place. We indicate the position of a point by placing a dot. In plane analytical geometry, points are defined as ordered pairs of real numbers, say, (x, y) with reference to the coordinate system.

Generally, a horizontal line is called the x-axis; and the line vertical to the x-axis is called the y-axis. Intersection of these two axes is called the origin. Any point P in the plane can be located by a unique ordered pair of numbers(x, y) where x gives the distance between the point P and the y-axis and y denote the distance between the point P and the x-axis. Note that if x is negative it lies left of $y - axis$, similarly if y is negative it lies below the x-axis. In applications, often letters other than x and y are used, and different scales are chosen in the horizontal and vertical directions.

Definition 6.2

The path traced out by a moving point under certain conditions is called the **locus** of that point. Alternatively, when a point moves in accordance with a geometrical law, its path is called locus. The plural of locus is loci.

The following illustrations shows some cases of loci and its different uses.

Illustration 6.1: In cricket, when a ball is bowled by a bowler, the path traced out by the ball is the locus of the ball. Whenever there is dispute between batsmen and the fielders for leg before wicket (LBW) decisions, the locus of the ball solves the crises, raised by the players for review, through the third umpire. The likely path of the ball can be projected forward, through the batsman's legs, to see whether it would have hit the stumps or not. Consultation of the third umpire, for conventional slow motion or Hawk-Eye, the probable decision will be taken. This method is currently sanctioned in international cricket.
https://www.hawkeyeinnovations.com/sports

Figure 6.3

Illustration 6.2: Suppose P be a point on the rim (circumference) of a circular wheel. When the circle is rolling without slipping along a straight line, the locus of the point P on the rim is shown in figure. The path traced out by the point P is known as cycloid. (Try yourself by taking a point inside the circle. Find the names of the curve from the web site: www.mathworld.wolfram.com

https://www.geogebra.org/b/bd2ADu2I

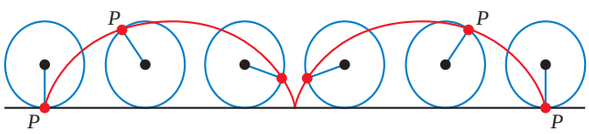
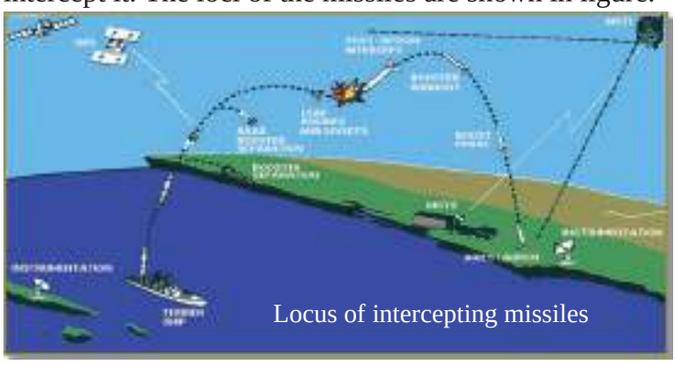


Figure 6.4

Illustration 6.3: A missile is launched from the army ship to attack and another from the land to intercept it. The loci of the missiles are shown in figure.

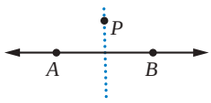
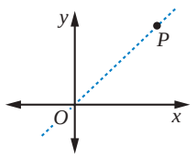
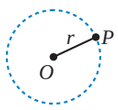


Locus of missiles play a vital role in many wars. During the Gulf War (2 Aug1990-28 Feb 1991), Iraq attacked Israeli cities with Scud missiles. To defend from Scud attack, Israel used Patriot missiles to shoot down enemy missiles. To launch a satellite or space shuttle successfully, the determination of path plays an crucial role in space research.

Figure 6.5

An equation in the two variables x and y will ordinarily be satisfied by infinitely many pair of real value of x and y. Every such pair is called a real solution of the equation. Each real solution of the equation will have its graph. The collection of all these graphs is called the locus of the given equation.

The following table shows some important loci in mathematics

A moving point ***P*** under the given condition	Graph	Name of the path
A point P moves such that it is equidistant from two fixed points A and B	 **Figure 6.6**	Perpendicular bisector of the line segment ***AB***
A point P moves such that it is equidistant from two fixed lines ox and oy	 **Figure 6.7**	Angle bisector of the angle$\angle xoy$
A point P moves equidistant from a fixed point O	 **Figure 6.8**	Circle

Now let us discuss the ways of finding the locus of the points. The equation of the locus is the relation that exists between the coordinates of all the points strictly lying on the path.

Procedure for finding the equation of the locus of a point

(i) If we are finding the equation of the locus of a point P, assign coordinates, say (h, k) to P

(ii) Express the given conditions as equations in terms of the known quantities and unknown parameters.

(iii) Eliminate the parameters, so that the resulting equation contains only h, k and known quantities.

(iv) Replace h by x, and k by y, in the resulting equation. The resulting equation is the equation of the locus of point P.

Example 6.1 Find the locus of a point which moves such that its distance from the x-axis is equal to the distance from the y-axis.

Solution:

Let $P(h, k)$ be a point on the locus.
Let A and B be the foot of the perpendiculars drawn from the point P on the x-axis and the y-axis respectively.
Therefore P is $(OA, OB) = (BP, AP) = (h, k)$
Given that $AP = BP$

$$\Rightarrow k = h$$

replacing h and k by substituting $h = x$ and $k = y$
The locus of P is, $y = x$, is a line passing through the origin

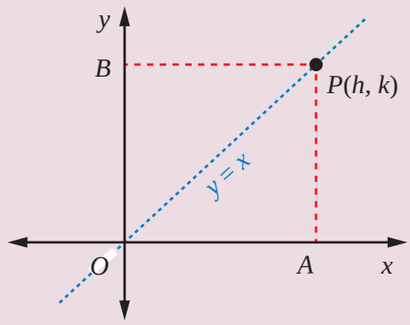


Figure 6.9

Example 6.2 Find the path traced out by the point $\left(ct, \frac{c}{t}\right)$, here $t \neq 0$ is the parameter and c is a constant

Solution:

Let $P(h, k)$be a point on the locus. From the given information, we have $h = ct$ and $k = \frac{c}{t}$.
To eliminate t, taking product of these two equations

$$(h)(k) = (ct)\left(\frac{c}{t}\right) \quad \Rightarrow \quad hk = c^2$$

Therefore, the required locus is $xy = c^2$

Example 6.3 Find the locus of a point P moves such that its distances from two fixed points $A(1, 0)$ and $B(5, 0)$, are always equal.

Solution:

Given that $A(1, 0)$ and $B(5, 0)$
Let $P(h, k)$ be any point on the required path.
From the information we have $AP = BP$
That is

$$\sqrt{(h-1)^2 + (k-0)^2} = \sqrt{(h-5)^2 + (k-0)^2} \Rightarrow h = 3$$

Therefore the locus of P is $x = 3$, which is a straight line parallel to the y-axis.

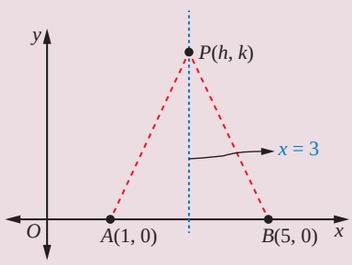


Figure 6.10

Example 6.4 If θ is a parameter, find the equation of the locus of a moving point, whose coordinates are $(a\sec\theta,\ b\tan\theta)$.

Solution:

Let $P\ (h, k)$ be any point on the required path. From the given information we have

$$\begin{aligned} h &= a\sec\theta \text{ and } k = b\tan\theta \\ &\Rightarrow \frac{h}{a} = \sec\theta \text{ and } \frac{k}{b} = \tan\theta \end{aligned}$$

To eliminate the parameter θ, squaring and subtracting, we get

$$\begin{aligned} \left(\frac{h}{a}\right)^2 - \left(\frac{k}{b}\right)^2 &= \sec^2\theta - \tan^2\theta \\ \left(\frac{h}{a}\right)^2 - \left(\frac{k}{b}\right)^2 &= 1 \end{aligned}$$

Therefore the locus of the given point is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

https://www.geogebra.org/geometry

Whenever the parameters are in trigonometric form, try to use trigonometric identities to eliminate θ

$$\boxed{\sin^2\theta + \cos^2\theta = 1,\ \sec^2\theta - \tan^2\theta = 1,\ \operatorname{cosec}^2\theta - \cot^2\theta = 1.}$$

Example 6.5 A straight rod of the length 6 units, slides with its ends A and B always on the x and y axes respectively. If O is the origin, then find the locus of the centroid of ΔOAB .

Solution:

Let the coordinates of the points O, A and B are$(0, 0)$, $(a, 0)$ and $(0, b)$ respectively.

Observed that the points A and B are moving points.

Let(h, k) be a centroid of ΔOAB.
Centroid of ΔOAB is

$$\left(\frac{0+a+0}{3}, \frac{0+0+b}{3}\right) = (h, k).$$

$$\frac{a}{3} = h \Rightarrow a = 3h, \qquad \frac{b}{3} = k \Rightarrow b = 3k$$

From right ΔOAB, $\quad OA^2 + OB^2 = AB^2$

$$(3h)^2 + (3k)^2 = (6)^2 \quad \Rightarrow \quad h^2 + k^2 = 4$$

Locus of (h, k) is a circle, $x^2 + y^2 = 4$.

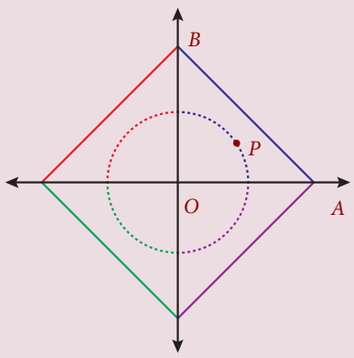


Figure 6.11

Example 6.6 If θ is a parameter, find the equation of the locus of a moving point, whose coordinates are $(a(\theta - \sin\theta)\,,\ \ a(1-\cos\theta))$.

Solution:

Let $P\ (h,\ k)$ be any point on the required path. From the given information we have

$$h = a(\theta - \sin\theta) \qquad (6.1)$$
$$k = a(1 - \cos\theta) \qquad (6.2)$$

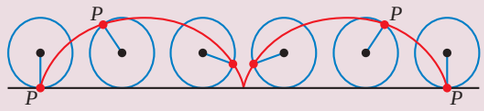


Figure 6.12

Let us find the value of θ and $\sin\theta$ from equation (6.2)

$$k = a(1-\cos\theta)$$

$$\cos\theta = \frac{a-k}{a} \ \Rightarrow\ \theta = \cos^{-1}\left(\frac{a-k}{a}\right) \text{ and } \sin\theta = \frac{\sqrt{2ak-k^2}}{a}$$

Substituting above values in (6.1) we get $h = a\cos^{-1}\left(\frac{a-k}{a}\right) - \sqrt{2ak-k^2}$

The locus of $(h,\ k)$ is

$$x = a\cos^{-1}\left(\frac{a-y}{a}\right) - \sqrt{2ay-y^2} \qquad (6.3)$$

https://www.geogebra.org/b/bd2ADu2I#material/zCKMj8kE

Though, the parametric form given above is converted to Cartesian form, in some cases the parametric form may be more useful to work with than the cartesian form.

Exercise - 6.1

1. Find the locus of P, if for all values of α, the co-ordinates of a moving point P is

   $(i)\ (9\cos\alpha\,,\ 9\sin\alpha)\ \ (ii)\ (9\cos\alpha\,,\ 6\sin\alpha)$.
2. Find the locus of a point P that moves at a constant distant of (i) two units from the x-axis (ii) three units from the y-axis.
3. If θ is a parameter, find the equation of the locus of a moving point, whose coordinates are $x = a\cos^3\theta$, $y = a\sin^3\theta$.
4. Find the value of k and b, if the points $P(-3,1)$ and $Q(2,b)$ lie on the locus of $x^2 - 5x + ky = 0$.
5. A straight rod of length 8 units slides with its ends A and B always on the x and y axes respectively. Find the locus of the mid point of the line segmentAB
6. Find the equation of the locus of a point such that the sum of the squares of the distance from the points $(3,5)$, $(1,-1)$ is equal to 20
7. Find the equation of the locus of the point P such that the line segment AB, joining the points $A(1,-6)$ and $B(4,-2)$, subtends a right angle at P.
8. If O is origin and R is a variable point on $y^2 = 4x$, then find the equation of the locus of the mid-point of the line segment OR.
9. The coordinates of a moving point P are $\left(\frac{a}{2}(\text{cosec}\theta + \sin\theta)\ ,\ \frac{b}{2}(\text{cosec}\theta - \sin\theta)\right)$, where θ is a variable parameter. Show that the equation of the locus P is $b^2x^2 - a^2y^2 = a^2b^2$.
10. If $P(2,-7)$ is a given point and Q is a point on $2x^2 + 9y^2 = 18$, then find the equations of the locus of the mid-point of PQ.

11. If R is any point on the x-axis and Q is any point on the y-axis and P is a variable point on RQ with $RP = b$, P$Q = a$. then find the equation of locus of P.
12. If the points $P(6, 2)$ and $Q(-2, 1)$ and R are the vertices of a ΔPQR and R is the point on the locus $y = x^2 - 3x + 4$, then find the equation of the locus of centroid of ΔPQR
13. If Q is a point on the locus of $x^2 + y^2 + 4x - 3y + 7 = 0$, then find the equation of locus of P which divides segment OQ externally in the ratio 3:4,where O is origin.
14. Find the points on the locus of points that are 3 units from x-axis and 5 units from the point $(5, 1)$.
15. The sum of the distance of a moving point from the points $(4, 0)$ and $(-4, 0)$ is always 10 units. Find the equation of the locus of the moving point.

6.3 Straight Lines

Linear equations can be rewritten using the laws of elementary algebra into several different forms. These equations are often referred to as the "equations of the straight line."

In the general form the linear equation is written as:

$$ax + by + c = 0 \tag{6.4}$$

where a and b are not both equal to zero. The name "linear" comes from the fact that the set of solutions of such an equation forms a straight line in the plane. In this chapter "line", we mean a straight line unless otherwise stated.

There are many ways to write the equation of a line which can all be converted from one to another by algebraic manipulation. These forms are generally named by the type of information (data) about the line that is needed to write down the form. Some of the important data are **points, slope**, and **intercepts**

6.3.1 The relationship between the angle of inclination and slope

Definition 6.3

The angle of inclination of a straight line is the angle, say θ, made by the line with the x-axis measured in the counter clockwise (positive) direction.

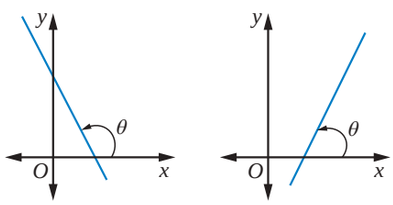


Figure 6.13

Definition 6.4

The slope or gradient of a straight line is a number that measures its "direction and steepness".

The slope of a line in the plane containing the x and y axes, is generally represented by the letter m. It can be measured in many ways as given below:

(i) When θ is the angle of inclination of the line with the x-axis measured in the counter clockwise direction then the slope

$m = \tan\theta.$

When θ is $\frac{\pi}{2}$, $\Rightarrow m = \tan\frac{\pi}{2}$ is undefined.

(ii) When (x_1, y_1) and (x_2, y_2) are any two points on the line with $x_2 \neq x_1$, then the slope is the change in the y coordinate divided by the corresponding change in the x coordinate.
This is described by the following equation.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{vertical\ change}{horizontal\ change}$$

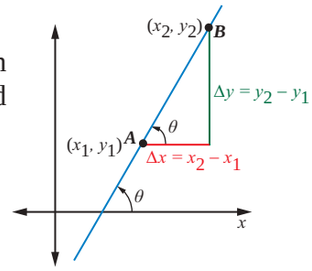


Figure 6.14

(iii) When the general form of the linear equation $ax + by + c = 0$ is given, then the slope of the line is

$$m = -\frac{a}{b}, \quad b \neq 0.$$

m is undefined when $b = 0$

The slope of a line can be a positive or negative or zero or undefined as shown below:

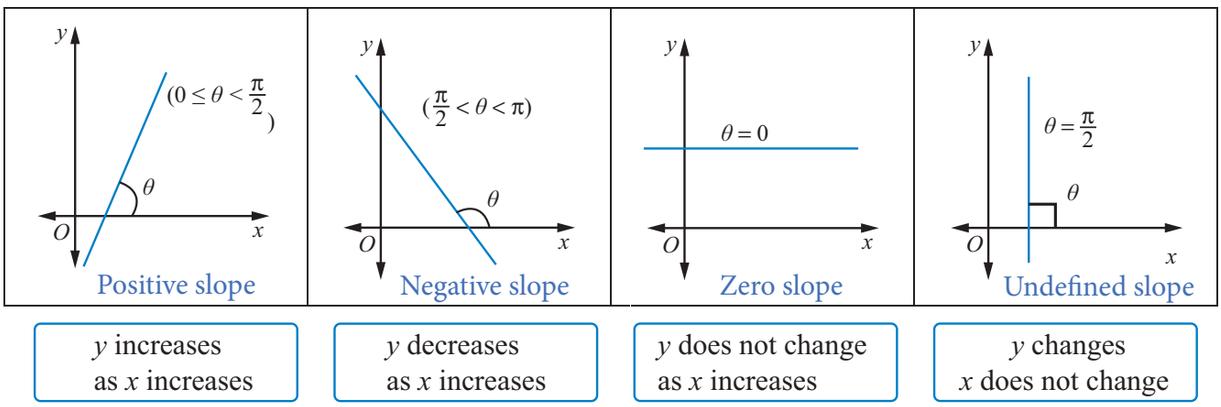


Figure 6.15

Definition 6.5

In a plane three or more points are said to be collinear if they lie on a same straight line.

Let A, $B$and C be any three points on a plane. If the slope of *AB* is equal to the slope of *BC* (or *AC*), then they are collinear.

6.3.2 Intercepts of a Line

Definition 6.6

The **intercept** of a line is the point at which the line crosses either the x-axis or the y-axis.

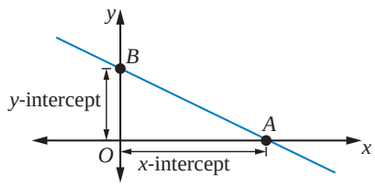


Figure 6.16

The x-intercept is a point where the y value is zero, and the y-intercept is a point where the x-value is zero.

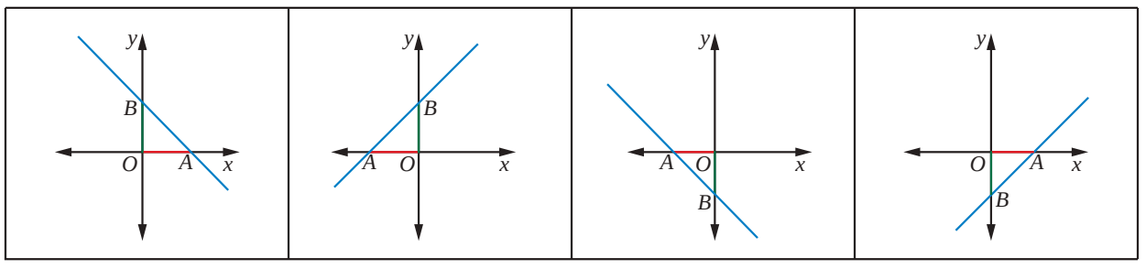
Therefore the intercepts of a line are the points where the line intersects, or crosses, the horizontal and vertical axes.

Therefore it is clear that

(i) the equation of the y-axis is $x = 0$.
(ii) the equation of the x-axis is $y = 0$.

In the figure *OA* is the x-intercept and *OB* is the y-intercept.

Different types of *x* and *y* Intercepts:



positive *x*-intercept positive *y*-intercept	negative *x*-intercept positive *y*-intercept	negative *x*-intercept negative *y*- intercept	positive *x*-intercept negative *y*- intercept

Figure 6.17

We have learnt the definition and detailed information about the points, slope and intercepts. Using these information, let us recall the different forms of an equation of a straight line.

6.3.3 Different Forms of an equation of a straight line

Two conditions are sufficient to determine uniquely the equation of a straight line. Using the combination of any two information from slope, intercepts and points, we can now form different types of straight lines such as

(i) Slope and intercept form
(ii) Point and Slope form
(iii) The two Point form
(iv) Intercepts form and two more special types are
(v) Normal form
(vi) Parametric form

Now let us look at an important way of describing the relationship between two quantities using the notion of a function.

(i) Slope and Intercept form

Proportional linear functions can be written in the form $y = mx$, where m is the slope of the line. Non proportional linear functions can be written in the form

$$y = m\,x + b, \qquad b \neq 0 \tag{6.5}$$

This is called the slope-intercept form of a straight line because m is the slope and b is the y-intercept.

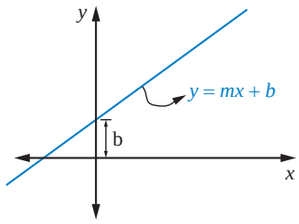


Figure 6.18

(1) when $b = 0$ and $m \neq 0$, the line passes through the origin and its equation is $y = mx$.
(2) when $b = 0$ and $m = 0$, the line coincides with the x-axis and its equation is $y = 0$.
(3) when $b \neq 0$ and $m = 0$, the line is parallel to the x-axis and its equation is $y = b$.

(ii) Point - Slope form:

Let m be the slope of the line and $A(x_1, y_1)$ be the given point on the line. Let $P(x, y)$ be any point other than A on the given line. Slope of the line joining $A(x_1, y_1)$ and $P(x, y)$ is given by $m = \dfrac{y - y_1}{x - x_1}$

$$\Rightarrow y - y_1 = m(x - x_1), \tag{6.6}$$

which is known as point-slope form.

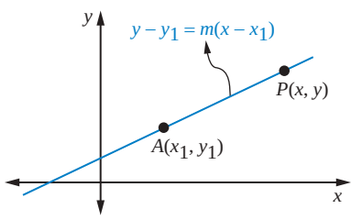


Figure 6.19

Since, the slope m is undefined for lines parallel to the y-axis, the point-slope form of the equation will not give the equation of a line through $A(x_1, y_1)$ parallel to the y-axis. However, this presents no difficulty, since for any such line the abscissa of any point on the line is x_1. Therefore, the equation of such a line is $x = x_1$.

(iii) Two Points form

If (x_1, y_1) and (x_2, y_2) are any two points on the line with $x_2 \neq x_1$ and $y_1 \neq y_2$, then the slope is $m = \dfrac{(y_2 - y_1)}{(x_2 - x_1)}$.

The equation using point- slope form, we get $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

Rewriting the above equation, we get

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \tag{6.7}$$

This equation is called two points form.

Two points form can also be represented in terms of the determinant as

$$\begin{vmatrix} x - x_1 & y - y_1 \\ x_2 - x_1 & y_2 - y_1 \end{vmatrix} = 0.$$

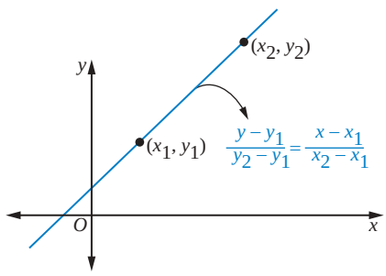


Figure 6.20

(iv) Intercepts form:

If the intercepts of a line on the x-axis and the y-axis are known then the equation of the line can also be found using intercepts. Suppose x-intercept $OA = a$ and y-intercept $OB = b$, where a and b are non-zero, then the line passes through two points $A(a, 0)$ and $B(0, b)$ is

$$\frac{y-0}{b-0} = \frac{x-a}{0-a}$$

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (6.8)$$

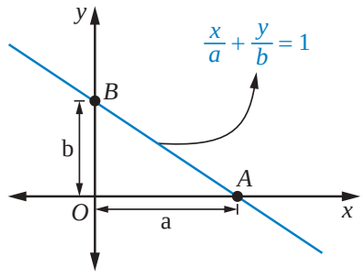


Figure 6.21

The above equation is called an intercept form.

Lines that pass through the origin or which are horizontal or vertical or violate the nonzero condition on a or b cannot be represented in this form.

In most of the cases this form is used to draw the graph of the line in easy way.

(v) Normal form:

Let A and B be the intercepts made by the line.

Let p be the length of the normal OP drawn from the origin to a line AB, which makes an angle α with the x-axis.

$$\text{In right } \Delta OPA, \frac{OP}{OA} = \cos\alpha \text{ and}$$

$$\text{in right } \Delta OPB, \frac{OP}{OB} = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$$

$$\Rightarrow \frac{1}{OA} = \frac{\cos\alpha}{p}$$

$$\text{and } \frac{1}{OB} = \frac{\sin\alpha}{p}$$

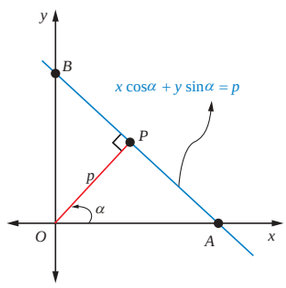


Figure 6.22

Using the above data in intercepts form

$$\frac{x}{OA} + \frac{y}{OB} = 1$$

$$\text{We get, } \frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1$$

$$\Rightarrow x\cos\alpha + y\sin\alpha = p \qquad (6.9)$$

is called the normal form of equation.

If p is positive in all positions of the line and if α is always measured from x-*axis* in the positive direction, this equation holds in every case as shown in the figure.

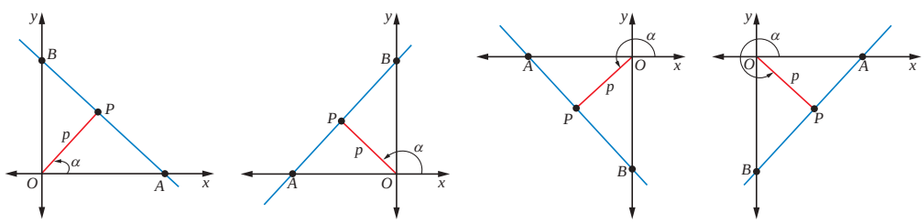


Figure 6.23

(vi) Parametric form:

Parametric equations of a straight line is of the form

$$x = ar + x_1 \text{ and } y = br + y_1$$

where a and b are constants and r is the parameter.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = r, (a \neq 0, \quad b \neq 0).$$

Suppose we have the equation of the line passing through the point $Q(x_1, y_1)$ and making an angle θ with x-axis. Let $P(x, y)$ be a point on the line at a distance r from Q. Drop perpendiculars *QN* and *PM* respectively from Q and P to the x-axis and perpendicular *QR* to *PM*.

From the right ΔQRP

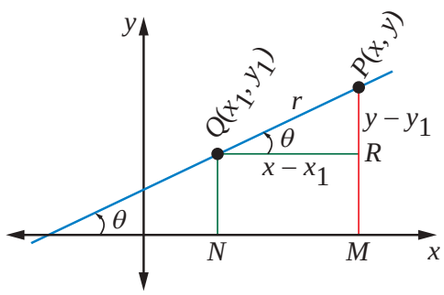


Figure 6.24

$$x - x_1 = QR = PQ \cos\theta = r\cos\theta$$

$$\text{Therefore } \frac{x - x_1}{\cos\theta} = r \quad (6.10)$$

Similarly, $y - y_1 = RP = QP \sin\theta = r\sin\theta$

$$\Rightarrow \frac{y - y_1}{\sin\theta} = r \quad (6.11)$$

From (6.10) and (6.11) we get

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \quad (6.12)$$

where the parameter r is the distance between (x_1, y_1) and any point (x, y) on the line. This is called the symmetric form or parametric form of the line.

The co-ordinates of any point on this line can be written as $(x_1 + r\cos\theta\,,\ y_1 + r\sin\theta)$. Clearly coordinates of the point depend on the value of r. This variable r is called parameter. Since r is a parameter the equations, $x = x_1 + r\cos\theta,\ y = y_1 + r\sin\theta$, is called the parametric equations of the line. The value of $'r'$ is positive for all points lying on the line one side of the given point and negative for all points lying on the line other side of the given point.

(vii) The general form of the equation of the straight line is

$$ax + by + c = 0, \quad \text{where } a,\ b \text{ and } c \text{ are all not zeros}$$

The below table summarizes the types of straight lines related to the given information.

S.No	Information given	Equation of the straight lines
1	Slope(m) and y-intercept (b)	$y = mx + b$
2	Slope (m) and point (x_1, y_1)	$y - y_1 = m(x - x_1)$
3	Two points(x_1, y_1) and (x_2, y_2)	$\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$
4	x-intercept (a) and y-intercept(b)	$\dfrac{x}{a} + \dfrac{y}{b} = 1$
5	Normal length (p) , angle (α)	$x\cos\alpha + y\sin\alpha = p$
6	Parametric form: parameter-r	$\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$
7	The general equation	$ax + by + c = 0$

If we have two variable quantities, then each can be represented by a variable. If the rate of change of one variable with respect to the other variable is constant, then the relationship between them is linear.

In linear equation, the choice of one as independent and other as a dependent may represent the physical reality or may be convenient fiction. The independent variable is normally plotted on the horizontal axis (x-axis), the dependent variable on the vertical axis (y-axis). That is the values of x are always independent and y is dependent on those values of x.

The number scales on the two axes need not be the same. Indeed, in many applications different quantities are represented by x and y. For example, x may represent the number of mobile phones sold and y the total revenue resulting from the sales. In such cases it is often desirable to choose different number scales to represent the different quantities. However, that the zero of both number scales coincide at the origin of the two-dimensional coordinate system.

From the given information, to solve the problem using the concepts of straight lines, we have to select suitably one of the six equations given above.

Example 6.7 Find the slope of the straight line passing through the points (5,7) and (7,5). Also find the angle of inclination of the line with the x-axis.

Solution:

Let (x_1, y_1) and (x_2, y_2) be (5,7) and (7,5) respectively. Let θ be the angle of inclination of the line with the x-axis

Slope of the line $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{5 - 7}{7 - 5} = -1$

We know that $m = \tan\theta$

That is, $\tan\theta = -1 \quad \Rightarrow \quad \theta = \dfrac{3\pi}{4}$ *or* 135°

Slope and angle of inclination of the line with the x-axis are respectively $m = -1$ and $\theta = \dfrac{3\pi}{4}$

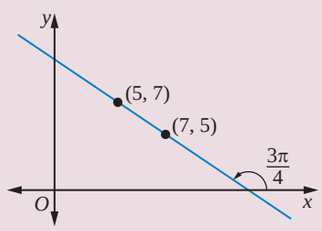


Figure 6.25

Example 6.8 Find the equation of a straight line cutting an intercept of 5 from the negative direction of the y-axis and is inclined at an angle 150° to the x-axis.

Solution:

Given that the negative y intercept is 5 i.e., $b = -5$ and $\theta = 150°$

slope $m = \tan 150° = \tan(180° - 30°) = -\tan 30° = -\dfrac{1}{\sqrt{3}}$

Slope and intercept form of the equation is $y = mx + b$.

That is $y = -\dfrac{1}{\sqrt{3}}x - 5$.

$$x + \sqrt{3}y + 5\sqrt{3} = 0$$

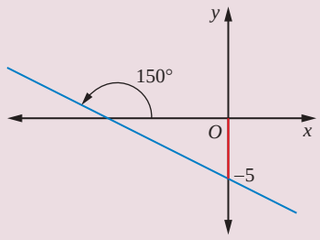


Figure 6.26

Example 6.9 Show the points $\left(0, -\dfrac{3}{2}\right)$, $(1, -1)$ and $\left(2, -\dfrac{1}{2}\right)$ are collinear.

Solution:

Let A, B and C be $\left(0, -\dfrac{3}{2}\right)$, $(1, -1)$ and $\left(2, -\dfrac{1}{2}\right)$ respectively.

The slope of *AB* is $\dfrac{-1 + \frac{3}{2}}{1 - 0} = \dfrac{1}{2}$

The slope of *BC* is $\dfrac{-\frac{1}{2} + 1}{2 - 1} = \dfrac{1}{2}$

Thus, the slope of *AB* is equal to slope of *BC*.
Hence, A, B and C are lying on the same line.

If the rate of change of one variable with respect to the other variable is constant, then this constant rate of change can be taken as slope.(such as speed, constant increase or constant decrease...). Also equations of straight lines depend on the coordinate axes how we define it. Thus in real world problems the equations are not necessarily identical but the path and distance will always be the same.

Example 6.10 The Pamban Sea Bridge is a railway bridge of length about 2065 m constructed on the PalkStrait, which connects the Island town of Rameswaram to Mandapam, the main land of India. The Bridge is restricted to a uniform speed of only 12.5 m/s. If a train of length 560 m starts at the entry point of the bridge from Mandapam, then

(i) find an equation of the motion of the train.
(ii) when does the engine touch island
(iii) when does the last coach cross the entry point of the bridge
(iv) what is the time taken by a train to cross the bridge.

Solution:

Let the x-axis be the time in seconds the y-axis be the distance in metres. Let the engine be at the origin O. Therefore the length of the train 560 m is the negative y-intercept.

The uniform speed 12.5 m/s is the slope of the motion of the train. $\left(\text{speed} = \dfrac{\text{distance}}{\text{time}}\right)$

Since we are given slope and y-intercept, the equation of the line is

$$y = mx + b \qquad (6.13)$$

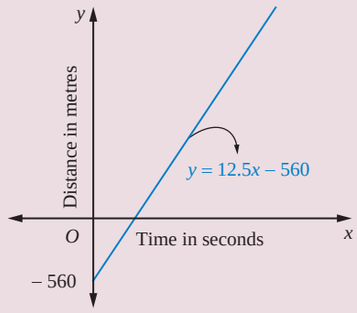


Figure 6.27

(i) The equation of the motion of the train,

$$\begin{aligned} \text{when } m &= 12.5 \text{ and } b = -560, \\ \text{is } y &= 12.5x - 560 \end{aligned}$$

(ii) When the engine touches the other side of the bridge (island)

$$\begin{aligned} y &= 2065 \text{ and } b = 0 \\ 2065 &= 12.5x \\ x &= 165.2 \text{ seconds.} \end{aligned}$$

(iii) When $y = 0$, the last coach cross the entry point of the bridge,

$$\begin{aligned} 0 &= 12.5x - 560 \\ x &= 44.8 \text{ seconds.} \end{aligned}$$

(iv) When $y = 2065$, the time taken for the train to cross the other end of the bridge is given by

$$\begin{aligned} 2065 &= 12.5x - 560 \\ x &= 210 \text{ seconds.} \end{aligned}$$

(One may take the tail of the train as the origin and can find the equation of the straight line. It need not be identical with the above equation, but the path traced out by the train, distance, time, etc.,will be the same. Try it.)

Example 6.11 Find the equations of the straight lines, making the y- intercept of 7 and angle between the line and the y-axis is 30°

Solution:

There are two straight lines making 30° with the y-axis.

From the figure, it is clear that the two lines make the angles $60°$ and $120°$ with the x-axis

$$\begin{aligned} \text{Let } m_1 \text{ be } \tan 60° &= \sqrt{3} \text{ and} \\ m_2 \text{ be } \tan 120° &= \tan(180° - 60°) = -\tan 60° = -\sqrt{3} \\ m_1 = \sqrt{3}, m_2 &= -\sqrt{3} \text{ and } b = 7 \\ \text{Equations of lines are } y &= m_1 x + b \text{ and } y = m_2 x + b \\ y &= \sqrt{3}x + 7 \text{ and } y = -\sqrt{3}x + 7 \end{aligned}$$

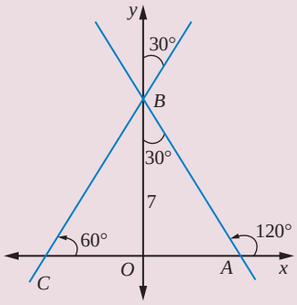


Figure 6.28

Whenever two points are given, one can apply two points form or point and slope form. Also when two intercepts are given, one can apply intercepts form or two points form The following example, is solved in chapter V, using the concepts of sequence and series. Let us solve this problem here, using the concepts of straight lines.

Example 6.12 The seventh term of an arithmetic progression is 30 and tenth term is 21.

(i) Find the first three terms of an A.P.
(ii) Which term of the A.P. is zero (if exists)
(iii) Find the relation ship between Slope of the straight line and common difference of A.P.

Solution:

Since there is a constant increase or decrease in arithmetic progression, it is a linear function. Let the x-axis be the number of the term and the y-axis be the value of the term. Let (x_1, y_1) and (x_2, y_2) be (7,30) and (10,21) respectively, using the equation

$$\begin{aligned} y - y_1 &= m(x - x_1) \text{ we get} \\ y - 30 &= \frac{21 - 30}{10 - 7}(x - 7) \\ y &= -3x + 51 \end{aligned} \tag{6.14}$$

(i) Substituting $x = 1, 2$ and 3 in the equation (**??**) we get the first three terms of AP as 48, 45, and 42.
(ii) Substituting $y = 0$ in equation (**??**) we get

$$0 = -3x + 51 \quad \Rightarrow x = 17.$$

That is seventeenth term of A.P. is zero.
(iii) clearly the slope of the line -3 is equal to the common difference A.P.

From this example, slope of the line is equal to common difference of A.P. (Try to prove it)

Example 6.13 The quantity demanded of a certain type of Compact Disk is 22,000 units when a unit price is ₹ 8. The customer will not buy the disk, at a unit price of ₹ 30 or higher. On the other side the manufacturer will not market any disk if the price is ₹ 6 or lower. However, if the price ₹ 14 the manufacturer can supply 24,000 units. Assume that the quantity demanded and quantity supplied are linearly proportional to the price. Find (i) the demand equation (ii) supply equation (iii) the market equilibrium quantity and price. (iv) The quantity of demand and supply when the price is ₹ 10.

Solution:

Let the x-axis represent the number of units in thousands and the y-axis represent the price in rupees per unit.

(i) For demand function, let (x_1, y_1) and (x_2, y_2) be (22,8) and (0,30) respectively.
Using two point form, we get the demand function as

$$\frac{y-8}{30-8} = \frac{x-22}{0-22}$$
$$\Rightarrow y_D = -x + 30 \text{ (demand function).}$$

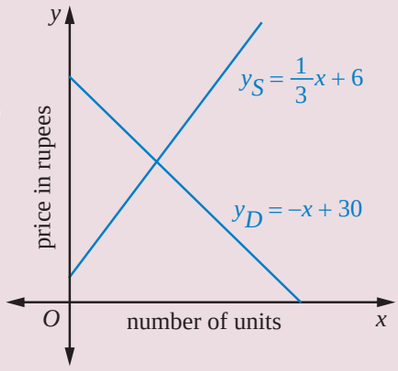


Figure 6.29

(ii) For supply function, let (x_1, y_1) and (x_2, y_2) be (0,6) and (24,14) respectively.
Using two point form, we get the supply function as

$$\frac{y-6}{14-6} = \frac{x-0}{24-0}$$
$$y_S = \frac{1}{3}x + 6 \text{(supply function).}$$

(iii) At the market equilibrium the demand equals to supply,
That is, $y_D = y_S \quad \Rightarrow -x + 30 = \frac{1}{3}x + 6$

$$x = 18 \text{ and } y = 12.$$

Market equilibrium price is Rs12 and number of quantity is 18,000 units.

(iv) when the price $y = 10$, from the demand function $y_D = -x + 30$, we get $x = 20$.
That is, the demand is 20,000 units.
Similarly from the supply function $y_S = \frac{1}{3}x + 6$, we get $x = 12$.
Hence, the supply is 12,000 units.

Example 6.14 Find the equation of the straight line passing through $(-1, 1)$ and cutting off equal intercepts, but opposite in signs with the two coordinate axes.

Solution:

Let the intercepts cut off from the axes be of lengths a and $-a$.

$\therefore$ Equation of the line is of the form

$$\frac{x}{a} - \frac{y}{a} = 1 \Rightarrow x - y = a.$$

Since it passes through $(-1, 1)$

$\Rightarrow (-1) - (1) = a \Rightarrow a = -2.$

Equation of the line is $x - y + 2 = 0$.

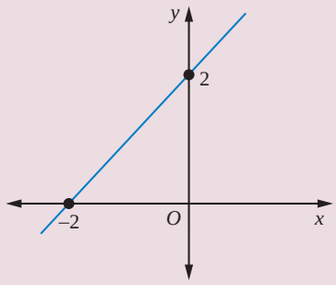


Figure 6.30

Example 6.15 A straight line L with negative slope passes through the point $(9,\ 4)$ cuts the positive coordinate axes at the points P and Q. As L varies, find the minimum value of $|OP| + |OQ|$, where O is the origin.

Solution:

Let m be the slope of the line L. Since it passes through the point $(9,\ 4)$ the equation of the line L is $y - 4 = m(x - 9)$.

The points P and Q are respectively $(9 - \frac{4}{m},\ 0)$ and $(0,\ 4 - 9m)$.

$$\begin{aligned}
(m < 0)\ |OP| + |OQ| &= \left|9 - \frac{4}{m}\right| + |4 - 9m| \\
&= \left|9 + \frac{4}{k}\right| + |4 + 9k| \quad (m < 0, \text{ take } m = -k,\ k > 0) \\
&= (9 + \frac{4}{k}) + (4 + 9k) \quad \text{(all terms are positive)} \\
&= (4 + 9) + (\frac{4}{K} + 9K) \\
&\geq 13 + 2\sqrt{\frac{4}{K} \times 9K} \quad (\textit{A}\text{rithmetic mean} \geq \textit{G}\text{eometric mean}) \\
|OP| + |OQ| &\geq 25
\end{aligned}$$

Therefore, the minimum absolute value of $|OP| + |OQ|$ is 25.

Example 6.16 The length of the perpendicular drawn from the origin to a line is 12 and makes an angle $150°$ with positive direction of the x-axis. Find the equation of the line.

Solution:

Here, $p = 12$ and $\alpha = 150°$, So the equation of the required line is of the form

$$\begin{aligned}
x\cos\alpha + y\sin\alpha &= p \\
\text{That is, } x\cos 150° + y\sin 150° &= 12 \\
\Rightarrow \sqrt{3}x - y + 24 &= 0
\end{aligned}$$

Example 6.17 Area of the triangle formed by a line with the coordinate axes, is 36 square units. Find the equation of the line if the perpendicular drawn from the origin to the line makes an angle of $45°$ with positive the x-axis.

Solution:

Let p be the length of the perpendicular drawn from the origin to the required line.
The perpendicular makes $45°$ with the x-axis.
The equation of the required line is of the form,

$$\begin{aligned} x\cos\alpha + y\sin\alpha &= p \\ \Rightarrow x\cos 45° + y\sin 45° &= p \\ \Rightarrow x + y &= \sqrt{2}p \end{aligned}$$

This equation cuts the coordinate axes at $A(\sqrt{2}p, 0)$ and $B(0, \sqrt{2}p)$.
Area of the ΔOAB is

$$\frac{1}{2} \times \sqrt{2}p \times \sqrt{2}p = 36 \quad \Rightarrow p = 6 \qquad (\because\ p \text{ is positive})$$

Therefore the equation of the required line is

$$x + y = 6\sqrt{2}$$

Example 6.18 Find the equation of the lines make an angle $60°$ with positive x-axis and at a distance $5\sqrt{2}$ units measured from the point $(4, 7)$, along the line $x - y + 3 = 0$.

Solution:

The angle of inclination of the line $x - y + 3 = 0$ is $45°$, and a point on the line is $(4, 7)$.
Using parametric form

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$

the above equation can be written as

$$\frac{x-4}{\frac{1}{\sqrt{2}}} = \frac{y-7}{\frac{1}{\sqrt{2}}} = \pm 5\sqrt{2} \text{ (for either side of (4,7) at a distance } r = \pm 5\sqrt{2})$$

That is $x - 4 = y - 7 = \pm 5$.
The points on the lines at a distance $5\sqrt{2}$ units either side of $(4, 7)$ are $(4 + 5, 7 + 5)$ and $(4 - 5, 7 - 5)$.
The points on the lines are $(9, 12)$ and $(-1, 2)$ and the given slope is $m = \tan 60° = \sqrt{3}$.
Therefore the required equations, using slope and a point form, we get

$$\begin{aligned} \sqrt{3}x - y + (12 - 9\sqrt{3}) &= 0 \text{ and} \\ \sqrt{3}x - y + (2 + \sqrt{3}) &= 0 \end{aligned}$$

6.3.4 General form to other forms

$Ax + By + C = 0$, where A, B and C being real numbers and A and B cannot be simultaneously equal to zero, can be expressed in terms of A, B and C of the general form into other forms.

(i) Slope and intercept form: $(B \neq 0)$: The given equation can be written as

$$y = -\frac{A}{B}x - \frac{C}{B} \quad \Rightarrow \text{ slope} = -\frac{A}{B} \text{ and } y\text{-intercept} = -\frac{C}{B}$$

(ii) Intercepts form: The given equation can be written as

$$\frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1 \ (A,\ B \text{ and } C \text{ are all non-zero})$$

Comparing with intercept form, we get

$$x \text{ -intercept } (a) = \frac{-C}{A} \text{ and } y \text{ -intercept } (b) = \frac{-C}{B}$$

(iii) Normal form: Here A and B are not equal to zero,

Comparing $Ax + By + C = 0$, with $x\cos\theta + y\sin\theta = p$

We get $\frac{-A}{\sqrt{A^2+B^2}}x + \frac{-B}{\sqrt{A^2+B^2}}y = \frac{|C|}{\sqrt{A^2+B^2}}$

Here $\cos\alpha = \frac{-A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \frac{-B}{\sqrt{A^2+B^2}}$ and $p = \frac{|C|}{\sqrt{A^2+B^2}}$

Example 6.19 Express the equation $\sqrt{3}x - y + 4 = 0$ in the following equivalent form:

(i) Slope and Intercept form
(ii) Intercept form
(iii) Normal form

Solution:

(i) Slope and intercept form

It is given that

$$\begin{aligned} \sqrt{3}x - y + 4 &= 0 \\ \Rightarrow y &= \sqrt{3}x + 4 \end{aligned} \tag{6.15}$$

Comparing the above equation with the equation $y = mx + b$, we have

$$\text{Slope} = \sqrt{3} \text{ and } y\text{-intercept} = 4$$

(ii) Intercept form

$$\begin{aligned} \sqrt{3}x - y + 4 &= 0 \Rightarrow \sqrt{3}x - y = -4 \\ \frac{-\sqrt{3}}{4}x + \frac{y}{4} &= 1 \\ \text{That is } \frac{x}{\left(-\frac{4}{\sqrt{3}}\right)} + \frac{y}{4} &= 1 \end{aligned} \tag{6.16}$$

Comparing the above equation with the equation $\frac{x}{a} + \frac{y}{b} = 1$

$$\text{We get, } x\text{-intercept} = -\frac{4}{\sqrt{3}} \text{ and } y\text{-intercept} = 4$$

(iii) Normal form:

$$\begin{aligned} \sqrt{3}x - y + 4 &= 0 \\ (-\sqrt{3})x + y &= 4, \end{aligned} \tag{6.17}$$

Comparing the above equation with the equation $Ax + By + C = 0$.

Here $A = -\sqrt{3}$, and $B = 1$, $\sqrt{A^2 + B^2} = 2$

Therefore, dividing the above equation by 2, we get

$$\frac{-\sqrt{3}x}{2} + \frac{y}{2} = 2 \tag{6.18}$$

Comparing the above equation with the equation $x\cos\alpha + y\sin\alpha = p$

If we take

$$\begin{aligned} \cos\alpha &= \frac{-\sqrt{3}}{2} \text{ and } \sin\alpha = \frac{1}{2} \text{ and } p = 2 \\ \Rightarrow \alpha &= 150^\circ = \frac{5\pi}{6} \text{ and length of the normal } (p) = 2 \end{aligned}$$

The normal form is $x\cos\dfrac{5\pi}{6} + y\sin\dfrac{5\pi}{6} = 2$

To express the given equation to the required form, some times, it is more convenient to use property the proportionality of the coefficients of like terms.

Example 6.20 Rewrite $\sqrt{3}x + y + 4 = 0$ in to normal form.

Solution:

The required form $x\cos\alpha + y\sin\alpha = p$
Given form $-\sqrt{3}x - y = 4$ ($\because p$ is always positive)
Since both represent the same equation, the coefficients are proportional. We get,

$$\frac{\cos\alpha}{-\sqrt{3}} = \frac{\sin\alpha}{-1} = \frac{p}{4}$$

$$\frac{\cos\alpha}{-\sqrt{3}} = \frac{\sin\alpha}{-1} = \frac{p}{4} = \frac{\sqrt{\cos^2\alpha + \sin^2\alpha}}{\sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2}} = \frac{1}{2}$$

(In componendo and dividendo, whenever a term is square off, then it should be square root off it.)

$$\cos\alpha = \frac{-\sqrt{3}}{2}, \quad \sin\alpha = \frac{-1}{2} \text{ and } p = \frac{4}{2}$$

$$\alpha = 210^\circ = \frac{7\pi}{6} \text{ and } p = 2$$

Normal form of the equation is

$$x\cos\frac{7\pi}{6} + y\sin\frac{7\pi}{6} = 2$$

Finding the shortest path between two points on a curved surface can often be difficult. However, the length of a path on the surface of a cylinder is not changed if the curved surface is flattened. For the following problem, by unrolling the hollow cylinder and flattening it into a rectangle, a single reflection allows us to determine the ant's path.

Example 6.21

Consider a hollow cylindrical vessel, with circumference 24 cm and height 10 cm. An ant is located on the outside of vessel 4 cm from the bottom. There is a drop of honey at the diagrammatically opposite inside of the vessel, 3 cm from the top. (i) What is the shortest distance the ant would need to crawl to get the honey drop?. (ii) Equation of the path traced out by the ant. (iii) Where the ant enter in to the cylinder?. Here is a picture that illustrates the position of the ant and the honey.

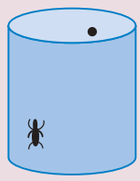


Figure 6.31

Solution:

By unrolling the hollow cylinder and flattening it into a rectangle, and with a single reflection allows us to determine the ant's path, as shown the figure. Let the base line x-axis in cm. and the vertical line through A (initial position of the ant) be the y-axis. Let H be the position of honey drop and E be the entry point of ant inside the vessel. From the given information we have

Let $A(x_1, y_1)$ and $H(x_2, y_2)$ be (0,4) and (12,13) respectively

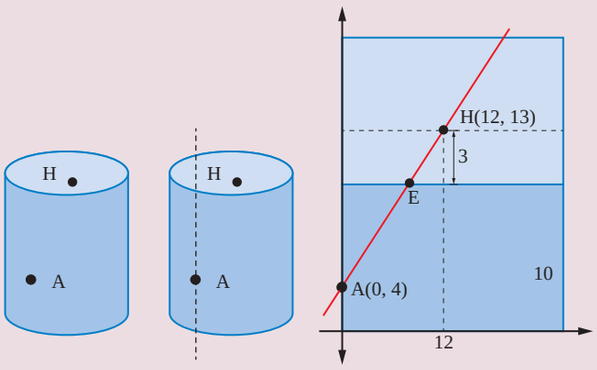


Figure 6.32

(i) The shortest distance between A and H is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{12^2 + 9^2} = 15$$

the ant would need to crawl 15 cm to get the honey.

(ii) The equation of the path *AH* is $\dfrac{y-4}{13-4} = \dfrac{x-0}{12-0}$

$$y = 0.75x + 4 \tag{6.19}$$

(iii) At the entry point E, $y = 10 \;\Rightarrow x = 8$

$$E = (8,\ 10)$$

Taking the orign at different location, different form of equation can be obtained, but the path and distance are the same as above.

Exercise - 6.2

1. Find the equation of the lines passing through the point (1,1)
   (i) with y-intercept (-4)
   (ii) with slope 3
   (iii) and (-2, 3)
   (iv) and the perpendicular from the origin makes an angle 60° with x- axis.
2. If $P(r, c)$ is mid point of a line segment between the axes, then show that $\frac{x}{r} + \frac{y}{c} = 2$.
3. Find the equation of the line passing through the point $(1, 5)$ and also divides the co-ordinate axes in the ratio 3:10.
4. If p is length of perpendicular from origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.
5. The normal boiling point of water is 100°C or 212°F and the freezing point of water is $0°C$ or 32°F. (i) Find the linear relationship between C and F
   Find (ii) the value of C for 98.6°F and (iii) the value of F for 38°C
6. An object was launched from a place P in constant speed to hit a target. At the 15th second it was 1400 m away from the target and at the 18th second 800m away. Find (i) the distance between the place and the target (ii) the distance covered by it in 15 seconds.(iii) time taken to hit the target.
7. Population of a city in the years 2005 and 2010 are 1,35,000 and 1,45,000 respectively.
   Find the approximate population in the year 2015. (assuming that the growth of population is constant)
8. Find the equation of the line, if the perpendicular drawn from the origin makes an angle 30° with x-axis and its length is 12.
9. Find the equation of the straight lines passing through (8, 3) and having intercepts whose sum is 1
10. Show that the points (1, 3), (2, 1) and $\left(\frac{1}{2}, 4\right)$ are collinear, by using (i) concept of slope (ii) using a straight line and (iii) any other method
11. A straight line is passing through the point $A(1, 2)$ with slope $\frac{5}{12}$. Find points on the line which are 13 units away from A.
12. A 150 m long train is moving with constant velocity of 12.5 m/s. Find (i) the equation of the motion of the train, (ii) time taken to cross a pole. (iii) The time taken to cross the bridge of length 850 m is?
13. A spring was hung from a hook in the ceiling. A number of different weights were attached to the spring to make it stretch, and the total length of the spring was measured each time is shown in the following table.

Weight (kg)	2	4	5	8
Length (cm)	3	4	4.5	6

   (i) Draw a graph showing the results.
   (ii) Find the equation relating the length of the spring to the weight on it.
   (iii) What is the actual length of the spring.
   (iv) If the spring has to stretch to 9 cm long, how much weight should be added?
   (v) How long will the spring be when 6 kilograms of weight on it?

14. A family is using Liquefied petroleum gas (LPG) of weight 14.2 kg for consumption. (Full weight 29.5kg includes the empty cylinders tare weight of 15.3kg.). If it is used with constant rate then it lasts for 24 days. Then the new cylinder is replaced (i) Find the equation relating the quantity of gas in the cylinder to the days. (ii) Draw the graph for first 96days.

15. In a shopping mall there is a hall of cuboid shape with dimension $800 \times 800 \times 720$ units, which needs to be added the facility of an escalator in the path as shown by the dotted line in the figure. Find (i)the minimum total length of the escalator. (ii) the heights at which the escalator changes its direction. (iii) the slopes of the escalator at the turning points.

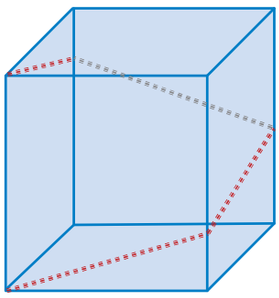


Figure 6.33

6.4 Angle between two straight lines

Two straight lines in a plane would either be parallel or coincide or intersect. Normally when two straight lines intersect, they form two angles at the point of intersection. One is an acute angle and another is an obtuse angle or equal. Both these angles would be supplements(Sum equals 180°) of each other. By definition, when we say 'angle between two straight lines' we mean the acute angle between the two lines.

$$\text{Let } y = m_1x + c_1 \text{ and}$$
$$y = m_2x + c_2$$

be the equations of two straight lines and let these two lines make angles $\theta_1$and θ_2 with x- axis.

$$\text{Then } m_1 = \tan\theta_1 \text{ and}$$
$$m_2 = \tan\theta_2$$

If ϕ (phi) is the angle between these two straight lines, then

$$\phi = \theta_2 - \theta_1 \Rightarrow \tan\phi = \tan(\theta_2 - \theta_1)$$

$$\Rightarrow \tan\phi = \frac{m_2 - m_1}{1 + m_2m_1}$$

$$\Rightarrow \phi = \tan^{-1}\frac{m_2 - m_1}{1 + m_2m_1}$$

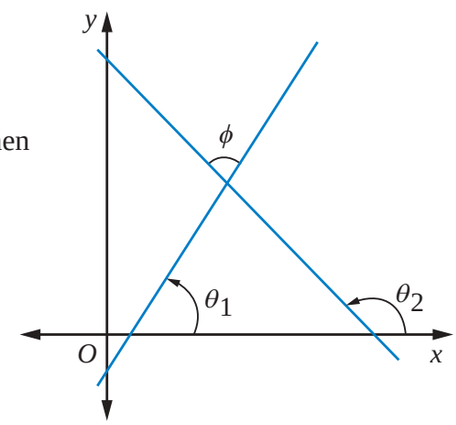


Figure 6.34

If $\dfrac{m_2 - m_1}{1 + m_2m_1}$ is positive then ϕ is the acute angle and if it is negative ϕ is the obtuse angle between the two lines. Therefore, the acute angle ϕ is $\tan^{-1}\left|\dfrac{m_2 - m_1}{1 + m_2m_1}\right|$

6.4.1 Condition for Parallel Lines

Lines in the same plane that do not intersect are called parallel lines.

Let $y = m_1x + c_1$ and $y = m_2x + c_2$ be the equations of two straight lines

If these two lines are parallel, then the angle between lines is zero or π (pi)

If ϕ (phi) is the angle between the lines then, $\phi = 0$

$$\Rightarrow \tan\phi = 0$$

$$\Rightarrow \frac{m_2 - m_1}{1 + m_2m_1} = 0 \Rightarrow m_2 - m_1 = 0 \Rightarrow m_2 = m_1$$

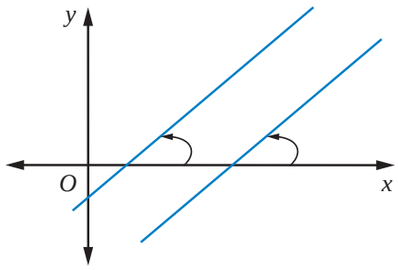


Figure 6.35

That is parallel lines have the same slope. If two non-vertical lines have the same slope, then they are parallel. All vertical lines are parallel.

If the equation of the two lines are in general form as $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the condition for lines to be parallel is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \quad \text{or} \quad a_1b_2 = a_2b_1.$$

(i) The lines parallel to $ax + by + c = 0$ are of the form $ax + by = k$.
(ii) The line parallel to $ax + by + c = 0$ and passing through a point (x_1, y_1), then its equation is $ax + by = ax_1 + by_1$.

6.4.2 Condition for perpendicular Lines

If the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular, then the angle between lines is $\frac{\pi}{2}$

If ϕ is the angle between the lines then

$$\tan\phi = \frac{m_2 - m_1}{1 + m_2m_1} \Rightarrow \cot\phi = \frac{1 + m_2m_1}{m_2 - m_1}$$

$$\cot\frac{\pi}{2} = \frac{1 + m_2m_1}{m_2 - m_1} \Rightarrow 1 + m_1m_2 = 0$$

$$\Rightarrow m_1m_2 = -1$$

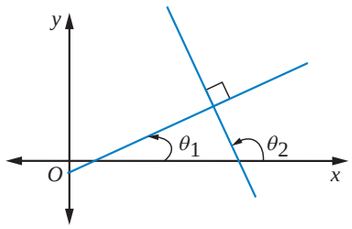


Figure 6.36

If the equation of the two lines are in general form as $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the condition for lines to be perpendicular is $a_1a_2 + b_1b_2 = 0$

(i) The perpendicular line to $ax + by + c = 0$ are of the form $bx - ay = k$.
(ii) The perpendicular line to $ax + by + c = 0$ and passes through the point (x_1, y_1), then the required equation is $bx - ay = bx_1 - ay_1$.

Form of lines	Condition for parallel	Condition for perpendicular
$y = m_1x + c_1$ and $y = m_2x + c_2$	$m_2 = m_1$	$m_1m_2 = -1$
$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$	$a_1b_2 = a_2b_1$	$a_1a_2 + b_1b_2 = 0$

Important Note: If $m_1m_2 = -1$, then the two lines are perpendicular. But the converse is not true, because the lines are parallel to the axes, the result cannot be applied, even when the lines are perpendicular.

6.4.3 Position of a point with respect to a straight line

Any line $ax + by + c = 0$ $(c \neq 0)$,divides the whole plane in to two parts:

(i) one containing the origin called origin side of the line and
(ii) the other not containing the origin called non-origin side of the line.
A point $P(x_1, y_1)$ is on the origin side or non-origin side of the line $ax + by + c = 0$ $(c \neq 0)$, according as $ax_1 + by_1 + c$ and c are of the same sign or opposite sign.
If $c > 0$, then $P(x_1, y_1)$ is on the origin side or non origin side of the line $ax + by + c = 0$, according as $ax_1 + by_1 + c$ is positive or negative.

After rewriting the equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, such that both $c_1 > 0$ and $c_2 > 0$, and if

(i) $a_1a_2 + b_1b_2 < 0$, then the angle between them is acute
(ii) $a_1a_2 + b_1b_2 > 0$, then the angle between them is obtuse.

6.4.4 Distance Formulas

Let us develop formulas to find the distance between

(i) two points
(ii) a point to a line
(iii) two parallel lines

(i) The distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The above formula was already proved in lower class.

(ii) The distance from a point $P(x_1, y_1)$ to a line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

Proof. Let AB be the given line,

$$ax + by + c = 0 \tag{6.20}$$

$P(x_1, y_1)$ be the given point.

Draw a line *CD* parallel to *AB* through P and drop a perpendicular from $P(x_1, y_1)$ to *AB* to meet at M. Also drop a perpendicular from the origin to the line AB to meet at R, and meeting *CD* at Q.

Let $\angle AOR = \alpha$

Therefore the normal form of the the line AB is

$$x \cos\alpha + y \sin\alpha = p \tag{6.21}$$

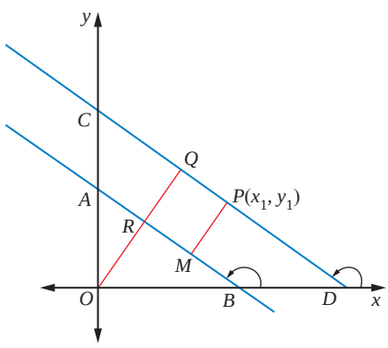


Figure 6.37

Since the equations (6.20) and (6.21) represent the same equations the coefficients of like terms are proportional.

$$\frac{\cos\alpha}{a} = \frac{\sin\alpha}{b} = \frac{p}{-c} \Rightarrow \frac{\cos\alpha}{a} = \frac{\sin\alpha}{b} = \frac{p}{-c} = \frac{\sqrt{\cos^2\alpha + \sin^2\alpha}}{\pm\sqrt{a^2+b^2}}$$

$$\cos\alpha = \frac{\pm a}{\sqrt{a^2+b^2}}, \quad \sin\alpha = \frac{\pm b}{\sqrt{a^2+b^2}} \text{ and } p = \frac{\mp c}{\sqrt{a^2+b^2}}$$

Normal equation of CD is $\quad x\cos\alpha + y\sin\alpha = p'$ (6.22)

Since it passes through $P(x_1, y_1)$, (6.22) $\Rightarrow$ $p' = x_1\cos\alpha + y_1\sin\alpha$

$$\begin{aligned}\text{Required distance} &= |PM| = |QR| = |OQ - OR| = |p' - p| \\ &= |(x_1\cos\alpha + y_1\sin\alpha) - p| \\ &= \left|\pm\frac{ax_1}{\sqrt{a^2+b^2}} \pm \frac{by_1}{\sqrt{a^2+b^2}} \pm \frac{c}{\sqrt{a^2+b^2}}\right| = \left|\pm\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|\end{aligned}$$

$$\text{Required distance} = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

(iii) The distance between two parallel lines $a_1x + b_1y + c_1 = 0$ and $a_1x + b_1y + c_2 = 0$ is

$D = \dfrac{|c_2 - c_1|}{\sqrt{a^2+b^2}}$ (It can be proved using above result by taking point (x_1, y_1) as the origin)

- The coordinates of the nearest point Q(foot of the perpendicular) on the line $ax + by + c = 0$ from the point $P(x_1, y_1)$ can be found from

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2} \text{(using parametric form)} \quad (6.23)$$

- The coordinates of the image of the point $\bar{P}(x_1, y_1)$ with respect to the line $ax + by + c = 0$ are given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2} \quad (6.24)$$

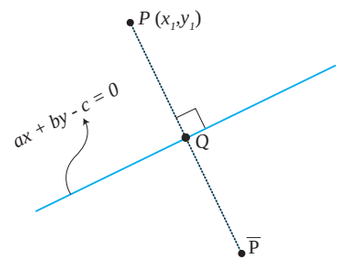


Figure 6.38

Example 6.22 Find the equations of a parallel line and a perpendicular line passing through the point (1, 2) to the line $3x + 4y = 7$.

Solution:

Parallel line to $3x + 4y = 7$ is of the form $3x + 4y = 3x_1 + 4y_1$ Let (x_1, y_1) be (1, 2)

$$\begin{aligned}\Rightarrow 3x + 4y &= 3(1) + 4(2) \\ 3x + 4y &= 11\end{aligned}$$

Perpendicular line to $3x + 4y = 7$ is of the form

$$\begin{aligned}4x - 3y &= 4x_1 - 3y_1 \\ \text{Here } (x_1, y_1) &= (1, 2) \\ \Rightarrow 4x - 3y &= 4(1) - 3(2) \\ 4x - 3y &= -2\end{aligned}$$

$\therefore$ The parallel and perpendicular lines are respectively

$$\begin{aligned} 3x + 4y &= 11 \\ 4x - 3y &= -2 \end{aligned}$$

Example 6.23 Find the distance

(i) between two points (5, 4) and (2, 0)
(ii) from a point (1, 2) to the line $5x + 12y - 3 = 0$
(iii) between two parallel lines $3x + 4y = 12$ and $6x + 8y + 1 = 0$.

Solution:

(i) Distance between two points $(x_1 y_1) = (5, 4)$ and $(x_2, y_2) = (2, 0)$ is

$$\begin{aligned} D &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{3^2 + 4^2} \\ D &= 5 \end{aligned}$$

(ii) Distance between the point (x_1, y_1) and the line $ax + by + c = 0$ is

$$D = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

$\therefore$ The distance between the point (1, 2) and the line $5x + 12y - 3 = 0$ is

$$\begin{aligned} D &= \left| \frac{5(1) + 12(2) - 3}{\sqrt{5^2 + 12^2}} \right| \\ D &= 2 \end{aligned}$$

(iii) Distance between two parallel lines $a_1x + b_1y + c_1 = 0$ and $a_1x + b_1y + c_2 = 0$ is

$$D = \frac{|c_1 - c_2|}{\sqrt{a_1^2 + b_1^2}}$$

Given lines can be written as $3x + 4y - 12 = 0$ and $3x + 4y + \frac{1}{2} = 0$
Here $a_1 = 3, b_1 = 4, c_1 = -12, c_2 = \frac{1}{2}$

$$D = \frac{|c_1 - c_2|}{\sqrt{a_2^1 + b_2^1}} = \left| \frac{-12 - \frac{1}{2}}{\sqrt{3^2 + 4^2}} \right| = \frac{25}{2 \times 5} = 2\ .5 \text{ units.}$$

Example 6.24 Find the nearest point on the line $2x + y = 5$ from the origin.

Solution:

The required point is the foot of the perpendicular from the origin on the line $2x + y = 5$.
The line perpendicular to the given line, through the origin is $x - 2y = 0$.
Solving the equations $2x + y = 5$ and $x - 2y = 0$, we get $x = 2, y = 1$
Hence the nearest point on the line from the origin is (2, 1).

Alternate method:

Using the formula (6.23)

$$\frac{x-x_1}{a}=\frac{y-y_1}{b} = -\frac{(ax_1+by_1+c)}{a^2+b^2}$$
$$\frac{x-0}{2}=\frac{y-0}{1} = -\frac{(2(0)+1(0)-5)}{2^2+1^2}$$
$$\Rightarrow \frac{x}{2} = \frac{y}{1}=1 \Rightarrow (2,\ 1)$$

Example 6.25 Find the equation of the bisector of the acute angle between the lines $3x+4y+2=0$ and $5x+12y-5=0$.

Solution:

First, let us make the constant term positives in both the equations.

The angle bisectors of the given equations are

$$\frac{3x+4y+2}{\sqrt{3^2+4^2}}=\pm\frac{-5x-12y+5}{\sqrt{5^2+12^2}} \quad \text{(moving point is equidistance from the lines)}$$

Since $a_1a_2+b_1b_2=-15-48<0$, the equation of bisector of the acute angle between the lines is $\frac{3x+4y+2}{5}=+\frac{-5x-12y+5}{13} \Rightarrow 64x+112y+1=0$

Example 6.26 Find the points on the line $x+y=5$, that lie at a distance 2 units from the line $4x+3y-12=0$

Solution:

Any point on the line $x+y=5$ is $x=t, y=5-t$

The distance from $(t, 5-t)$ to the line $4x+3y-12=0$ is given by 2 units.

$$\therefore \frac{4(t)+3(5-t)-12}{\sqrt{4^2+3^2}} = 2$$
$$\Rightarrow \frac{|t+3|}{5} = 2$$
$$\Rightarrow t+3 = \pm 10 \Rightarrow t=-13,\ t=7$$

$\therefore$ The points $(-13, 18)$ and $(7, -2)$.

Example 6.27 A straight line passes through a fixed point $(6,\ 8)$. Find the locus of the foot of the perpendicular drawn to it from the origin O.

Solution:

Let the point (x_1, y_1) be $(6, 8)$. and $P(h, k)$ be a point on the required locus.

Family of equations of the straight lines passing through the fixed point (x_1, y_1) is

$$y-y_1=m(x-x_1) \ \Rightarrow\ y-8=m(x-6) \tag{6.25}$$

Since OP is perpendicular to the line (6.25)

$$m \times \left(\frac{k-0}{h-0}\right) = -1 \Rightarrow m = -\frac{h}{k}$$

Also $P(h, k)$ lies on (6.25) $\Rightarrow k - 8 = -\frac{h}{k}(h-6)$

$$\Rightarrow k\,(k-8) = -h(h-6) \Rightarrow h^2 + k^2 - 6h - 8k = 0$$

Locus of $P(h, k)$ is $x^2 + y^2 - 6x - 8y = 0$

This result shows from the fact that the angle in a semi circle is a right angle.

6.4.5 Family of lines

All lines follow a specific condition are called a family of lines. The following example shows some families of straight lines.(where $m.h,$ and k are arbitrary constants).

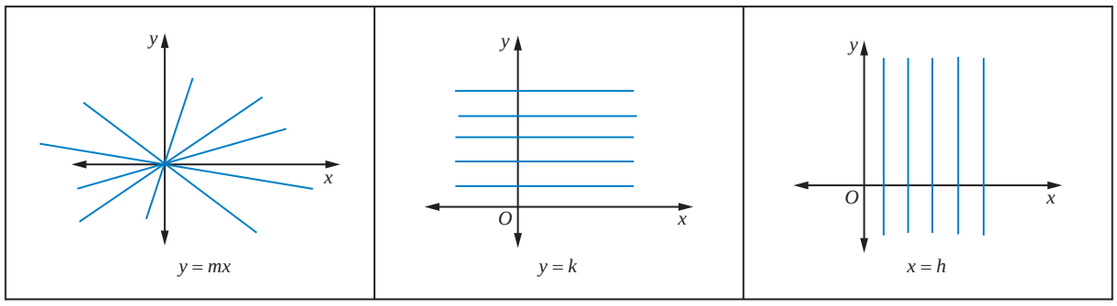


Figure 6.39

It may seem that the equation of a straight line $ax + by + c = 0$ contains three arbitrary constants. In fact, it is not so. On dividing it by b (or a, whichever is non-zero), we get

$$\frac{a}{b}x + y + \frac{c}{b} = 0, \quad \text{which can be written as}$$
$$Ax + y + C = 0 \quad \text{where} \quad A = \frac{a}{b} \text{ and } C = \frac{c}{b}.$$

The above equation can be written as slope and intercept form.

It follows that the equation of a straight line contains two arbitrary constants, and the number of these arbitrary constants cannot be decreased further. Thus, the equation of every straight line contains two arbitrary constants; consequently, two conditions are needed to determine the equation of a straight line uniquely.

One condition yields a linear relation among two arbitrary constants and hence each arbitrary constant determines the other. Therefore, the lines which satisfy one condition contain a **single arbitrary constant.** Such a system of lines is called **one parameter family of lines** and the unknown arbitrary constant is called, the **parameter.**

Let us now discuss the three types of families of straight lines, using $y = mx + b$. First two types are one parameter families and third one is two parameters families

(i) when m is arbitrary and b is a fixed constant.
(ii) when b is arbitrary and m is a fixed constant.
(iii) when both m and b are arbitrary

6.4.6 One parameter families

(i) **when m is arbitrary and b is a fixed constant**

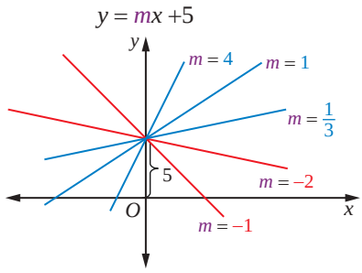


Figure 6.40

Let us find the family of equations of straight lines for the line $y = mx+b$ by considering m is arbitrary constant and b is a fixed constant say $b = 5$. Therefore the equation, for different real values of m, represents a family of lines with y-intercept 5 units. A few members of this family are shown in figure. For example, in this diagram the slope m takes $-1,\ -2,\ \frac{1}{3},\ 1$ and 4

Example 6.28 Find the equations of the straight lines in the family of the lines $y = mx + 2$, for which m and the x-coordinate of the point of intersection of the lines with $2x+3y = 10$ are integers.

Solution:

To find the equations of straight lines for the family of line $y = mx + 2$, we have to determine the values of the parameter m.

The point of intersection of the lines $y = mx + 2$ and $2x + 3y = 10$ is

$$\left(\frac{4}{3m+2}, \frac{10m+4}{3m+2}\right)$$

It is given that the slope m and the x-coordinate are integers.

$\therefore\ \dfrac{4}{3m+2}$ is an integer $\Rightarrow 3m + 2$ is a divisor of 4 $(\pm 1,\ \pm 2$ and $\pm 4)$

$\therefore\ 3m + 2 = \pm 1,\quad 3m + 2 = \pm 2,\quad 3m + 2 = \pm 4$, where m is an integer

Solving we get, $m = \{-2, -1, 0\}$

The equations are, $y = -2x + 2$, $y = -x + 2$ and $y = 2$

(ii) **when b is arbitrary and m is a fixed constant**.

As discussed above, suppose b is arbitrary constant and m is a fixed constant say $m = -2$,the equation $y = mx + b$ becomes $y = -2x + b$. For different real values of b, a family of lines can be obtained with slope -2. A few members of this family are shown in the figure. For example, in this diagram b can take values $-3,\ -1,\ 0,\ 1,\ 2,\ 3$ and 4.

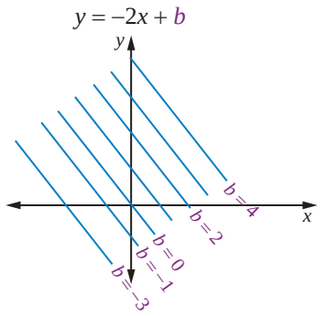


Figure 6.41

Two special cases family of parallel lines and family of perpendicular lines are given below

Family of parallel lines: Family of parallel lines to $ax+by+c=0$ is of the form $ax+by+\lambda=0$. For different values of λ (call it *lambda*), we get different lines parallel to $ax+by+c=0$.

Family of perpendicular lines: Family of perpendicular lines to $ax+by+c=0$ is of the form $bx-ay+\lambda=0$. For different values of λ, we get different lines perpendicular to $ax+by+c=0$.

6.4.7 Two parameters families

(iii) when both m and b are arbitrary

Suppose both m and b are arbitrary constants in $y = mx + b$, we cannot visualize the family easily in graph. But some cases like $y - y_1 = m(x - x_1)$ for different real values of m, a family of lines can be visualized. Suppose the slope m takes -2, -4, $\frac{1}{3}$, 1 and 3, lines which pass through the fixed point (x_1, y_1) except the vertical line $x = x_1$ as shown in the diagram.

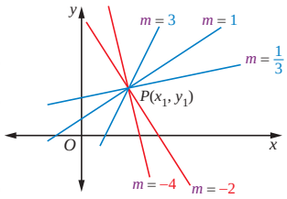


Figure 6.42

6.4.8 The family of equation of straight lines through the point of intersection of the two given lines.

Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$,

be the equation of two given lines. The family of equations of straight lines through the point of intersection of the above lines is $L_1+\lambda L_2 = 0$ where λ is a parameter. That is, for different real values of λ we get different equations.

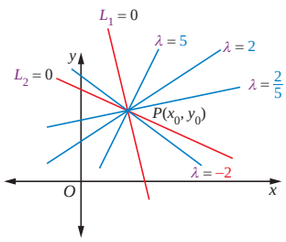


Figure 6.43

Example 6.29 Find the equation of the line through the intersection of the lines $3x+2y+5=0$ and $3x - 4y + 6 = 0$ and the point (1,1).

Solution:

The family of equations of straight lines through the point of intersection of the lines is of the form $(a_1x + b_1y + c_1) + \lambda\,(a_2x + b_2y + c_2) = 0$

That is, $(3x + 2y + 5) + \lambda\,(3x - 4y + 6) = 0$

Since the required equation passes through the point (1,1), the point satisfies the above equation

Therefore $\{3 + 2(1) + 5\} + \lambda\,\{3(1) - 4(1) + 6\} = 0 \Rightarrow \lambda = -2$

Substituting $\lambda = -2$ in the above equation we get the required equation as $3x - 10y + 7 = 0$ (verify the above problem by using two points form)

Example 6.30 Suppose the Government has decided to erect a new Electrical Power Transmission Substation to provide better power supply to two villages namely A and B. The substation has to to be on the line l. The distances of villages A and B from the foot of the perpendiculars P and Q on the line l are 3 km and 5 km respectively and the distance between P and Q is 6 km. (i) What is the smallest length of cable required to connect the two villages. (ii) Find the equations of the cable lines that connect the power station to two villages. (Using the knowledge in conjunction with the principle of reflection allows for approach to solve this problem.)

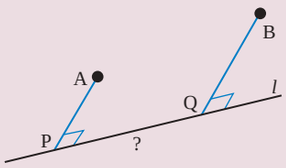


Figure 6.44

Solution:

Take conveniently PQ as x-axis, PA as y-axis and P is the origin (instead of conventional origin O). Therefore, the coordinates are $P(0,0)$, $A(0,3)$ and $B(6,5)$

If the image of A about the x-axis is $\overline{A}$, then $\overline{A}$ is $(0,-3)$.

The required R is the point of intersection of the line $\overline{A}B$ and x-axis.

AR and BR are the path of the cable (road)

The shortest length of the cable is $AR + BR = BR + R\overline{A} = B\overline{A} = \sqrt{(6-0)^2 + (5+3)^2} = 10$ km

Equation of the line $\overline{A}B$ is $y - (-3) = \dfrac{5-(-3)}{6-0}(x-0)$

$$4x - 3y = 9$$

When $y = 0$, R is $\left(\dfrac{9}{4}, 0\right)$

That is the substation should be located at a distance of 2.25 km from P.

The equation of AR is $4x + 3y = 9$

The equations of the cable lines (roads) of RA and RB are $4x - 3y = 9$ and $4x + 3y = 9$.

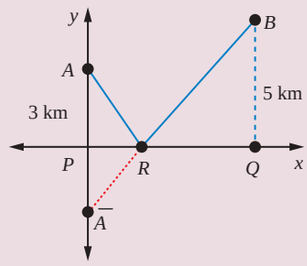


Figure 6.45

Example 6.31 A car rental firm has charges ₹ 25 with 1.8 free kilometers, and ₹ 12 for every additional kilometer. Find the equation relating the cost y to the number of kilometers x. Also find the cost to travel 15 kilometers

Solution:

Given that up to 1.8 kilometers the fixed rent is ₹ 25.

The corresponding equation is

$$y = 25, \quad 0 \le x \le 1.8 \tag{6.26}$$

Also ₹ 12 for every additional kilometer after 1.8 kilometers

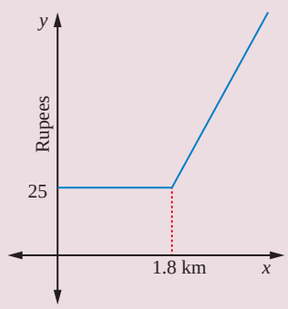


Figure 6.46

The corresponding equation is

$$y = 25 + 12\,(x - 1.8)\,, \quad x > 1.8 \tag{6.27}$$

The combined equation of (6.26) and (6.27), we get

$$y = \begin{cases} 25, & 0 \le x \le 1.8 \\ 25 + 12(x - 1.8), & x > 1.8 \end{cases} \tag{6.28}$$

When $x = 15$, from (6.27), we get cost to travel 15 kilometers is ₹ 183.40.

If you connect from your mobile phone to book a call taxi it automatically connects you to the first available cab in your area wherever you are, as long as there's taxi service. What type of coordinate system used to locate and navigate? www.mapbox.com

Example 6.32 If a line joining two points (3, 0) and (5, 2) is rotated about the point (3, 0) in counter clockwise direction through an angle 15°, then find the equation of the line in the new position

Solution:

Let P (3, 0) and Q (5, 2) be the given points.

Slope of $PQ = \dfrac{y_2 - y_1}{x_2 - x_1} = 1 \Rightarrow$ the angle of inclination of the line $PQ = \tan^{-1}(1) = \dfrac{\pi}{4} = 45°$

$\therefore$ The slope of the line in new position is

$$m = \tan(45° + 15°) \Rightarrow \text{ Slope} = \tan(60°) = (\sqrt{3})$$

$\therefore$ Equation of the straight line passing through (3, 0) and with the slope $\sqrt{3}$ is

$$\begin{aligned} y - 0 &= \sqrt{3}(x - 3) \\ \sqrt{3}x - y - 3\sqrt{3} &= 0 \end{aligned}$$

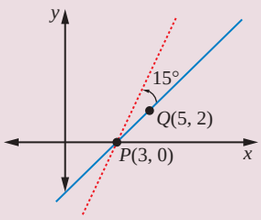


Figure 6.47

Exercise - 6.3

1. Show that the lines are $3x + 2y + 9 = 0$ and $12x + 8y - 15 = 0$ are parallel lines.
2. Find the equation of the straight line parallel to $5x - 4y + 3 = 0$ and having x-intercept 3.
3. Find the distance between the line $4x + 3y + 4 = 0$, and a point (i) $(-2, 4)$ (ii) $(7, -3)$
4. Write the equation of the lines through the point $(1, -1)$
   (i) parallel to $x + 3y - 4 = 0$
   (ii) perpendicular to $3x + 4y = 6$
5. If $(-4, 7)$ is one vertex of a rhombus and if the equation of one diagonal is $5x - y + 7 = 0$, then find the equation of another diagonal.
6. Find the equation of the lines passing through the point of intersection lines $4x - y + 3 = 0$ and $5x + 2y + 7 = 0$, and (i) through the point $(-1, 2)$ (ii) Parallel to $x - y + 5 = 0$ (iii) Perpendicular to $x - 2y + 1 = 0$
7. Find the equations of two straight lines which are parallel to the line $12x + 5y + 2 = 0$ and at a unit distance from the point (1, − 1).
8. Find the equations of straight lines which are perpendicular to the line $3x + 4y - 6 = 0$ and are at a distance of 4 units from (2, 1).
9. Find the equation of a straight line parallel to $2x + 3y = 10$ and which is such that the sum of its intercepts on the axes is 15.

10. Find the length of the perpendicular and the co-ordinates of the foot of the perpendicular from $(-10, -2)$ to the line $x + y - 2 = 0$
11. If p_1 and p_2 are the lengths of the perpendiculars from the origin to the straight lines $x \sec\theta + y \operatorname{cosec}\theta = 2a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$, then prove that $p_1^2 + p_2^2 = a^2$.
12. Find the distance between the parallel lines
    (i) $12x + 5y = 7$ and $12x + 5y + 7 = 0$
    (ii) $3x - 4y + 5 = 0$ and $6x - 8y - 15 = 0$.
13. Find the family of straight lines (i) Perpendicular (ii) Parallel to $3x + 4y - 12 = 0$.
14. If the line joining two points A(2,0) and B(3,1) is rotated about A in anticlockwise direction through an angle of 15°, then find the equation of the line in new position.
15. A ray of light coming from the point (1,2) is reflected at a point A on the x-axis and it passes through the point (5,3). Find the co-ordinates of the point A.
16. A line is drawn perpendicular to $5x = y + 7$. Find the equation of the line if the area of the triangle formed by this line with co-ordinate axes is 10 sq. units.
17. Find the image of the point $(-2, 3)$ about the line $x + 2y - 9 = 0$.
18. A photocopy store charges ₹ 1.50 per copy for the first 10 copies and ₹ 1.00 per copy after the 10th copy. Let x be the number of copies, and let y be the total cost of photocopying. (i) Draw graph of the cost as x goes from 0 to 50 copies. (ii) Find the cost of making 40 copies
19. Find atleast two equations of the straight lines in the family of the lines $y = 5x + b$, for which b and the x-coordinate of the point of intersection of the lines with $3x - 4y = 6$ are integers.
20. Find all the equations of the straight lines in the family of the lines $y = mx - 3$, for which m and the x-coordinate of the point of intersection of the lines with $x - y = 6$ are integers.

6.5 Pair of Straight Lines

The equations of two or more lines can be expressed together by an equation of degree higher than one. As we see that a linear equation in x and y represents a straight line, the product of two linear equations represent two straight lines, that is a pair of straight lines. Hence we study pair of straight lines as a quadratic equations in x and y.

Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$, be separate equations of two straight lines. If $P(x_1, y_1)$ is a point on L_1, then it satisfies the equaiton $L_1 = 0$. Similarly, if $P(x_1, y_1)$ is on L_2 then $L_2 = 0$. If $P(x_1, y_1)$ lies either on $L_1 = 0$ or $L_2 = 0$, then $P(x_1, y_1)$ satisfies the equation $(L_1)(L_2) = 0$, and no other point satisfies $L_1 \cdot L_2 = 0$. Therefore the equation $L_1 \cdot L_2 = 0$ represents the pair of straight lines $L_1 = 0$ and $L_2 = 0$.

For example, consider the two equations

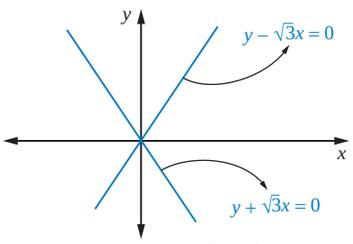
$$y + \sqrt{3}x = 0 \text{ and } y - \sqrt{3}x = 0.$$

The above two equations represent the equation of two straight lines passing through the origin with slopes $-\sqrt{3}$ and $\sqrt{3}$ respectively.

Combining the above equation, we get

$$\left(y + \sqrt{3}x\right)\left(y - \sqrt{3}x\right) = 0$$

$\Rightarrow y^2 - 3x^2 = 0$, represents the pair of straight lines



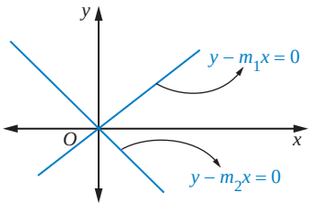
Pair of straight lines $y^2 - 3x^2 = 0$

Figure 6.48

6.5.1 Pair of Lines Passing through the Origin

We first consider a simple case. Both the lines in this pair pass through the origin. Thus, their equations can be written as

$$y - m_1 x = 0$$
$$y - m_2 x = 0$$



Pair of straight lines

Figure 6.49

Combined equation of these two lines is

$$\begin{aligned} (y - m_1 x)(y - m_2 x) &= 0 \qquad (6.29) \\ y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 &= 0 \end{aligned}$$

The above equation suggests that the general equation of a pair of straight lines passing through the origin with slopes m_1 and m_2, $ax^2 + 2hxy + by^2 = 0$ is **a homogenous equation of degree two**, implying that the degree of each term is 2.

Nature of the homogenous equations tells us whether the lines pass through the origin.

Example 6.33 Separate the equations $5x^2 + 6xy + y^2 = 0$

Solution:

We factorize this equation straight away as

$$\begin{aligned} 5x^2 + 6xy + y^2 &= 0 \\ 5x^2 + 5xy + xy + y^2 &= 0 \\ 5x(x + y) + y(x + y) &= 0 \\ (5x + y)(x + y) &= 0 \end{aligned}$$

So that the lines are $5x + y = 0$, and $x + y = 0$

Alternate method: since the given equation is a homogeneous equation, divide the given equation

$$\begin{aligned} 5x^2 + 6xy + y^2 &= 0 \text{ by } x^2 \\ \text{We get } 5 + 6\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 &= 0 \\ \text{Substitute } \frac{y}{x} &= m \text{ (slope of the lines for homogenous equation)} \end{aligned}$$

The above equation becomes $m^2 + 6m + 5 = 0$

$$\begin{aligned} \text{Factorizing, we get} \qquad (m + 1)(m + 5) &= 0 \\ \Rightarrow m = -1, \quad m &= -5 \\ \Rightarrow \frac{y}{x} = -1, \quad \frac{y}{x} &= -5 \end{aligned}$$

That is, the lines are $x + y = 0, \qquad 5x + y = 0$

Example 6.34 If exists, find the straight lines by separating the equations $2x^2 + 2xy + y^2 = 0$.

Solution:

Since the given equation is a homogeneous equation, divide the given equation $2x^2 + 2xy + y^2 = 0$ by x^2 and substituting $\frac{y}{x} = m$

We get $m^2 + 2m + 2 = 0$

The values of m (slopes) are not real (complex number), therefore no line will exist with the joint equation $2x^2 + 2xy + y^2 = 0$

We sometimes say that this equation represents imaginary lines.

Note that in the entire plane, only (0, 0) satisfies this equation

6.5.2 Angle between Pair of Straight Lines

Consider the equation of a pair of straight lines passing through the origin as:

$$ax^2 + 2hxy + by^2 = 0 \tag{6.30}$$

Let m_1 and m_2 be the slopes of these two lines.

By dividing (6.30) by x^2 and substituting $\frac{y}{x} = m$

$$\text{we get,} \quad bm^2 + 2hm + a = 0$$

This quadratic in m will have its roots as m_1 and m_2.

Thus, $m_1 + m_2 = \frac{-2h}{b}$ and $m_1 m_2 = \frac{a}{b}$

If the angle between the two lines is θ Then

$$\begin{aligned}
\tan\theta &= \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right| \\
&= \left|\frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_2 m_1}\right| \\
&= \left|\frac{\sqrt{\left(\frac{-2h}{b}\right)^2 - 4\frac{a}{b}}}{1 + \frac{a}{b}}\right| \\
\tan\theta &= \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|
\end{aligned}$$

As a consequence of this formula, we can conclude that

1. The lines are real and distinct, if m_1 and m_2 are real and distinct, that is if $h^2 - ab > 0$
2. The lines are real and coincident, if m_1 and m_2 are real and equal, that is if $h^2 - ab = 0$
3. The lines are not real (imaginary), if m_1 and m_2 are not real, that is if $h^2 - ab < 0$

Also, we see that the lines represented by (6.30), are parallel (since both pass through the origin, the lines are coincident lines) if $\tan\theta = 0$, that is $h^2 - ab = 0$, and perpendicular if $\cot\theta = 0$ that is $a + b = 0$

Pair of straight lines	Condition for parallel	Condition for perpendicular
$ax^2 + 2hxy + by^2 = 0$	$h^2 - ab = 0,$	$a + b = 0$

Example 6.35 Find the equation of the pair of lines through the origin and perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$

Solution:

Let m_1 and m_2 be the slopes of these two lines.

$$y - m_1x = 0 \text{ and } y - m_2x = 0 \tag{6.31}$$

Combined equation of these two lines is

$$\begin{aligned}(y - m_1x)(y - m_2x) &= 0\\ y^2 - (m_1 + m_2)xy + m_1m_2x^2 &= 0\end{aligned} \tag{6.32}$$

Given that

$$ax^2 + 2hxy + by^2 = 0 \tag{6.33}$$

$$\text{Thus, } m_1 + m_2 = \frac{-2h}{b} \quad \text{and} \quad m_1m_2 = \frac{a}{b} \tag{6.34}$$

The lines perpendicular to (6.31) are

$$y + \frac{1}{m_1}x = 0 \quad \text{and} \quad y + \frac{1}{m_2}x = 0$$

The combined equation is

$$\begin{aligned}(m_1y + x)(m_2y + x) &= 0\\ m_1m_2y^2 + (m_1 + m_2)xy + x^2 &= 0\end{aligned}$$

By using (6.34) $\quad \frac{a}{b}y^2 - \frac{2h}{b}xy + x^2 = 0$

$$\text{The required equation is } ay^2 - 2hxy + bx^2 = 0 \tag{6.35}$$

6.5.3 Equation of the bisectors of the angle between the lines $ax^2 + 2hxy + by^2 = 0$

Let the equations of the two straight lines be $y - m_1x = 0$ and $y - m_2x = 0$

$$\therefore\ m_1 + m_2 = -\frac{2h}{b} \quad \text{and} \quad m_1m_2 = \frac{a}{b}$$

We know that the equation of bisectors is the locus of points from which the perpendicular drawn to the two straight lines are equal.

Let $P(p, q)$ be any point on the locus of bisectors.

The perpendiculars from $P(p, q)$ to the line $y - m_1x = 0$ is equal to the perpendicular from $P(p, q)$ to $y - m_2x = 0$

$$\pm\frac{q - m_1p}{\sqrt{1 + m_1^2}} = \pm\frac{q - m_2p}{\sqrt{1 + m_2^2}}$$

$$\text{That is,} \qquad (q - m_1p)^2\left(1 + m_2^2\right) = (q - m_2p)^2\left(1 + m_1^2\right)$$

Simplifying we get $$p^2 - q^2 = 2pq\left(\frac{1 - m_1m_2}{m_1 + m_2}\right)$$

$$\Rightarrow p^2 - q^2 = 2pq\left(\frac{1 - \frac{a}{b}}{-\frac{2h}{b}}\right)$$

That is $$\frac{p^2 - q^2}{a - b} = \frac{pq}{h}$$

$\therefore$ The locus of $P(p, q)$ is $$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \tag{6.36}$$

Example 6.36 Show that the straight lines $x^2 - 4xy + y^2 = 0$ and $x + y = 3$ form an equilateral triangle.

Solution:

Let the line $x + y = 3$ intersects the pair of line $x^2 - 4xy + y^2 = 0$ at A and B. The angle between the lines $x^2 - 4xy + y^2 = 0$ is

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \frac{2\sqrt{4 - 1}}{2} = \sqrt{3}$$

$$\Rightarrow \theta = \tan^{-1}\sqrt{3} = 60^\circ$$

The angle bisectors of the angle *AOB* are given by

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

$$\Rightarrow x^2 - y^2 = 0$$

$$\Rightarrow x + y = 0 \quad \text{and} \quad x - y = 0$$

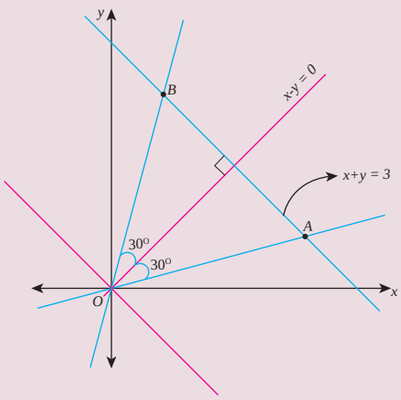


Figure 6.50

The angle bisectors $x - y = 0$ is perpendicular to the given line through *AB*,

that is $x + y = 3 \Rightarrow \Delta OAB$ is isosceles.

$$\Rightarrow \angle ABO = \angle BAO = 60^\circ$$

therefore the given lines form an equilateral triangle.

Example 6.37 If the pair of lines represented by $x^2 - 2cxy - y^2 = 0$ and $x^2 - 2dxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $cd = -1$.

Solution:

Given that the pair of straight lines,

$$x^2 - 2cxy - y^2 = 0 \tag{6.37}$$

$$x^2 - 2dxy - y^2 = 0 \tag{6.38}$$

The equation of the angle bisectors of (6.37) is

$$\frac{x^2-y^2}{2} = \frac{xy}{-c}$$

$$\Rightarrow \quad cx^2+2xy-cy^2 = 0 \tag{6.39}$$

Given that the angle bisector of (6.37) is (6.38)
Therefore equations

$$x^2-2dxy-y^2 = 0$$
$$cx^2+2xy-cy^2 = 0$$

represent the same equation as the angle bisector of (6.37)

Comparing the like terms of the above two equations, we get

$$\frac{1}{c} = \frac{-2d}{2} = \frac{-1}{-c}$$
$$\Rightarrow \quad cd = -1$$

6.5.4 General form of Pair of Straight Lines

Consider the equations of two arbitrary lines $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$

The combined equation of the two lines is

$$(l_1x+m_1y+n_1)(l_2x+m_2y+n_2)=0$$

If we multiply the above two factors together, we get a more general equation to a pair of straight lines has the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \tag{6.40}$$

The above equation is **a non homogenous equation of degree two.**

in which
$$a=l_1l_2 \qquad 2g=l_1n_2+l_2n_1$$
$$b=m_1m_2 \qquad 2f=m_1n_2+m_2n_1$$
$$c=n_1n_2 \qquad 2h=l_1m_2+l_2m_1$$

An equation of the form (6.40) will always represent a pair of straight lines, provided it must able to be factorized into two linear factors of the form $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$.

Condition that the general second degree equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ should represent a pair of straight lines

Let us rearrange the equation of the pair of straight lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ as a quadratic in x, we have

$$ax^2+2(hy+g)x+\left(by^2+2fy+c\right)=0$$

$$x=\frac{-(hy+g)\pm\sqrt{(hy+g)^2-a(by^2+2fy+c)}}{a}$$

$$ax+hy+g=\pm\sqrt{(hy+g)^2-a(by^2+2fy+c)}$$

That is, $$ax+hy+g=\pm\sqrt{(h^2-ab)y^2+2(gh-af)y+g^2-ac}$$

Since each of the above equations represents a straight line, they must be of the first degree in x and y. Therefore the expression under the radical sign should be a perfect square and the condition for this is

$$4\left(gh-af\right)^2-4\left(h^2-ab\right)\left(g^2-ac\right)=0$$

Simplifying and dividing by a,

We get, $abc+2fgh-af^2-bg^2-ch^2=0$ or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$ (Expansion of Determinant will be studied in the next chapter)

Results without Proof:

(i) Two straight lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ are

parallel if $\dfrac{a}{h}=\dfrac{h}{b}=\dfrac{g}{f}$ or $bg^2=af^2$

(ii) If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of parallel straight lines, then

the distance between them is $2\sqrt{\dfrac{(g^2-ac)}{a(a+b)}}$ or $2\sqrt{\dfrac{(f^2-bc)}{b(a+b)}}$

The relationship between the equations of pair of straight lines

$$ax^2+2hxy+by^2=0 \qquad (6.41)$$

$$ax^2+2hxy+by^2+2gx+2fy+c=0) \qquad (6.42)$$

The slope of the above pair of straight lines (6.41) and (6.42) are depending only on the coefficients of

x^2, xy and y^2.

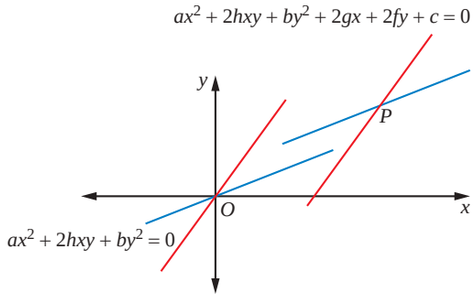


Figure 6.51

(i) If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, then $ax^2+2hxy+by^2=0$ represents a pair of straight lines through the origin parallel to the first pair. The point of intersection (6.41) is (0, 0) and the point of intersection (6.42) is

$$P\left(\frac{hf-bg}{ab-h^2},\quad \frac{gh-af}{ab-h^2}\right)$$

(ii) If θ be the angle between the straight lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$, then it will have the same value as the angle between the two lines represented by $ax^2+2hxy+by^2=0$

$$\text{Thus } \theta=\tan^{-1}\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$$

(iii) If the two straight lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ are at right angles, then the two lines represented by $ax^2+2hxy+by^2=0$ are also at right angles and the condition is $a+b=0$

Example 6.38 If the equation $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, find (i) the value of λ and the separate equations of the lines (ii) point of intersection of the lines (iii) angle between the lines

Solution:

(i) general equation is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

given equation is $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$

Comparing the given equation with the general equation of the second degree we have

$$a = \lambda, b = 12, c = -3, h = -5, g = \frac{5}{2}, f = -8$$

Now applying the condition for pair of straight lines

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

i.e. $\lambda(12)(-3) + 2(-8)\left(\frac{5}{2}\right)(-5) - \lambda(-8)^2 - 12\left(\frac{5}{2}\right)^2 - (-3)(-5)^2 = 0$

$\Rightarrow -36\lambda + 200 - 64\lambda - 75 + 75 = 0, \quad \Rightarrow \lambda = 2$

$\therefore$ The equation is $2x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$.
Let us first factorize the terms of second degree terms from the above equation,

$$\text{we get} \qquad 2x^2 - 10xy + 12y^2 \equiv (x - 2y)(2x - 6y)$$

$$\therefore\ 2x^2 - 10xy + 12y^2 + 5x - 16y - 3 \equiv (x - 2y + c_1)(2x - 6y + c_2)$$

Equating like terms, we get

$$2c_1 + c_2 = 5, \quad 3c_1 + c_2 = 8, \quad c_1c_2 = -3$$

Solving first two equation, we get $c_1 = 3, c_2 = -1$.

$\therefore$ The separate equations of the lines are

$$x - 2y + 3 = 0 \quad \text{and} \quad 2x - 6y - 1 = 0$$

(ii) Point of intersection of the lines is given by solving the two equation of the lines, we get

$$(x, y) = \left(-10, -\frac{7}{2}\right) \left(\text{or use the formula } \left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)\right)$$

(iii) Angle between the lines is given by

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$$= \frac{2\sqrt{25 - 24}}{2 + 12} = \frac{1}{7}$$

$$\therefore\ \theta = \tan^{-1}\left(\frac{1}{7}\right)$$

Example 6.39 A student when walks from his house, at an average speed of 6 kmph, reaches his school ten minutes before the school starts. When his average speed is 4 kmph, he reaches his school five minutes late. If he starts to school every day at 8.00 A.M, then find (i) the distance between his house and the school (ii) the minimum average speed to reach the school on time and time taken to reach the school (iii) the time the school gate closes (iv) the pair of straight lines of his path of walk.

Solution:

Let x-axis be the time in hours and y-axis be the distance in kilometer.

From the given information, we have

$$y = 6(x - \frac{10}{60}) \;\Rightarrow\; y = 6x - 1 \tag{6.43}$$

$$y = 4(x + \frac{5}{60}) \;\Rightarrow\; y = 4x + \frac{1}{3} \tag{6.44}$$

Solving the above two equations, we get $(x, y) = \left(\frac{2}{3}, 3\right)$

$$x = \frac{2}{3} \text{ hour} = 40 \text{ minutes},\; y = 3 \text{ km}$$

(i) the distance between house and the school is 3km

(ii) the minimum average speed to reach the school on time is

$$\frac{60}{40} \times 3 = 4.5 \text{ kmph}$$

and time taken is hours $\frac{2}{3}$ or 40 minutes

(iii) the school gate closes at 8.40 AM

(iv) the pair of straight lines of his path of walk to school is

$$(6x - y - 1)\left(4x - y + \frac{1}{3}\right) = 0$$

$$72x^2 - 30xy + 3y^2 - 6x + 2y - 1 = 0 \tag{6.45}$$

Draw the graph by using the site: https://www.geogebra.org/graphing
https://www.geogebra.org/m/fhS6HUtP

Example 6.40 If one of the straight lines of $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$, then show that $ap^2 + 2hpq + bq^2 = 0$.

Solution:

Let m_1 and m_2 be the slopes of the pair of lines $ax^2 + 2hxy + by^2 = 0$ and m be the slope of $px + qy = 0$

Therefore, $m_1 + m_2 = -\frac{2h}{b}$, $m_1 m_2 = \frac{a}{b}$ and $m = -\frac{p}{q}$

since one of the straight lines of $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$,

$$
\begin{aligned}
m\,m_1 = -1 \quad \text{or} \quad m\,m_2 &= -1 \\
\Rightarrow \quad (m\,m_1 + 1) = 0 \ \text{ or } \ (m\,m_2 + 1) &= 0 \\
\Rightarrow \quad (m\,m_1 + 1)(m\,m_2 + 1) &= 0 \\
\Rightarrow \quad (m_1 m_2)m^2 + (m_1 + m_2)m + 1 &= 0 \\
\Rightarrow \quad \left(\frac{a}{b}\right)\left(-\frac{p}{q}\right)^2 + \left(-\frac{2h}{b}\right)\left(-\frac{p}{q}\right) + 1 &= 0 \\
\Rightarrow \quad ap^2 + 2hpq + bq^2 &= 0
\end{aligned}
$$

The pair of straight lines through the origin is a homogeneous equation of degree two

Example 6.41 Show that the straight lines joining the origin to the points of intersection of $3x - 2y + 2 = 0$ and $3x^2 + 5xy - 2y^2 + 4x + 5y = 0$ are at right angles

Solution:

The straight lines joining the origin and the points of intersection of given equations is a second degree homogeneous equation.

Following steps show, the way of homogenizing the $3x^2 + 5xy - 2y^2 + 4x + 5y = 0$ with $3x - 2y + 2 = 0$

$$
\begin{aligned}
3x^2 + 5xy - 2y^2 + (4x + 5y)(1) &= 0 \ \text{ and } \ \frac{(3x - 2y)}{-2} = 1 \\
3x^2 + 5xy - 2y^2 + (4x + 5y)\left(\frac{3x - 2y}{-2}\right) &= 0 \\
(-2)(3x^2 + 5xy - 2y^2) + (4x + 5y)(3x - 2y) &= 0
\end{aligned}
$$

On simplification,

We get, $2x^2 - xy - 2y^2 = 0. \quad \Rightarrow \ a = 2, \ b = -2 \ \Rightarrow \ a + b = 0$

Since sum of the co-efficient of x^2 and y^2 is equal to zero, the lines are at right angles

Exercise - 6.4

1. Find the combined equation of the straight lines whose separate equations are $x - 2y - 3 = 0$ and $x + y + 5 = 0$.
2. Show that $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$ represents a pair of parallel lines.
3. Show that $2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$ represents a pair of perpendicular lines.
4. Show that the equation $2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$ represents a pair of intersecting lines. Show further that the angle between them is $\tan^{-1}(5)$.
5. Prove that the equation to the straight lines through the origin, each of which makes an angle α with the straight line $y = x$ is $x^2 - 2xy \ \sec 2\alpha + y^2 = 0$
6. Find the equation of the pair of straight lines passing through the point $(1, 3)$ and perpendicular to the lines $2x - 3y + 1 = 0$ and $5x + y - 3 = 0$

7. Find the separate equation of the following pair of straight lines
   (i) $3x^2 + 2xy - y^2 = 0$
   (ii) $6(x-1)^2 + 5(x-1)(y-2) - 4(y-2)^2 = 0$
   (iii) $2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$
8. The slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is twice that of the other, show that $8h^2 = 9ab$.
9. The slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is three times the other, show that $3h^2 = 4ab$.
10. A ΔOPQ is formed by the pair of straight lines $x^2 - 4xy + y^2 = 0$ and the line PQ. The equation of PQ is $x + y - 2 = 0$. Find the equation of the median of the triangle ΔOPQ drawn from the origin O.
11. Find p and q, if the following equation represents a pair of perpendicular lines

$$6x^2 + 5xy - py^2 + 7x + qy - 5 = 0$$

12. Find the value of k, if the following equation represents a pair of straight lines. Further, find whether these lines are parallel or intersecting, $12x^2 + 7xy - 12y^2 - x + 7y + k = 0$
13. For what value of k does the equation $12x^2 + 2kxy + 2y^2 + 11x - 5y + 2 = 0$ represent two straight lines.
14. Show that the equation $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$ represents a pair of parallel lines. Find the distance between them.
15. Show that the equation $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$ represents a pair of parallel lines. Find the distance between them.
16. Prove that one of the straight lines given by $ax^2 + 2hxy + by^2 = 0$ will bisect the angle between the co-ordinate axes if $(a+b)^2 = 4h^2$
17. If the pair of straight lines $x^2 - 2kxy - y^2 = 0$ bisect the angle between the pair of straight lines $x^2 - 2lxy - y^2 = 0$, Show that the later pair also bisects the angle between the former.
18. Prove that the straight lines joining the origin to the points of intersection of $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ and $3x - 2y - 1 = 0$ are at right angles.

Exercise - 6.5

2E7RDH

Choose the correct or more suitable answer

1. The equation of the locus of the point whose distance from y-axis is half the distance from origin is
   (1) $x^2 + 3y^2 = 0$ (2) $x^2 - 3y^2 = 0$ (3) $3x^2 + y^2 = 0$ (4) $3x^2 - y^2 = 0$
2. Which of the following equation is the locus of $(at^2, 2at)$
   (1) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (3) $x^2 + y^2 = a^2$ (4) $y^2 = 4ax$
3. Which of the following point lie on the locus of $3x^2 + 3y^2 - 8x - 12y + 17 = 0$
   (1) $(0, 0)$ (2) $(-2, 3)$ (3) $(1, 2)$ (4) $(0, -1)$
4. If the point (8,-5) lies on the locus $\frac{x^2}{16} - \frac{y^2}{25} = k$, then the value of k is
   (1) 0 (2) 1 (3) 2 (4) 3

5. Straight line joining the points $(2,3)$ and $(-1,4)$ passes through the point (α,β) if

   (1) $\alpha+2\beta=7$ (2) $3\alpha+\beta=9$ (3) $\alpha+3\beta=11$ (4) $3\alpha+\beta=11$

6. The slope of the line which makes an angle 45° with the line $3x-y=-5$ are

   (1) 1, -1 (2) $\frac{1}{2}$, -2 (3) 1, $\frac{1}{2}$ (4) 2, $-\frac{1}{2}$

7. Equation of the straight line that forms an isosceles triangle with coordinate axes in the I-quadrant with perimeter $4+2\sqrt{2}$ is

   (1) $x+y+2=0$ (2) $x+y-2=0$ (3) $x+y-\sqrt{2}=0$ (4) $x+y+\sqrt{2}=0$

8. The coordinates of the four vertices of a quadrilateral are ($-$2,4), ($-$1,2), (1,2) and (2,4) taken in order.The equation of the line passing through the vertex ($-$1,2) and dividing the quadrilateral in the equal areas is

   (1) $x+1=0$ (2) $x+y=1$ (3) $x+y+3=0$ (4) $x-y+3=0$

9. The intercepts of the perpendicular bisector of the line segment joining (1, 2) and (3,4) with coordinate axes are

   (1) 5, -5 (2) 5, 5 (3) 5, 3 (4) 5, -4

10. The equation of the line with slope 2 and the length of the perpendicular from the origin equal to $\sqrt{5}$ is

    (1) $x-2y=\sqrt{5}$ (2) $2x-y=\sqrt{5}$ (3) $2x-y=5$ (4) $x-2y-5=0$

11. A line perpendicular to the line $5x-y=0$ forms a triangle with the coordinate axes. If the area of the triangle is 5 sq. units, then its equation is

    (1) $x+5y\pm5\sqrt{2}=0$ (2) $x-5y\pm5\sqrt{2}=0$ (3) $5x+y\pm5\sqrt{2}=0$ (4) $5x-y\pm5\sqrt{2}=0$

12. Equation of the straight line perpendicular to the line$x-y+5=0$, through the point of intersection the y-axis and the given line

    (1) $x-y-5=0$ (2) $x+y-5=0$ (3) $x+y+5=0$ (4) $x+y+10=0$

13. If the equation of the base opposite to the vertex (2, 3) of an equilateral triangle is $x+y=2$, then the length of a side is

    (1) $\sqrt{\frac{3}{2}}$ (2) 6 (3) $\sqrt{6}$ (4) $3\sqrt{2}$

14. The line $(p+2q)x+(p-3q)y=p-q$ for different values of p and q passes through the point

    (1) $\left(\frac{3}{2},\frac{5}{2}\right)$ (2) $\left(\frac{2}{5},\frac{2}{5}\right)$ (3) $\left(\frac{3}{5},\frac{3}{5}\right)$ (4) $\left(\frac{2}{5},\frac{3}{5}\right)$

15. The point on the line $2x-3y=5$ is equidistance from $(1,2)$ and $(3,4)$ is

    (1) $(7,3)$ (2) $(4,1)$ (3) $(1,-1)$ (4) $(-2,3)$

16. The image of the point (2, 3) in the line $y=-x$ is

    (1) ($-$3, $-$2) (2) ($-$3, 2) (3) ($-$2, $-$3) (4) (3, 2)

17. The length of $\perp$ from the origin to the line $\frac{x}{3}-\frac{y}{4}=1$, is

    (1) $\frac{11}{5}$ (2) $\frac{5}{12}$ (3) $\frac{12}{5}$ (4) $-\frac{5}{12}$

18. The y-intercept of the straight line passing through (1,3) and perpendicular to $2x-3y+1=0$ is

    (1) $\frac{3}{2}$ (2) $\frac{9}{2}$ (3) $\frac{2}{3}$ (4) $\frac{2}{9}$

19. If the two straight lines $x + (2k - 7)y + 3 = 0$ and $3kx + 9y - 5 = 0$ are perpendicular then the value of k is

(1) $k = 3$ (2) $k = \frac{1}{3}$ (3) $k = \frac{2}{3}$ (4) $k = \frac{3}{2}$

20. If a vertex of a square is at the origin and its one side lies along the line $4x + 3y - 20 = 0$, then the area of the square is

(1) 20 sq. units (2) 16 sq. units (3) 25 sq. units (4) 4 sq.units

21. If the lines represented by the equation $6x^2 + 41xy - 7y^2 = 0$ make angles α and β with x- axis, then $\tan\alpha \tan\beta =$

(1) $-\frac{6}{7}$ (2) $\frac{6}{7}$ (3) $-\frac{7}{6}$ (4) $\frac{7}{6}$

22. The area of the triangle formed by the lines $x^2 - 4y^2 = 0$ and $x = a$ is

(1) $2a^2$ (2) $\frac{\sqrt{3}}{2}a^2$ (3) $\frac{1}{2}a^2$ (4) $\frac{2}{\sqrt{3}}a^2$

23. If one of the lines given by $6x^2 - xy + 4cy^2 = 0$ is $3x + 4y = 0$,, then c equals to

(1) -3 (2) -1 (3) 3 (4) 1

24. θ is acute angle between the lines $x^2 - xy - 6y^2 = 0$, then $\frac{2\cos\theta + 3\sin\theta}{4\sin\theta + 5\cos\theta}$ is

(1) 1 (2) $-\frac{1}{9}$ (3) $\frac{5}{9}$ (4) $\frac{1}{9}$

25. One of the equation of the lines given by $x^2 + 2xy\cot\theta - y^2 = 0$ is

(1) $x - y\cot\theta = 0$ (2) $x + y\tan\theta = 0$ (3) $x\cos\theta + y(\sin\theta + 1) = 0$

(4) $x\sin\theta + y(\cos\theta + 1) = 0$

Summary

The types of straight lines related to the information.

S.No	Information given	Equation of the lines
1	Slope(m) and y-intercept (b)	$y = mx + b$
2	Slope (m) and point (x_1, y_1)	$y - y_1 = m(x - x_1)$
3	Two points(x_1, y_1) and (x_2, y_2)	$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$
4	x-intercept (a) and y-intercept (b)	$\frac{x}{a} + \frac{y}{b} = 1$
5	Normal length (p) , angle (α)	$x \cos\alpha + y \sin\alpha = p$
6	Parametric form: parameter-r	$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$
7	The general equation	$ax + by + c = 0$

Form of lines	Condition for parallel	Condition for perpendicular
$y = m_1x + c_1$ and $y = m_2x + c_2$	$m_2 = m_1$	$m_1m_2 = -1$
$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$	$a_1b_2 = a_2b_1$	$a_1a_2 + b_1b_2 = 0$

A point $P(x_1, y_1)$is on the origin side or non origin side of the line $ax + by + c = 0$ $(c \neq 0)$, according as $ax_1 + by_1 + c$ and c are of the same sign or opposite sign.

The distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance from a point $P(x_1, y_1)$ to a line $ax + by + c = 0$ is $\left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$

The distance between two parallel lines $a_1x + b_1y + c_1 = 0$ and $a_1x + b_1y + c_2 = 0$ is $\frac{|c_2 - c_1|}{\sqrt{a^2 + b^2}}$

The line parallel to $ax + by + c = 0$ through a point (x_1, y_1), is $ax + by = ax_1 + by_1$ and the perpendicular line is $bx - ay = bx_1 - ay_1$

The coordinates of the image of the point (x_1, y_1) with respect to the line $ax + by + c = 0$ can be obtained by the line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$$

Pair of straight lines	Condition for parallel	Condition for perpendicular
$ax^2 + 2hxy + by^2 = 0$	$h^2 - ab = 0,$	$a + b = 0$

The equation of the bisectors of the angle between the lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

.

The condition that the general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ should represent a pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

(i) The angle between them is $\theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$

If $a + b = 0$, then the lines are perpendicular.

(ii) The point of intersection

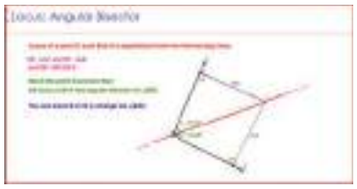
$$P\left(\frac{hf - bg}{ab - h^2}, \quad \frac{gh - af}{ab - h^2}\right)$$

(iii) Two straight lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are parallel if $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$ or $bg^2 = af^2$

(iv) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel straight lines, then the distance between them is $2\sqrt{\frac{(g^2 - ac)}{a(a + b)}}$ or $2\sqrt{\frac{(f^2 - bc)}{b(a + b)}}$

ICT CORNER-6(a)

Expected Outcome ⇒

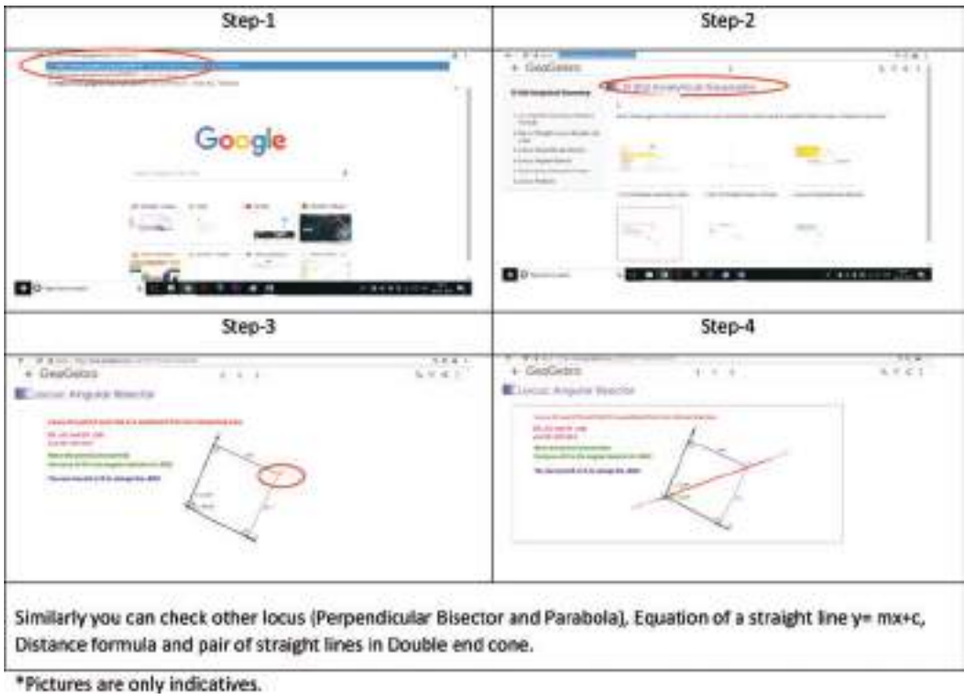


Step – 1
Open the Browser and type the URL Link given below (or) Scan the QR Code.

Step – 2
GeoGebra work book called "XI STD Analytical Geometry" will open. There are more than 5 worksheets given. Select the one you want. For example, select "Locus: **Angular Bisector**"

Step – 3
Locus: Angular Bisector work sheet will open. In the page move the point D to see the locus of the point D

Step – 4
The point D moves such that its perpendicular distance from two fixed lines are equal. Justify the locus.

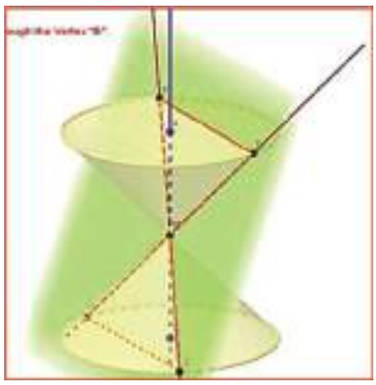


*Pictures are only indicatives.

Browse in the link
Analytical Geometry GeoGebra Workbook: `https://ggbm.at/QNaNghJZ`

ICT CORNER-6(b)

Expected Outcome ⇒

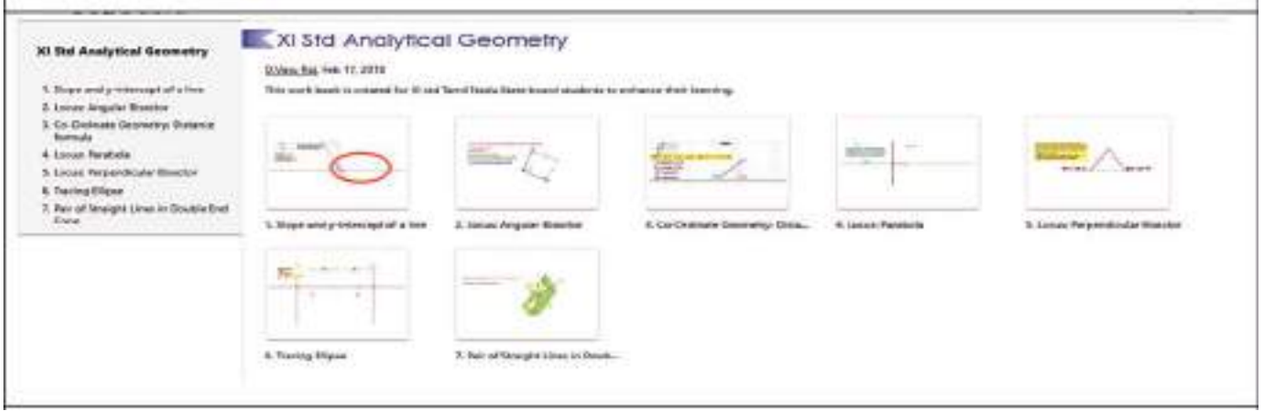


Step–1
Open the Browser and type the URL Link given below (or) Scan the QR Code.

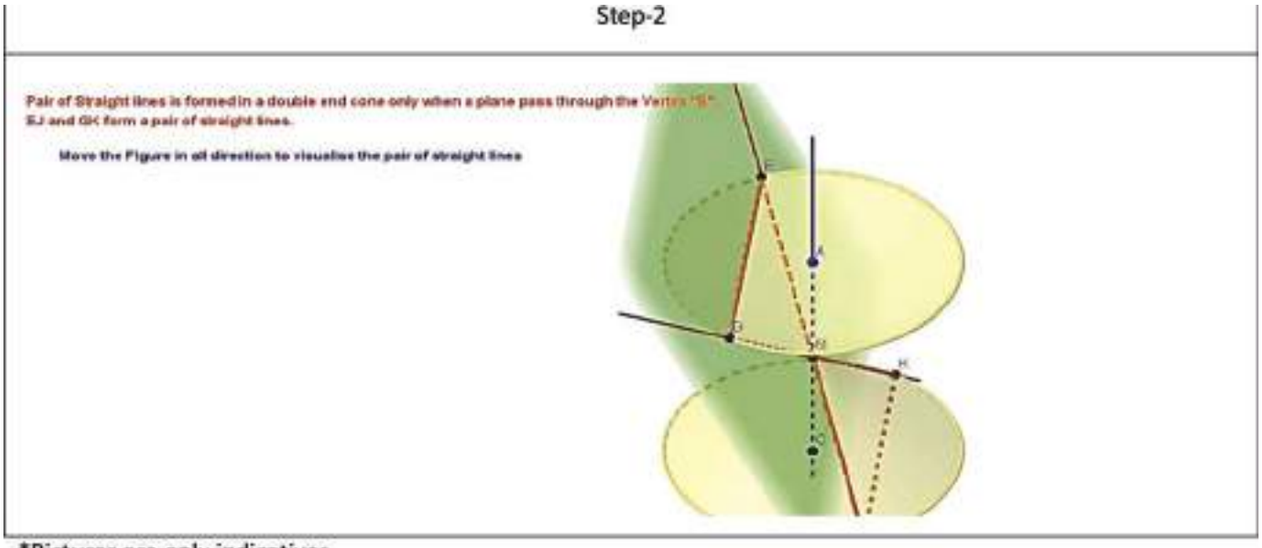
Step–2
GeoGebra work book called “XI STD Analytical Geometry” will open. There are 7 worksheets given. Select the one you want. For example, select “Pair of Straight Line in Double End Cone”

Step–3
A plane cuts the Double end Cone at 4 points at its edges and also it passes through the Vertex. Now If you move by holding the points E or D the plane moves. You can observe at any position you get a pair of straight line. You can rotate the picture in any direction. Explore.

Step-1



Step-2



*Pictures are only indicatives.

Browse in the link
Analytical Geometry GeoGebra Workbook: `https://ggbm.at/QNaNghJZ`

Answers

Exercise 1.1

(1) (i) $\{2, 3, 5, 7\}$ (ii) $\{1\}$ (iii) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (iv) $\{-5\}$

(2) $\{x \in \mathbb{R} : x^2 = 1\}$

(3) (i) finite (ii) infinite (iii) infinite (iv) infinite (v) infinite

(5) not true (6) 0 (7) 128 (8) $\{0, 1, 2, 3\}$

(9) $A = \{x, y, z\}$ and $B = \{1, 2\}$ (10) $\{(-1, 0), (-1, 1), (0, 2), (1, 2)\}$

Exercise 1.2

(1) (i) reflexive; not symmetric; transitive (ii) not reflexive; symmetric; not transitive

(iii) reflexive; not symmetric ;not transitive (iv) reflexive; symmetric

(v) R is an empty set; not reflexive; symmetric; transitive

(2) (i) (c, c) and (d, d) (ii) (c, a)

(iii) nothing to include (iv) $(c, c), (d, d), (c, a)$ to be included

(3) (i) (c, c) (ii) (c, a) (iii) nothing (iv) (c, c) and (c, a)

(5) $\{(3, 8), (6, 6), (9, 4), (12, 2)\}$; not reflexive; not symmetric; transitive; not an equivalence relation

(7) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1),$

$(4, 2), (5, 1)\}$; not reflexive; symmetric; not transitive; not equivalence

(8) smallest set is $\{(a, a), (b, b), (c, c)\}$; largest set is $A \times A$

Exercise 1.3

(1) Yes; inverse is not a function

(2) $f(-4) = 8$, $f(1) = 0$, $f(-2) = 6$, $f(7) = 0$, $f(0) = 0$

(3) $f(-3) = 1$, $f(5) = 38$, $f(2) = 1$, $f(-1) = -5$, $f(0) = -3$

(4) (i) a function; not one–to–one and not onto (ii) not a function

(5) (i) $\{(1, a), (2, a), (3, a), (4, a)\}$ (ii) not possible (iii) not possible

(iv) $\{(1, a), (2, b), (3, c), (4, d)\}$

(6) $\mathbb{R} - \{n\pi + (-1)^n \frac{\pi}{6}\}, n \in \mathbb{Z}$ (7) $\emptyset$

(8) $(-\infty, -\frac{1}{3}] \cup [1, \infty)$ (9) $\mathbb{R} - \{0\}, \mathbb{R} - \{0\}$

(10) for all x, $(f \circ g) = \mathbf{O}, (g \circ f) = \mathbf{O}$ (12) $f^{-1}(\text{x}) = \dfrac{x+5}{3}$ (13) $x > 0$

(15) total cost $= 0.43m + 50$, airfare = ₹738

(16) $(A + S)(\text{x}) = 55,000 + 0.09x$, total family income = ₹$14,05,000$

(17) $(g \circ f)(\text{x}) = 62.115x$

(18) day revenue = $(200 - x)x$; total cost =$100(200 - x)$;

total profit $= (200 - x)x - 100(200 - x) in rupees$

(19) $f^{-1}(\text{x}) = \dfrac{9x}{5} + 32$ (20) $f^{-1}(\text{x}) = \dfrac{x+4}{3}$

Exercise 1.5

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(3)	(2)	(4)	(1)	(1)	(4)	(2)	(2)	(3)	(2)	(2)	(3)	(3)	(2)	(4)
16	17	18	19	20	21	22	23	24	25					
(3)	(4)	(3)	(2)	(4)	(4)	(1)	(4)	(2)	(3)					

Exercise 2.1

(1) $\sqrt{7} \in \mathbb{R} - \mathbb{Q}$, $-\dfrac{1}{4} \in \mathbb{Q}$, $0 \in \mathbb{Z}, \mathbb{Q}$,

$3.14 \in \mathbb{Q}$, $4 \in \mathbb{N}, \mathbb{Z}, \mathbb{Q}$, $\dfrac{22}{7} \in \mathbb{Q}$

(3) yes, $4 + \sqrt{3}$, $2 + \sqrt{3}$ (4) yes, $2 + \sqrt{3}, 2 - \sqrt{3}$ (5) $\dfrac{1}{2^{1001}}$

Exercise 2.2

(1)(i) $-4 < x < 10$ (ii) $\mathbb{R}$ (iii) $\dfrac{11}{3} \le x \le \dfrac{13}{3}$ (iv) $-7 < x < 7$

(2) $\left(-\infty, \dfrac{5}{12}\right) \cup \left(\dfrac{7}{12}, \infty\right)$ (3) $\left(-\infty, \dfrac{-7}{3}\right] \cup \left[\dfrac{7}{3}, \infty\right)$ (4) $-\dfrac{15}{2} \le x \le \dfrac{11}{2}$

(5) $-\dfrac{3}{10} < x < \dfrac{7}{10}$ (6) no solution

Exercise 2.3

(1)(i) $[-1, 4)$ (ii) $[-3, 5]$ (iii) $(-\infty, 3)$ (iv) $(-\infty, 5)$

(2)(i) $1, 2, 3, 4$ (ii) $\ldots, -3, -2, -1, 0, 1, 2, 3, 4$

(3)(i) $\left(-\infty, -\dfrac{9}{2}\right]$ (ii) $\ldots, -7, -6, -5$ (iii) no solution

(5) 93 (6) between120ℓ and 300ℓ (7) $(11, 13), (13, 15), (15, 17), (17, 19)$

(8) $t = 9$seconds, 11seconds (9) less than10 hours (10) less than ₹$21,000$ or greater than ₹33000

Exercise 2.4

(1) $x^2 - 4x - 21 = 0$ (2) $-\dfrac{2}{5}(x^2 - 2x - 4)$ (3) $x^2 + \dfrac{\sqrt{2}}{3}x + \dfrac{1}{3} = 0$

(6) (i) $b = 0$ (ii) $3b^2 = 16ac$ (iii) $c = a$

(8) (i) real and distinct (ii) real and distinct (iii) real and distinct

(9) (i) not intersect (ii) intersect at two points (iii) touch at only one point

(10) $\left(x+\frac{5}{2}\right)^2-\left(\frac{3}{2}\right)^2$

Exercise 2.5

(1) $\left[-3,\frac{5}{2}\right]$ (2) $[1,2]$

Exercise 2.6

(1) $-\frac{5}{2}$ and $\frac{5}{2}$ (2) $\frac{3+\sqrt{53}}{2}$ and $\frac{3-\sqrt{53}}{2}$ (3) $x=\pm 2$ (4) $x=-\frac{3}{5}$ or -1

Exercise 2.7

(1) $(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$ (2) $a=5$

Exercise 2.8

(1) $(0,1)\cup(2,\infty)$ (2) $\left(-\infty,\frac{3}{2}\right)\cup(2,4)$ (3) $(-3,-2]\cup[2,5)$

Exercise 2.9

(1) $\frac{1}{2a(x-a)}-\frac{1}{2a(x+a)}$ (2) $\frac{7}{3(x-2)}+\frac{2}{3(x+1)}$

(3) $\frac{1}{6(x-1)}+\frac{2}{15(x+2)}+\frac{-3x+1}{10(x^2+1)}$ (4) $\frac{1}{(x-1)^2}+\frac{1}{(x-1)^3}$

(5) $\frac{1}{2(x^2-1)}-\frac{2}{2(x^2+1)}$ (6) $\frac{1}{x}-\frac{2}{x^2+1}$

(7) $1+\frac{13}{x-3}-\frac{7}{x-2}$ (8) $x+\frac{9}{x+2}-\frac{13}{x+3}$

(9) $-\frac{14}{9(x+1)}-\frac{11}{3(x+1)^2}+\frac{14}{9(x-2)}$ (10) $\frac{4}{x+1}+\frac{2x-3}{x^2+1}$

(11) $2+\left(\frac{2}{x+3}-\frac{1}{x-1}\right)$ (12) $\frac{3}{x+1}+\frac{4-3x}{x^2+1}$

Exercise 2.11

(1)(i) 25 (ii) $\frac{1}{8}$ (iii) $\frac{1}{100}$ (iv) $\frac{1}{9}$ (v) $\frac{1}{3}$

(2) 8 (3) $\frac{1}{\sqrt{2}}$ (4) $\frac{1}{2}$ (5) 2

(6) $\dfrac{21+7\sqrt{2}+3\sqrt{6}+2\sqrt{3}}{7}$ (7) 5 (8) $\dfrac{2+2\sqrt{6}}{5}$

Exercise 2.12

(1) $\log_b y = x$, $(0,\infty)$, $(-\infty,\infty)$

(2) $\dfrac{5}{6}$ (3) 64 (4) 2 (11) $2\sqrt{2}$ (12) -10

Exercise 2.13

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(2)	(1)	(2)	(3)	(2)	(2)	(3)	(1)	(2)	(2)	(2)	(3)	(3)	(2)	(3)
16	17	18	19	20										
(1)	(3)	(1)	(1)	(4)										

Exercise 3.1

(1)(i) first quadrant (ii) second quadrant (iii) fourth quadrant (iv) fourth quadrant (v) second quadrant

(2)(i) 35° (ii) 165° (iii) 70° (iv) 90° (v) 270° (8) $k \in [-1,1]$

(9) $\sec\theta = \dfrac{p^2+1}{2p}$; $\tan\theta = \dfrac{p^2-1}{2p}$; $\sin\theta = \dfrac{p^2-1}{p^2+1}$ (12) $\left(c^2+bd\right)^2 = (ad+cb)^2 + \left(ac-b^2\right)^2$

Exercise 3.2

(1)(i) $\dfrac{\pi}{6}$ radians (ii) $\dfrac{3\pi}{4}$ radians (iii) $-\dfrac{41\pi}{36}$ radians (iv) $\dfrac{5\pi}{6}$ radians (v) $\dfrac{11\pi}{6}$ radians

(2)(i) 60° (ii) 20° (iii) 72° (iv) 420° (v) 200°

(3) $r \approx 31.82$ meters (4) $s = \dfrac{20\pi}{3} = 20.92$cm (5) $\theta = 12^\circ 36'$ (6) $s = 7.16$feet

(7) $r_1 : r_2 = 5 : 4$

(8) Angle of sector $\approx 65^\circ 27' 16''$ (9) 6000° (10) 14° (11) $\dfrac{3\pi}{4}$

Exercise 3.3

(1)(i) $\dfrac{\sqrt{3}}{2}$ (ii) $-\dfrac{1}{2}$ (iii) $\dfrac{1}{2}$ (iv) $-\dfrac{1}{\sqrt{3}}$ (v) $-\dfrac{1}{\sqrt{3}}$ (vi) $\sqrt{3}$ (vii) $\dfrac{\sqrt{3}}{2}$

(2) $\sin\theta = \dfrac{2\sqrt{6}}{7}$; $\cos\theta = \dfrac{5}{7}$; $\tan\theta = \dfrac{2\sqrt{6}}{5}$;

$\operatorname{cosec}\theta = \dfrac{7}{2\sqrt{6}}$; $\sec\theta = \dfrac{7}{5}$; $\cot\theta = \dfrac{5}{2\sqrt{6}}$

(3)(i) $\sin\theta = -\frac{\sqrt{3}}{2}$; $\operatorname{cosec}\theta = -\frac{2}{\sqrt{3}}$; $\sec\theta = -2$; $\tan\theta = \sqrt{3}$; $\cot\theta = \frac{1}{\sqrt{3}}$

(ii) $\sin\theta = \frac{\sqrt{5}}{3}$; $\operatorname{cosec}\theta = \frac{3}{\sqrt{5}}$; $\sec\theta = \frac{3}{2}$; $\tan\theta = \frac{\sqrt{5}}{2}$; $\cot\theta = \frac{2}{\sqrt{5}}$

(iii) $\cos\theta = \frac{\sqrt{5}}{3}$; $\operatorname{cosec}\theta = -\frac{3}{2}$; $\sec\theta = \frac{3}{\sqrt{5}}$; $\tan\theta = -\frac{2}{\sqrt{5}}$; $\cot\theta = -\frac{\sqrt{5}}{2}$

(iv) $\sec\theta = -\sqrt{5}$; $\cos\theta = -\frac{1}{\sqrt{5}}$; $\cot\theta = -\frac{1}{2}$; $\sin\theta = \frac{2}{\sqrt{5}}$; $\operatorname{cosec}\theta = \frac{\sqrt{5}}{2}$;

(v) $\cos\theta = \frac{5}{13}$; $\sin\theta = -\frac{12}{13}$; $\operatorname{cosec}\theta = -\frac{13}{12}$; $\tan\theta = -\frac{12}{5}$; $\cot\theta = -\frac{5}{12}$;

(5) $\theta = 60°, 120°, 240°, 300°$,

Exercise 3.4

(1)(i) $\sin(x+y) = \frac{220}{221}$ (ii) $\cos(x-y) = \frac{171}{221}$ (iii) $\tan(x+y) = \frac{220}{21}$

(2)(i) $\sin(A+B) = \frac{187}{205}$ (ii) $\cos(A-B) = \frac{156}{205}$ (3) $\cos(x-y) = \frac{4}{5}$

(4) $\sin(x-y) = -\frac{87}{425}$ (5) $\cos 105° = \frac{1-\sqrt{3}}{2\sqrt{2}}$; $\sin 105° = \frac{\sqrt{3}+1}{2\sqrt{2}}$;

$\tan\left(\frac{7\pi}{12}\right) = -(2+\sqrt{3})$

(7) $4x^2 - 2\sqrt{6}x + 1 = 0$ (16) 0 (22) 1 (24) $\frac{2}{11}$

Exercise 3.5

(1)(i) $\frac{161}{289}$ (ii) $-\frac{7}{25}$ (iii) $\frac{3713}{4225}$ (2)(i) $\frac{2\sqrt{3}}{5}$, $\frac{1}{3\sqrt{2}}$

Exercise 3.6

(1)(i) $\frac{1}{2}[\sin 63° + \sin 7°]$ (ii) $\frac{1}{2}[\sin 6x + \sin 2x]$ (iii) $\frac{1}{2}[\sin 12\theta + \sin 8\theta]$

(iv) $\frac{1}{2}[\cos 7\theta + \cos 3\theta]$ (v) $\frac{1}{2}[\cos\theta - \cos 9\theta]$ (2)(i) $2\cos 55° \sin 20°$

(ii) $2\cos 40° \cos 25°$ (iii) $2\sin 45° \cos 5°$ (iv) $2\sin 55° \sin 20°$

Exercise 3.8

(1)(i) $\theta = -\frac{\pi}{4}$; $\theta = n\pi + (-1)^n\left(-\frac{\pi}{4}\right)$. $n \in \mathbb{Z}$

(ii) $\theta = \frac{\pi}{6}$; $\theta = n\pi + \frac{\pi}{6}$, $n \in \mathbb{Z}$

(iii) $\theta = -\frac{\pi}{6}$; $\theta = n\pi + \left(\frac{-\pi}{6}\right)$, $n \in \mathbb{Z}$

(2)(i) $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ (ii) $x = \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$

(iii) $x = \frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ (iv) $x = 0, \pi$

(3)(i) $x = (2n+1)\frac{\pi}{6}$ or $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$, $n \in \mathbb{Z}$

(ii) $\theta = n\pi + (-1)^n \frac{\pi}{6}$ or $\theta = n\pi + (-1)^n \frac{\pi}{2}$ $n \in \mathbb{Z}$

(iii) $\theta = (2n+1)\frac{\pi}{4}$ or $\theta = 2n\pi$, $n \in \mathbb{Z}$

(iv) $\theta = \frac{n\pi}{3}$ or $\theta = n\pi \pm \frac{\pi}{3}$, $n \in \mathbb{Z}$

(v) $\theta = 2n\pi$ or $\theta = \frac{2n\pi}{3} + \left(\frac{-\pi}{6}\right)$, $n \in \mathbb{Z}$

(vi) $\theta = (8n+1)\frac{\pi}{4}$, $n \in \mathbb{Z}$

(vii) $\theta = 2n\pi + \frac{\pi}{6} \pm \frac{\pi}{3}$, $n \in \mathbb{Z}$

(viii) $\theta = 2n\pi - \frac{\pi}{2} \pm \frac{2\pi}{3}$, $n \in \mathbb{Z}$

(ix) $\theta = \frac{n\pi}{3} + \frac{\pi}{18}$, $n \in \mathbb{Z}$

(x) $\theta = n\pi \pm \frac{\pi}{10}$, $n \in \mathbb{Z}$

(xi) $x = 2n\pi \pm \frac{\pi}{3}$ $n \in \mathbb{Z}$

Exercise 3.9

(2) $\angle A = 75°$ (9) $40m, 40m, 40m$ (10) $4m,\ 4m,\ 4m$ and $4\sqrt{3}\ sq.meter.$

Exercise 3.10

(1) no such triangle exist (3) $\angle A = 15°, \angle B = 105°$ (7) $\frac{5\sqrt{2}}{\sqrt{3}-1}$ km (8) $2\sqrt{13}$ km

(9) $3 + \sqrt{73}$ km (10) 7 km

(11) Total Cost: 155800 and Perimeter $180 + 20\sqrt{27}$feet

(12) $x = 100$ km (13) $\sqrt{5 - 2\sqrt{2}}$ km

(14) $2\sqrt{6} + 2\sqrt{3} + 6$ km (15) $AB = 10\sqrt{3}$ km

Exercise 3.11

(1)(i) $\theta = \frac{\pi}{4}$ (ii) $\theta = \frac{\pi}{6}$ (iii) $\theta = -\frac{\pi}{2}$ (iv) $\theta = \frac{3\pi}{4}$ (v) $\theta = \frac{\pi}{3}$

Exercise 3.12

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(4)	(1)	(1)	(1)	(4)	(4)	(1)	(2)	(4)	(2)	(3)	(2)	(3)	(3)	(2)
16	17	18	19	20										
(4)	(1)	(1)	(1)	(1)										

Exercise 4.1

(1)(i)	17	(ii)	6	(iii)	20	(iv)	720	(v)	120		
(2)(i)	151200	(ii)	24	(3)(i)	12	(ii)	24	(4)(i)	64	(ii)	24
(5)(i)	90	(ii)	64	(6)(i)	144	(ii)	80	(7)(i)	48	(ii)	90
(8)(i)	9000	(ii)	4536	(iii)	4464	(9)(i)	36	(ii)	60	(10)	13
(11)	400	(12)(i)	42	(ii)	78	(13)(i)	4^6	(ii)	3^{10}	(iii)	10^{12}
(14)(i)	720	(ii)	144	(iii)	4	(iv)	144	(v)	220	(vi)	$(n+3)(n+2)$
(15)(i)	15	(ii)	120	(iii)	$\frac{n(n-1)}{2}$	16.(i)	4	(ii)	100		

Exercise 4.2

(1) 10 (2) 4 (3)(i) 336 (ii) 172800 (4) 720

(5)(i) 4^{10} (ii) $3^4 \times 5^6$ (iii) 11! (6)(i) 4^5 (ii) 15^5

(7) 144 (8)(i) 14! (ii) $9! \times 6!$ (iii) $8! \times {}^9P_6$ (9) 34650

(10) 1260 (11) 6912 (12) 60 (13)(i) 2^8 (ii) 28

(14)(i) 43200 (ii) 151200 (iii) 19807200 (iv) 151200 (15)(i) 180
(ii) 60 (iii) 30 (16)(i) 379 (ii) 135 (17) 120, NIGHT

(18) 7 (19) 399960 (20) 571956

Exercise 4.3

(1) 1 (2) 3 (3) 10, 3 (6) 20

(9) (i) ${}^{14}C_7 = 3432$ (ii) ${}^{15}C_2 = 105$ (iii) ${}^{20}C_2 = 190$ (iv) ${}^{100}C_5$ (v) ${}^5C_3 \times {}^4C_2 \times {}^2C_1 = 120$

(10) (i) $2^4 = 16$ (ii) $2^5 = 32$ (iii) 2^n

(11) (i) ${}^{25}C_3$ (ii) ${}^{25}P_3$

(12) ${}^{10}P_2 \times {}^8C_4 = 6300$ (13)(i) ${}^{10}C_3 = 120$ (ii) ${}^{10}C_5 = 252$

(14) ${}^5C_2 \times {}^{20}C_3 = 11400$ (i) ${}^4C_1 \times {}^{20}C_3 = 4560$ (ii) ${}^5C_2 \times {}^{19}C_3 = 9690$

(15) ${}^7C_3 = 35$ (16) 4512 (17) 546 (18)(i) 280 (ii) 336 (iii) 736

(19) 485 (20) 64 (21) 2454

(22) ${}^{15}C_3 = 455$ (23) 364 (24)(i) 50 (ii) 161 (25) 15

Exercise 4.5

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(2)	(2)	(1)	(2)	(4)	(2)	(1)	(4)	(2)	(2)	(3)	(4)	(2)	(3)	(1)
16	17	18	19	20	21	22	23	24	25					
(4)	(2)	(3)	(4)	(3)	(2)	(2)	(2)	(1)	(2)					

Exercise 5.1

(1)(i) $8x^6 - 36x^3 + 54 - \frac{27}{x^3}$ (ii) $2[16x^8 + 216x^4(1-x^2) + 81(1-x^2)^2]$

(2)(i) 108243216 (ii) 96059601 (iii) 4782969

(3) $(1.01)^{10^6} > 10000$ (4) 10 (5) $15, x^6$ term is not possible.

(6) 26235 (7) $-\frac{40}{27}$ (8) $'01'$

(13) $n = 15$ (14) $n = 55$ (15) $n = 7$ or 14

Exercise 5.2

(1)(i) G.P (ii) None of them. (iii) G.P

(iv) None of them (v) None of them (vi) None of them (vii) A.G.P

(2)(i) $2, 2, 4, 4, 6, 6, \ldots$ (ii) $1, 2, 3, 5, 8, 13, \ldots$ (iii) $1, 2, 3, 6, 11, 20, \ldots$

(3)(i) $a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$ (ii) $a_n = \frac{n}{n+1}, \forall n \in \mathbb{N}$ (iii) $a_n = \frac{2n-1}{2n}, \forall n \in \mathbb{N}$

(iv) $a_n = \begin{cases} 7-n & \text{if } n \text{ is odd} \\ 8+n & \text{if } n \text{ is even} \end{cases}$

(4) $12, 18, 27$ (5) $t_n = \frac{1}{n^2} - \frac{1}{(n+1)^2}$ (8) $5, 45$

Exercise 5.3

(1) $a = \frac{133}{25}, d = -\frac{2}{75}, S_{20} = \frac{304}{3}$ (2) $S_{17} = 527$

(3)(i) $S_n = \frac{8}{81}[10(10^n - 1) - 9n]$ (ii) $S_n = \frac{6}{81}[10(10^n - 1) - 9n]$

(4) $S_n = \frac{4}{9}(4^n - 1) - \frac{n}{3}$ (5) $\frac{3n-2}{3^{n-1}}, \frac{3^n - (3n-2)}{2 \cdot 3^{n-1}} + \frac{3^{n-1} - 1}{4 \cdot 3^{n-3}}$

(6) $n = 15$ (8) 20 months (9) 2480 metres (10) $120, 480, 30(2)^n$

(11) $500\left(\frac{11}{10}\right)^{10} = 1296.87$ (12) 15th day

Exercise 5.4

(1)(i) $\frac{1}{5}\left[1-\frac{x}{5}+\frac{x^2}{25}-\frac{x^3}{125}+\dots\right]$ $|x|<5$

(ii) $\frac{2}{9}\left[1-2\left(\frac{4x}{3}\right)+3\left(\frac{4x}{3}\right)^2-4\left(\frac{4x}{3}\right)^3+\dots\right]$ $|x|<\frac{3}{4}$

(iii) $5^{\left(\frac{2}{3}\right)}\left[1+\frac{2}{15}(\mathrm{x})^2-\frac{1}{225}x^4+\frac{4}{81\times 125}x^6+\dots\right]$ $x^2<5$

(iv) $2^{\left(-\frac{2}{3}\right)}\left[1-\frac{x}{3}+\frac{5}{36}x^2-\frac{5}{81}x^3+\dots\right]$ $|x|<2$

(2) $(1001)^{\left(\frac{1}{3}\right)}$ $\approx$ 10.00333

(5)(i) $1+5x+\frac{25x^2}{2}+\frac{125x^3}{6}+\frac{625x^4}{24}+\frac{625x^5}{24}+\dots$

(ii) $1-2x+2x^2-\frac{4x^3}{3}+\frac{2x^4}{3}-\frac{4x^5}{15}+\dots$

(iii) $1+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{48}x^3+\frac{1}{384}x^4+\dots$

(6)(i) $4x-8x^2+\frac{64x^3}{3}-\frac{64x^4}{3}+\dots$ for $|x|<\frac{1}{4}$

(ii) $-2x-\frac{4x^2}{2}-\frac{8x^3}{3}-\frac{16x^4}{4}-\dots$ for $|x|<\frac{1}{2}$

(iii) $2[3x+\frac{27x^3}{3}+\frac{243x^5}{5}+\frac{2187x^7}{7}+\dots]$ for $|x|<\frac{1}{3}$

(iv) $-2[2x+\frac{8x^3}{3}+\frac{32x^5}{5}+\frac{128x^7}{7}+\dots]$ for $|x|<\frac{1}{2}$

(8) $\left(\frac{15}{16}\right)^{\frac{1}{8}}\simeq 0.99196$ (9) $\frac{28}{3}$ (10) $\frac{1}{2}\log_e^{10}$

Exercise 5.5

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(4)	(4)	(4)	(4)	(2)	(3)	(1)	(3)	(4)	(1)	(1)	(4)	(4)	(2)	(3)
16	17	18	19	20										
(2)	(2)	(3)	(3)	(2)										

Exercise 6.1

(1)(i) $x^2+y^2=81$ (ii) $\frac{x^2}{81}+\frac{y^2}{36}=1$ (2)(i) $y=\pm 2$ (ii) $x=\pm 3$

(3) $x^{2/3}+y^{2/3}=a^{2/3}$ (4) $k=-24,\quad b=-\frac{1}{4}$ (5) $x^2+y^2=16$

(6) $x^2+y^2-4x-4y+8=0$ (7) $x^2+y^2-5x+8y+16=0$

(8) $y^2=2x$ (10) $8x^2+36y^2-16x+252y+431=0$

(11) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (12) $9x^2-33x-3y+35=0$

(13) $x^2+y^2-12x+9y+63=0$ (14) $(5-\sqrt{21},3),(5+\sqrt{21},3),(8,-3),(2,-3)$

(15) $\frac{x^2}{25}+\frac{y^2}{9}=1$

Exercise 6.2

(1)(i) $y=5x-4$ (ii) $3x-y=2$ (iii) $2x+3y=5$ (iv) $x+\sqrt{3}y=(1+\sqrt{3})$

(3) $10x+3y=25$

(5)(i) $C=\frac{5}{9}(F-32)$ (or) $F=\frac{9}{5}C+32$ (ii) $C=37°$ (iii) $F=100.4°$

(6)(i) 4400metre (ii) $D=3000$metre (iii) $T=22seconds$ (7) $P=1,55,000$

(8) $\sqrt{(3)}x+y=24$ (9) $3x-4y=12, x-2y=2$ (11) $(13,7),(-11,-3)$

(12)(i) $y=12.5x-150$ (ii) 12seconds (iii) 80seconds

(13)(ii) $x-2y+4=0$ (iii) $2cm$ (iv) $14kg$ (v) $5cm$

(14)(i) $y=-\frac{71}{120}x+14.2, \quad 0\le x\le 24$

(ii) $y=f(\text{x})$ is a periodic function with period $24, f(\text{x})=f(x+24)$

(15)(i) The minimum length $=3280$units

(ii) $180, 360$ and 540units

(iii) The slope at each turning point is $\frac{9}{40}$

Exercise 6.3

(2) $5x-4y-15=0$ (3)(i) $\frac{8}{5}$ (ii) $\frac{23}{5}$

(4)(i) $x+3y+2=0$ (ii) $4x-3y-7=0$ (5) $x+5y-31=0$

(6)(i) $x+1=0$ (ii) $x-y=0$ (iii) $2x+y+3=0$

(7) $12x+5y+6=0,$ and $12x+5y-20=0$ (8) $4x-3y+15=0,$ and $4x-3y-25=0$

(9) $2x+3y-18=0$ (10) $7\sqrt{2},$ and $(-3,5)$

(12) (i)$\frac{14}{13}$, (ii)$\frac{5}{2}$ (13)(i) $4x-3y+k=0, k\in\mathbb{R}$ (ii) $3x+4y+k_1=0, k_1\in\mathbb{R}$

(14) $\sqrt{3}x-y-2\sqrt{3}=0$ (15) $A\left(\frac{13}{5},0\right)$ (16) $x+5y=\pm 10$

(17) $(0,7)$ (18)(i) $y=\begin{cases}1.50x, & 0\le x\le 10\\ x+5, & x>10\end{cases}$ (ii) ₹45

(19) $y=5x-7, \quad y=5x+10$

(20) $y+3=0,$ $2x+y+3=0,$ and $2x-y-3=0$

Exercise 6.4

(1) $x^2 - xy - 2y^2 + 2x - 13y - 15 = 0$

(6) $3x^2 - 13xy - 10y^2 + 33x + 73y - 126 = 0$ (7)(i) $x + y = 0, 3x - y = 0$

(ii) $3x + 4y - 11 = 0, 2x - y = 0$ (iii) $x + y - 5 = 0, 2x - 3y + 4 = 0$

(10) $y = x$

(11) $p = 6, q = 17(\text{or}) - \frac{67}{6}$ (12) $k = -1$

(13) $k = -5(\text{or}) - \frac{35}{4}$ (14) $\frac{8}{5}$ (15) $\sqrt{5}$

Exercise 6.5

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(4)	(4)	(3)	(4)	(3)	(2)	(2)	(4)	(2)	(3)	(1)	(2)	(3)	(4)	(2)
16	17	18	19	20	21	22	23	24	25					
(1)	(3)	(2)	(1)	(2)	(1)	(3)	(1)	(3)	(4)					

Glossary

Chapter 1 Sets, Relations and Functions

absolute value function or modulus function	மட்டு சார்பு அல்லது எண்ணளவுச் சார்பு
basic relations	அடிப்படைத் தொடர்புகள்
bijective function	இருபுறச் சார்பு
cardinality	செவ்வெண்மை
closed interval	மூடிய இடைவெளி
co-domain	துணைச் சார்பகம்
composition of functions	சார்புகளின் சேர்ப்பு
constant	மாறிலி
constant function	மாறிலிச் சார்பு
dependent variable	சார்ந்த மாறி
dilation	விரிதல்
disjoint set	வெட்டாக் கணம்
domain	சார்பகம்
empty relation	வெற்றுத் தொடர்பு
empty set or void set	வெற்றுகணம்
equivalence relation	சமானத் தொடர்பு
even function	இரட்டைப் படைச் சார்பு
exponential function	அடுக்குக் குறிச் சார்பு
extreme relations	உச்சத் தொடர்புகள்
finite set	முடிவுறு கணம்
greatest integer function	மீப்பெரு முழுஎண் சார்பு
horizontal line test	கிடை மட்டக் கோட்டுச் சோதனை
identity function	சமனிச் சார்பு
image	பிம்பம்
improper subset	தகா உட்கணம்
independent variable	சாரா மாறி
infinite set	முடிவுறா கணம்
inverse	நேர்மாறு
invertible	நேர்மாற்றுத்தன்மை

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linear function	நேரியச் சார்பு
logarithmic function	மடக்கைச் சார்பு
odd function	ஒற்றைப் படைச் சார்பு
one-to-one function	ஒன்றுக்கொன்றான சார்பு
onto function	மேற்கோர்த்தல் சார்பு
open interval	திறந்த இடைவெளி
power set	அடுக்கு கணம்
pre-image	முன் பிம்பம்
proper subset	தகு உட்கணம்
range	வீச்சகம்
rational function	விகிதமுறுச் சார்பு
real line	மெய்யெண் கோடு
real-valued function	மெய் மதிப்புச் சார்பு
reciprocal function	தலைகீழ் சார்பு
reflection	பிரதிபலிப்பு
reflexive relation	தற்சுட்டுத் தொடர்பு
set difference	கண வேறுபாடு
signum function	குறியீட்டுச் சார்பு
singleton set	ஒருறுப்பு கணம்
smallest integer function	மீச்சிறு முழு எண் சார்பு
step functions	படி நிலைச் சார்புகள்
subset	உட்கணம்
symmetric difference	சமச்சீர் வேறுபாடு
transitive relation	கடப்புத் தொடர்பு
translation	இடப்பெயர்ச்சி
trivial subset	வெள்ளிடை உட்கணம்
union of sets	சேர்ப்புக் கணம்
universal relation	அனைத்துத் தொடர்பு
variable	மாறி
vertical line test	நிலைக் குத்துக் கோட்டுச் சோதனை

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zero function	பூஜ்ஜியச் சார்பு
set	கணம்
super set	மேற்கணம் அல்லது மிகைக் கணம்

Chapter 2 Basic Algebra

conjugate	இணை
discriminant	தன்மைக் காட்டி
division algorithm	வகுத்தல் கோட்பாடு
exponents	அடுக்குக் குறி
inequality	அசமன்பாடு
partial fraction	பகுதிப் பின்னம்
polynomial expression	பல்லுறுப்புக் கோவை
quartic	நான்காம் படி
quintic	ஐந்தாம் படி
radical	படிமூலம்
rational inequality	விகிதமுறு அசமன்பாடு
remainder theorem	மீதித் தேற்றம்

Chapter 3 Trigonometry

allied angles	தொடர்புடைக் கோணங்கள்
angles in standard position	திட்டநிலையில்கோணங்கள்
altitude of a triangle	முக்கோணத்தின் உயரம்
centesimal system	நூற்றின் கூறுமுறை
chord	நாண்
circular system	வட்ட முறைஅமைப்பு
circum centre	சுற்றுவட்ட மையம்
circum circle	சுற்றுவட்டம்
circum radius	சுற்றுவட்டத்தின் ஆரம்
complementary angles	நிரப்புக் கோணங்கள்
compound angles	கூட்டுக் கோணங்கள்
conjugate angles	இணையியக் கோணங்கள்

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general solution	பொதுத் தீர்வு
interval	இடைவெளி
inverse trigonometric functions	நேர்மாறு முக்கோணவியல் சார்புகள்
periodicity	காலமுறைப் பண்பு
power reducing identities	அடுக்குக் குறைப்பு முற்றொருமைகள்
principal solution	முதன்மைத் தீர்வு
projection	வீழல்
quadrant	காற்பகுதி
radian	ரேடியன்
sector	வட்டக்கோணப்பகுதி
sub-multiple angles	உட்மடங்குக் கோணங்கள்
supplementary angles	மிகை நிரப்புக்கோணங்கள்
trigonometric identities	முக்கோணவியல் முற்றொருமைகள்
trigonometric ratios	முக்கோணவியல் விகிதங்கள்

Chapter 4 – Combinatorics and Mathematical Induction

cryptography	குறியாக்கவியல்
factorial	காரணியப் பெருக்கம்
heptagon	எழுகோணம்
inclusion-exclusion	சேர்த்தல்–நீக்கல்
inductive step	தொகுக்கும் நிலை
mathematical induction	கணிதத் தொகுத்தறிதல் முறை
pentagon	ஐங்கோணம்
permutation	வரிசைமாற்றம்
polygon	பலகோணம்
product rule	பெருக்கல்விதி
string method	கட்டுதல் முறை
sum rule	கூட்டல் விதி

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Chapter 5 Binomial Theorem, Sequences and Series

arithmetic mean	கூட்டுச் சராசரி
arithmetic progression	கூட்டுத் தொடர்முறை
arithmetic series	கூட்டுத் தொடர்
arithmetico–geometric progression	கூட்டுப் பெருக்குத் தொடர்முறை
arithmetico–geometric series	கூட்டுப் பெருக்குத்தொடர்
binomial coefficients	ஈருறுப்புக் கெழுக்கள்
binomial expansion	ஈருறுப்பு விரிவு
binomial series	ஈருறுப்புத் தொடர்
binomial theorem	ஈருறுப்புத் தேற்றம்
common difference	பொது வித்தியாசம்
common ratio	பொது விகிதம்
convergent series	ஒருங்குத் தொடர்
exponential series	அடுக்குக்குறித் தொடர்
finite sequence	முடிவுறுத் தொடர்முறை
finite series	முடிவுறு தொடர்
geometric mean	பெருக்குச் சராசரி
geometric progression	பெருக்குத் தொடர்முறை
geometric series	பெருக்குத் தொடர்
harmonic mean	இசைச் சராசரி
harmonic progression	இசைத் தொடர்முறை
infinite sequence	முடிவுறாத் தொடர்முறை
infinite series	முடிவுறாத் தொடர்
initial term	முதல் உறுப்பு
logarithmic series	மடக்கைத்தொடர்
partial sum	பகுதிக் கூடுதல்
rational exponent	விகிதமுறு அடுக்கு
telescopic summation	தொலைநோக்கிக் கூடுதல்

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Chapter 6 Two Dimensional Analytical Geometry

angle of inclination	சாய்வுக் கோணம்
arbitrary constant	மாறத்தக்க மாறிலி
bisector	இருசமவெட்டி
equilibrium	சமநிலை
fixed constant	நிலையான மாறிலி
fixed point	நிலைப்புள்ளி
general form	பொது வடிவம்
homogenous equation	சமப்படித்தான சமன்பாடு
intercept	வெட்டுத்துண்டு
locus	நியமப் பாதை (அ) இயங்குவரை
negative intercept	குறை வெட்டுத்துண்டு
negative slope	குறை சாய்வு
non–homogenous equation	அசமப்படித்தான சமன்பாடு
normal form	செங்குத்து வடிவம்
parameter	துணையலகு
parametric form	துணையலகு வடிவம்
positive intercept	மிகை வெட்டுத்துண்டு
positive slope	மிகை சாய்வு
symmetry	சமச்சீர்
two point form	இருபுள்ளிகள் வடிவம்

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Chapter 7 Matrices and Determinants

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"Mathematics is the music of reason"

- Sylvester

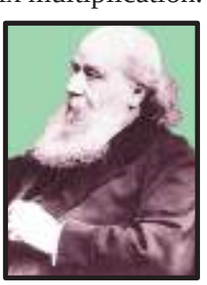
7.1 Introduction

The beginnings of matrices and determinants go back to the second century BC although traces can be seen back to the fourth century BC. However, it was not until near the end of the seventeenth century that the ideas reappeared and development really got underway. It is not surprising that the beginnings of matrices and determinants should arise through the study of systems of linear equations. The Babylonians studied problems which lead to simultaneous linear equations and some of these are preserved in clay tablets which survive till now.

The evolution of the theory of **'matrices'** is the result of attempts to obtain compact and simple methods for solving systems of linear equations. It also began with the study of transformations of geometric objects. In 1850, it was **James Joseph Sylvester** an English Mathematician and lawyer, coined the word **'Matrix'** (originally from Latin: M$\bar{a}$ter means Mother – Collin's Dictionary). Matrices are now one of the most powerful tools in mathematics.

Generally, a matrix is nothing but a rectangular array of objects. These matrices can be visualised in day-to-day applications where we use matrices to represent a military parade or a school assembly or vegetation.

The term **'determinant'** was first coined by **Carl F Gauss** in **Disquisitiones arithmeticae** (1801) while studying quadratic forms. But the concept is not the same as that of modern day determinant. In the same work Gauss laid out the coefficients of his quadratic forms in rectangular arrays where he described matrix multiplication.



Sylvester (1814 - 1897)

It was Cauchy (in 1812) who used *determinant* in its modern sense and studied it in detail. He reproved the earlier results and gave new results of his own on minors and adjoints. It was **Arthur Cayley** whose major contribution was in developing the algebra of matrices and also published the theory of determinants in 1841. In that paper he used two vertical lines on either side of the array to denote the determinant, a notation which has now become standard. In 1858, he published

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Memoir on the theory of matrices which was remarkable for containing the first abstract definition of a matrix. He showed that the coefficient arrays studied earlier for quadratic forms and for linear transformations were special cases of his general concept. They simplify our work to a great extent when compared with other straight forward methods which would involve tedious computation. The mathematicians **James Joseph Sylvester** (1814 – 1897), **William Rowan Hamilton** (1805 – 1865), and **Arthur Cayley** (1821 – 1895) played important roles in the development of matrix theory. English mathematician **Cullis** was the first to use modern bracket notation for matrices in 1913. The knowledge of matrices is absolutely necessary not only within the branches of mathematics but also in other areas of science, genetics, economics, sociology, modern psychology and industrial management.

Matrices are also useful for representing coefficients in systems of linear equations. Matrix notations and operations are used in electronic spreadsheet programs on computers, which in turn are used in different areas of business like budgeting, sales projection, cost estimation, and in science, for analyzing the results of an experiment etc.

Interestingly, many geometric operations such as magnification, rotation and reflection through a plane can also be represented mathematically by matrices. Economists use matrices for social accounting, input-output tables and in the study of inter-industry economics. Matrices are also used in communication theory and network analysis in electrical engineering. They are also used in Cryptography.

In this chapter, we now first discuss matrices and their various properties. Then we continue to study determinants, basic properties, minors and their cofactors. Here we now restrict the discussion up to determinants of order 3 only.

Learning Objectives

On completion of this chapter, the students are expected to

- visualise difficult problems in a simple manner in terms of matrices.
- understand different types of matrices and algebra of matrices.
- compute determinant values through expansion and using properties of determinants.
- apply the concepts of matrices and determinants to find the area of a triangle and collinearity of three points.

7.2 Matrices

A **matrix** is a rectangular array or arrangement of entries or elements displayed in rows and columns put within a square bracket [ ].

In general, the entries of a matrix may be real or complex numbers or functions of one variable (such as polynomials, trigonometric functions or a combination of them) or more variables or any other object. Usually, matrices are denoted by capital letters A, B, C, ... etc. In this chapter the entries of matrices are restricted to either real numbers or real valued functions on real variables.

General form of a matrix

If a matrix A has m rows and n columns, then it is written as

$$A = [a_{ij}]_{m\times n}, 1 \le i \le m, 1 \le j \le n.$$

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That is,

$$A=[a_{ij}]_{m\times n}=\begin{array}{c} \xrightarrow{\quad n \text{ columns} \quad} \\ \begin{array}{cccccc} \text{Column 1} & \text{Column 2} & & \text{Column } j & & \text{Column } n \\ \downarrow & \downarrow & & \downarrow & & \downarrow \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \begin{array}{l} \leftarrow \text{Row } 1 \\ \leftarrow \text{Row } 2 \\ \\ \leftarrow \text{Row } i \\ \\ \leftarrow \text{Row } m \end{array} \end{array} \Bigg\downarrow m \text{ rows}$$

Note that m and n are positive integers.

The following are some examples of matrices :

$$A=\begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 9 & -8 & 6 \end{bmatrix},\ B=\begin{bmatrix} 7 & -9 & 1.2 & 0 \\ \sin\frac{x}{4} & 2 & x^2 & 4 \\ \cos\frac{x}{2} & 1 & 3 & -6 \end{bmatrix}, \text{and } C=\begin{bmatrix} 1 & 5 & -7 \\ 3.4 & \frac{1}{2} & \sqrt{2} \\ e^2 & -3 & 4 \\ \sqrt{5} & 2 & a \end{bmatrix}.$$

In a matrix, the horizontal lines of elements are known as **rows** and the vertical lines of elements are known as **columns**. Thus A has 3 rows and 3 columns, B has 3 rows and 4 columns, and C has 4 rows and 3 columns.

Definition 7.1

If a matrix A has m rows and n columns then the **order or size** of the matrix A is defined to be $m \times n$ (read as m by n).

The objects $a_{11}, a_{12}, \ldots, a_{mn}$ are called **elements or entries** of the matrix $A=[a_{ij}]_{m\times n}$. The element a_{ij} is common to i^{th} row and j^{th} column and is called $(i, j)^{\text{th}}$ element of A. Observe that the ith row and j^{th} column of A are $1\times n$ and $m\times 1$ matrices respectively and are given by $[a_{i1}\ \ a_{i2}\ \ \ldots\ \ a_{in}]$ and $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

We shall now visualize the representation and construction of matrices for simplifying day-to-day problems.

Illustration 7.1

Consider the marks scored by a student in different subjects and in different terminal examinations. They are exhibited in a tabular form as given below :

	Tamil	English	Mathematics	Science	Social Science
Exam 1	48	71	80	62	55
Exam 2	70	68	91	73	60
Exam 3	77	84	95	82	62

This tabulation represents the above information in the form of matrix. What does the entry in the third row and second column represent?

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The above information may be represented in the form of a 3×5 matrix A as

$$A = \begin{bmatrix} 48 & 71 & 80 & 62 & 55 \\ 70 & 68 & 91 & 73 & 60 \\ 77 & \mathbf{84} & 95 & 82 & 62 \end{bmatrix}.$$

The entry 84 common to the third row and the second column in the matrix represents the mark scored by the student in English Exam 3.

Example 7.1

Suppose that a matrix has 12 elements. What are the possible orders it can have? What if it has 7 elements?

Solution

The number of elements is the product of number of rows and number of columns. Therefore, we will find all ordered pairs of natural numbers whose product is 12. Thus, all the possible orders of the matrix are $1\times12,\ 12\times1,\ 2\times6,\ 6\times2,\ 3\times4$ and 4×3.

Since 7 is prime, the only possible orders of the matrix are 1×7 and 7×1.

Example 7.2

Construct a 2×3 matrix whose $(i, j)^{\text{th}}$ element is given by

$$a_{ij} = \frac{\sqrt{3}}{2}|2i - 3j| \quad (1 \le i \le 2,\ 1 \le j \le 3).$$

Solution

In general, a 2×3 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

By definition of a_{ij}, we easily have $a_{11} = \frac{\sqrt{3}}{2}|2-3| = \frac{\sqrt{3}}{2}$ and other entries of the matrix A may be computed similarly. Thus, the required matrix A is $\begin{bmatrix} \frac{\sqrt{3}}{2} & 2\sqrt{3} & \frac{7\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \sqrt{3} & \frac{5\sqrt{3}}{2} \end{bmatrix}$.

7.2.1 Types of Matrices

Row, Column, Zero matrices

Definition 7.2

A matrix having only one row is called a **row matrix**.

For instance, $A = [A]_{1\times4} = [1 \quad 0 \quad -1.1 \quad \sqrt{2}]$ is a row matrix. More generally, $A = [a_{ij}]_{1\times n} = [a_{1j}]_{1\times n}$ is a row matrix of order $1\times n$.

Definition 7.3

A matrix having only one column is called a **column matrix**.

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For instance, $[A]_{4\times1} = \begin{bmatrix} x+1 \\ x^2 \\ 3x \\ 4 \end{bmatrix}$ is a column matrix whose entries are real valued functions of real variable x. More generally, $A=[a_{ij}]_{m\times1}=[a_{i1}]_{m\times1}$ is a column matrix of order $m\times1$.

Definition 7.4

A matrix $A=[a_{ij}]_{m\times n}$ is said to be a **zero matrix** or **null matrix** or **void matrix** denoted by O if $a_{ij}=0$ for all values of $1\le i\le m$ and $1\le j\le n$.

For instance, $[0]$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are zero matrices of order 1×1, 3×3 and 2×4 respectively.

A matrix A is said to be a **non-zero matrix** if at least one of the entries of A is non-zero.

Square, Diagonal, Unit, Triangular matrices

Definition 7.5

A matrix in which number of rows is equal to the number of columns, is called a **square matrix**. That is, a matrix of order $n\times n$ is often referred to as a square matrix of order n.

For instance, $A=\begin{bmatrix} a & b & c \\ d & c & f \\ g & h & l \end{bmatrix}$ is a square matrix of order 3.

Definition 7.6

In a square matrix $A=[a_{ij}]_{n\times n}$ of order n, the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the **principal diagonal** or simply the **diagonal** or **main diagonal** or **leading diagonal elements**.

Definition 7.7

A square matrix $A=[a_{ij}]_{n\times n}$ is called a **diagonal matrix** if $a_{ij}=0$ whenever $i\ne j$.

Thus, in a diagonal matrix all the entries except the entries along the main diagonal are zero. For instance,

$$A=\begin{bmatrix} 2.5 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 0.5 \end{bmatrix},\ B=\begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix},\ C=[6],\ \text{and } D=\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}$$

are diagonal matrices of order 3, 2, 1, and n respectively.

Is a square zero matrix, a diagonal matrix?

Definition 7.8

A diagonal matrix whose entries along the principal diagonal are equal, is called a **Scalar matrix**.

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That is, a square matrix $A=[a_{ij}]_{n\times n}$ is said to be a scalar matrix if $a_{ij}=\begin{cases} c & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$ where c is a fixed number. For instance,

$$A=\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, B=\begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}, C=\begin{bmatrix} \sqrt{3} \end{bmatrix} \text{ and } D=\begin{array}{c} \begin{bmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{bmatrix} \end{array}\begin{array}{c} R_1 \\ R_2 \\ \\ R_n \end{array}$$

are scalar matrices of order 3, 2, 1, and n respectively.

Observe that any square zero matrix can be considered as a scalar matrix with scalar 0.

Definition 7.9

A square matrix in which all the diagonal entries are 1 and the rest are all zero is called a **unit matrix**. Thus, a square matrix $A=[a_{ij}]_{n\times n}$ is said to be a unit matrix if $a_{ij}=\begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases}$.

We represent the unit matrix of order n as I_n. For instance,

$$I_1=[1], I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } I_n=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}\begin{array}{c} R_1 \\ R_2 \\ \\ R_n \end{array}$$

are unit matrices of order 1, 2, 3 and n respectively.

Note 7.1

Unit matrix is an example of a scalar matrix.

There are two kinds of triangular matrices namely upper triangular and lower triangular matrices.

Definition 7.10

A square matrix is said to be an **upper triangular matrix** if all the elements below the main diagonal are zero.

Thus, the square matrix $A=[a_{ij}]_{n\times n}$ is said to be an upper triangular matrix if $a_{ij}=0$ for all $i>j$. For instance,

$$\begin{bmatrix} 4 & 3 & 0 \\ 0 & 7 & 8 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -5 & 2 \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix} \text{ are all upper triangular matrices.}$$

Definition 7.11

A square matrix is said to be a **lower triangular matrix** if all the elements above the main diagonal are zero.

More precisely, a square matrix $A=[a_{ij}]_{n\times n}$ is said to be a lower triangular matrix if $a_{ij}=0$ for all $i<j$. For instance,

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$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & -5 & 7 \end{bmatrix}$, $\begin{bmatrix} -2 & 0 \\ 9 & -3 \end{bmatrix}$, and $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ are all lower triangular matrices.

Definition 7.12

A square matrix which is either upper triangular or lower triangular is called a **triangular matrix**.

Observe that a square matrix that is both upper and lower triangular simultaneously will turn out to be a diagonal matrix.

7.2.2 Equality of Matrices

Definition 7.13

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** (written as $A = B$) if and only if

(i) both A and B are of the same order

(ii) the corresponding entries of A and B are equal. That is, $a_{ij} = b_{ij}$ for all i and j.

For instance, if

$$\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 2.5 & -1 \\ \frac{1}{\sqrt{2}} & \frac{3}{5} \end{bmatrix}, \text{ then we must have } x = 2.5, y = -1, u = \frac{1}{\sqrt{2}} \text{ and } v = \frac{3}{5}.$$

Definition 7.14

Two matrices A and B are called **unequal** if either of condition (i) or (ii) of Definition 7.13 does not hold.

For instance, $\begin{bmatrix} 4 & -3 \\ 0 & 8 \end{bmatrix} \neq \begin{bmatrix} 8 & -5 \\ 0 & 4 \end{bmatrix}$ as the corresponding entries are not equal. Also $\begin{bmatrix} 4 & -3 \\ 0 & 8 \end{bmatrix} \neq \begin{bmatrix} 5 & -8 \\ 3 & 4 \\ 6 & 7 \end{bmatrix}$ as the orders are not the same.

Example 7.3

Find x, y, a, and b if $\begin{bmatrix} 3x+4y & 6 & x-2y \\ a+b & 2a-b & -3 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 5 & -5 & -3 \end{bmatrix}$.

Solution

As the orders of the two matrices are same, they are equal if and only if the corresponding entries are equal. Thus, by comparing the corresponding elements, we get

$3x + 4y = 2$, $x - 2y = 4$, $a + b = 5$, and $2a - b = -5$.

Solving these equations, we get $x = 2$, $y = -1$, $a = 0$, and $b = 5$.

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7.2.3 Algebraic Operations on Matrices

Basic operations on matrices are

(1) multiplication of a matrix by a scalar,

(2) addition/subtraction of two matrices, and

(3) multiplication of two matrices.

There is no concept of dividing a matrix by another matrix and thus, the operation $\frac{A}{B}$, where A and B are matrices, is not defined.

(1) Multiplication of a matrix by a scalar

For a given matrix $A=[a_{ij}]_{m\times n}$ and a scalar k, we define a new matrix $kA=[b_{ij}]_{m\times n}$, where $b_{ij}=ka_{ij}$ for all i and j.

For instance, if $A=\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, then $kA=\begin{bmatrix} ka & kb & kc \\ kd & ke & kf \end{bmatrix}$

In particular if $k=-1$, we obtain $-A=[-a_{ij}]_{m\times n}$. This $-A$ is called **negative** of the matrix A. Don't say $-A$ as a negative matrix.

(2) Addition and Subtraction of two matrices

If A and B are two matrices of the same order, then their sum denoted by $A + B$, is again a matrix of same order, obtained by adding the corresponding entries of A and B.

More precisely, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices, then the sum $A + B$ of A and B is a matrix given by

$A+B=[c_{ij}]_{m\times n}$ where $c_{ij}=a_{ij}+b_{ij}$ for all i and j.

Similarly subtraction $A - B$ is defined as $A-B=A+(-1)B$.

That is, $A-B=[d_{ij}]_{m\times n}$, where $d_{ij}=a_{ij}-b_{ij}$ $\forall$ i and j. (The symbol $\forall$ denotes for every or for all).

Note 7.2

(i) If A and B are not of the same order, then $A + B$ and $A - B$ are not defined.

(ii) The addition and subtraction can be extended to any finite number of matrices.

Example 7.4

Compute $A + B$ and $A - B$ if

$$A=\begin{bmatrix} 4 & \sqrt{5} & 7 \\ -1 & 0 & 0.5 \end{bmatrix} \text{ and } B=\begin{bmatrix} \sqrt{3} & \sqrt{5} & 7.3 \\ 1 & \frac{1}{3} & \frac{1}{4} \end{bmatrix}.$$

Solution

By the definitions of addition and subtraction of matrices, we have

$$A+B=\begin{bmatrix} 4+\sqrt{3} & 2\sqrt{5} & 14.3 \\ 0 & \frac{1}{3} & \frac{3}{4} \end{bmatrix} \text{ and } A-B=\begin{bmatrix} 4-\sqrt{3} & 0 & -0.3 \\ -2 & -\frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

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Example 7.5

Find the sum $A + B + C$ if A, B, C are given by

$$A=\begin{bmatrix}\sin^2\theta & 1\\ \cot^2\theta & 0\end{bmatrix},\ B=\begin{bmatrix}\cos^2\theta & 0\\ -\text{cosec}^2\theta & 1\end{bmatrix} \text{ and } C=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}.$$

Solution

By the definition of sum of matrices, we have

$$A+B+C=\begin{bmatrix}\sin^2\theta+\cos^2\theta+0 & 1+0-1\\ \cot^2\theta-\text{cosec}^2\theta-1 & 0+1+0\end{bmatrix}=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}.$$

Example 7.6

Determine $3B + 4C - D$ if $B, C,$ and D are given by

$$B=\begin{bmatrix}2 & 3 & 0\\ 1 & -1 & 5\end{bmatrix},\ C=\begin{bmatrix}-1 & -2 & 3\\ -1 & 0 & 2\end{bmatrix},\ D=\begin{bmatrix}0 & 4 & -1\\ 5 & 6 & -5\end{bmatrix}.$$

Solution

$$3B+4C-D=\begin{bmatrix}6 & 9 & 0\\ 3 & -3 & 15\end{bmatrix}+\begin{bmatrix}-4 & -8 & 12\\ -4 & 0 & 8\end{bmatrix}+\begin{bmatrix}0 & -4 & 1\\ -5 & -6 & 5\end{bmatrix}=\begin{bmatrix}2 & -3 & 13\\ -6 & -9 & 28\end{bmatrix}.$$

Example 7.7

Simplify :

$$\sec\theta\begin{bmatrix}\sec\theta & \tan\theta\\ \tan\theta & \sec\theta\end{bmatrix}-\tan\theta\begin{bmatrix}\tan\theta & \sec\theta\\ \sec\theta & \tan\theta\end{bmatrix}.$$

Solution

If we denote the given expression by A, then using the scalar multiplication rule, we get

$$A=\begin{bmatrix}\sec^2\theta & \sec\theta\tan\theta\\ \sec\theta\tan\theta & \sec^2\theta\end{bmatrix}-\begin{bmatrix}\tan^2\theta & \tan\theta\sec\theta\\ \sec\theta\tan\theta & \tan^2\theta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

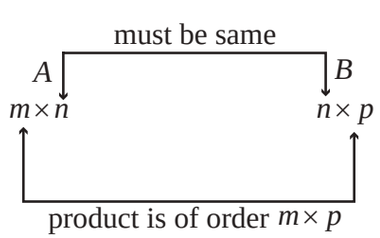
(3) Multiplication of matrices

Definition 7.15

A matrix A is said to be **conformable for multiplication** with a matrix B if the number of columns of A is equal to the number of rows of B.

That is, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ are given two matrices, then the product of matrices A and B is denoted by AB and its order is $m \times p$.

The order of AB is $m\times p=(\text{number of rows of } A)\times(\text{number of columns of } B)$.



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If $A=[a_1 \quad a_2 \quad \dots \quad a_n]_{1\times n}$ and $B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1}$,then AB is a matrix of order 1×1, that gives a single element which is defined by $AB=[a_1 \quad a_2 \quad \dots \quad a_n]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}=[a_1b_1+a_2b_2+\dots+a_nb_n]=\left[\sum_{k=1}^{n} a_kb_k\right]$.

For instance,

$$[1 \;\; 2 \;\; 3]\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}=[1(-2)+2(3)+3(5)]=[-2+6+15]=[19].$$

In general,

if $A=[a_{ij}]_{m\times n}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and $B=[b_{ij}]_{n\times p}=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$ then

$$AB=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

and the product $AB=[c_{ij}]_{m\times p}=\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$,

where $c_{ij}=[a_{i1} \quad a_{i2} \quad \dots \quad a_{in}]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}=\sum_{k=1}^{n} a_{ik}b_{kj}$, since c_{ij} is an element.

Example 7.8

If $A=\begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}$, compute A^2.

Solution

$$A^2=AA=\begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}=\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

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where $c_{11} = [0 \;\; c \;\; b]\begin{bmatrix} 0 \\ c \\ b \end{bmatrix} = 0\cdot 0 + c\cdot c + b\cdot b = c^2 + b^2$ and other elements c_{ij} may be computed similarly.

Finally, we easily obtain that

$$A^2 = \begin{bmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ab & c^2+0+a^2 & bc+0+0 \\ 0+ac+0 & bc+0+0 & b^2+a^2+0 \end{bmatrix} = \begin{bmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{bmatrix}.$$

Example 7.9

Solve for x if $[x \;\; 2 \;\; -1]\begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O.$

Solution

$$[x \;\; 2 \;\; -1]\begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O$$

That is, $$[x-2+1 \quad x-8+1 \quad 2x+2+2]\begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O$$

$$[x-1 \quad x-7 \quad 2x+4]\begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O$$

$$x(x-1)+2(x-7)+1(2x+4) = 0$$

$$x^2+3x-10 = 0 \Rightarrow x = -5, 2.$$

Note 7.3

We have the following important observations:

(1) If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$, and $m \neq p$, then the product AB is defined but not BA.

(2) The fundamental properties of real numbers namely,

$ab = ba \quad \forall a, b \in \mathbb{R}$

$ab = ac \Rightarrow b = c \;\forall\; a, b, c \in \mathbb{R},\; a \neq 0$

$ab = 0 \Rightarrow a = 0$ or $b = 0 \;\forall\; a, b \in \mathbb{R}.$

Can we discuss these in matrices also?

(i) Even if AB and BA are defined, then $AB = BA$ is not necessarily true.

For instance, we consider

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

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and observe that $AB \neq BA$, since

$$AB = \begin{bmatrix} 5 & 0 \\ 4 & -2 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 2 \\ 5 & 3 \end{bmatrix}$$

In this case we say that A and B do not commute (with respect to multiplication)

Observe that $AB = BA$ is also possible. For instance,

$$\text{if } A = \begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \text{ then, } AB = BA = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

In this case we say that A and B commute with respect to multiplication.

(ii) Cancellation property does not hold for matrix multiplication. That is, $A \neq O$, B, and C are three square matrices of same order $n \times n$ with $n > 1$, then $AB = AC$ does not imply $B = C$ and $BA = CA$ does not imply $B = C$.

As a simple demonstration of these facts, we observe that for instance,

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but $$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}.$$

(iii) It is possible that $AB = O$ with $A \neq O$ and $B \neq O$; Equivalently, $AB = O$ is not necessarily imply either $A = O$ or $B = O$. The following relation demonstrates this fact :

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(3) In general, for any two matrices A and B which are conformable for addition and multiplication, for the below operations, we have

- $(A \pm B)^2$ need not be equal to $A^2 \pm 2AB + B^2$
- $A^2 - B^2$ need not be equal to $(A + B)(A - B)$.

Example 7.10

If $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 0 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$ find AB and BA if they exist.

Solution

The order of A is 3×3 and the order of B is 3×2. Therefore the order of AB is 3×2.

A and B are conformable for the product AB. Call $C = AB$. Then,

c_{11} = (first row of A) (first column of B)

$$\Rightarrow \quad c_{11} = [1 \quad -1 \quad 2]\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1 + 1 + 2 = 4, \quad \text{since } c_{11} \text{ is an element.}$$

Similarly $c_{12} = 0, c_{21} = 0, c_{22} = 13, c_{31} = 7, c_{32} = 5.$

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Therefore, $AB = C = [c_{ij}] = \begin{bmatrix} 4 & 0 \\ 0 & 13 \\ 7 & 5 \end{bmatrix}$

The product BA does not exist, because the number of columns in B is not equal to the number of rows in A.

Example 7.11

A fruit shop keeper prepares 3 different varieties of gift packages. Pack-I contains 6 apples, 3 oranges and 3 pomegranates. Pack-II contains 5 apples, 4 oranges and 4 pomegranates and Pack –III contains 6 apples, 6 oranges and 6 pomegranates. The cost of an apple, an orange and a pomegranate respectively are ₹ 30, ₹ 15 and ₹ 45. What is the cost of preparing each package of fruits?

Solution

Cost matrix $A = [30 \;\; 15 \;\; 45]$, Fruit matrix $B = \begin{array}{c} \begin{array}{ccc} \text{P-I} & \text{P-II} & \text{P-III} \end{array} \\ \begin{bmatrix} 6 & 5 & 6 \\ 3 & 4 & 6 \\ 3 & 4 & 6 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Apples} \\ \text{Oranges} \\ \text{Pomegranates} \end{array}$

Cost of packages are obtained by computing AB. That is, by multiplying cost of each item in A (cost matrix A) with number of items in B (Fruit matrix B).

$$AB = [30 \;\; 15 \;\; 45] \begin{bmatrix} 6 & 5 & 6 \\ 3 & 4 & 6 \\ 3 & 4 & 6 \end{bmatrix} = [360 \;\; 390 \;\; 540]$$

Pack-I cost ₹ 360, Pack-II cost ₹ 390, Pack-III costs ₹ 540.

7.2.4 Properties of Matrix Addition, Scalar Multiplication and Product of Matrices

Let A, B, and C be three matrices of same order which are conformable for addition and a, b be two scalars. Then we have the following:

(1) $A + B$ yields a matrix of the same order

(2) $A + B = B + A$ (Matrix addition is commutative)

(3) $(A + B) + C = A + (B + C)$ (Matrix addition is associative)

(4) $A + O = O + A = A$ (O is additive identity)

(5) $A + (-A) = O = (-A) + A$ ($-A$ is the additive inverse of A)

(6) $(a + b)A = aA + bA$ and $a(A + B) = aA + aB$

(7) $a(bA) = (ab)A$, $1A = A$ and $0A = O$.

Properties of matrix multiplication

Using the algebraic properties of matrices we have,

- If A, B, and C are three matrices of orders $m \times n$, $n \times p$ and $p \times q$ respectively, then $A(BC)$ and $(AB)C$ are matrices of same order $m \times q$ and

  $A(BC) = (AB)C$ (Matrix multiplication is associative).

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- If A, B, and C are three matrices of orders $m \times n$, $n \times p$, and $n \times p$ respectively, then $A(B + C)$ and $AB + AC$ are matrices of the same order $m \times p$ and

  $A(B + C) = AB + AC$. (Matrix multiplication is left distributive over addition)
- If A, B, and C are three matrices of orders $m \times n$, $m \times n$, and $n \times p$ respectively, then $(A + B)C$ and $AC + BC$ are matrices of the same order $m \times p$ and

  $(A + B)C = AC + BC$. (Matrix multiplication is right distributive over addition).
- If A, B are two matrices of orders $m \times n$ and $n \times p$ respectively and α is scalar, then $\alpha(AB) = A(\alpha B) = (\alpha A)B$ is a matrix of order $m \times p$.
- If I is the unit matrix, then $AI = IA = A$ (I is called multiplicative identity).

7.2.5 Operation of Transpose of a Matrix and its Properties

Definition 7.16

The **transpose** of a matrix is obtained by interchanging rows and columns of A and is denoted by A^T.

More precisely, if $A = [a_{ij}]_{m\times n}$, then $A^T = [b_{ij}]_{n\times m}$, where $b_{ij} = a_{ji}$ so that the $(i, j)^{\text{th}}$ entry of A^T is a_{ji}.

For instance,

$$A = \begin{bmatrix} 1 & \sqrt{2} & 4 \\ -8 & 0 & 0.2 \end{bmatrix} \text{ implies } A^T = \begin{bmatrix} 1 & -8 \\ \sqrt{2} & 0 \\ 4 & 0.2 \end{bmatrix}$$

We state a few basic results on transpose whose proofs are straight forward.

For any two matrices A and B of suitable orders, we have

(i) $(A^T)^T = A$ (ii) $(kA)^T = kA^T$ (where k is any scalar)

(iii) $(A+B)^T = A^T + B^T$ (iv) $(AB)^T = B^T A^T$ (reversal law on transpose)

Example 7.12

If $A = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$,

verify (i) $(AB)^T = B^T A^T$ (ii) $(A+B)^T = A^T + B^T$ (iii) $(A-B)^T = A^T - B^T$ (iv) $(3A)^T = 3A^T$

Solution

(i) $$AB = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 2 & 22 \\ -2 & 9 & 9 \\ 7 & 1 & 14 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \qquad \text{... (1)}$$

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$$B^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \qquad \text{... (2)}$$

From (1) and (2), $\quad (AB)^T = B^T A^T$.

(ii) $$A + B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 1 \\ 3 & 0 & 9 \\ -1 & 5 & 3 \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \qquad \text{...(3)}$$

$$A^T + B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \qquad \text{... (4)}$$

From (3) and (4), $\quad (A+B)^T = A^T + B^T$.

(iii) $$A - B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 3 \\ -3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$(A - B)^T = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \qquad \text{... (5)}$$

$$A^T - B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \qquad \text{... (6)}$$

From (5) and (6) $\quad (A - B)^T = A^T - B^T$.

(iv) $$3A = \begin{bmatrix} 12 & 18 & 6 \\ 0 & 3 & 15 \\ 0 & 9 & 6 \end{bmatrix}$$

$$(3A)^T = \begin{bmatrix} 12 & 0 & 0 \\ 18 & 3 & 9 \\ 6 & 15 & 6 \end{bmatrix} = 3\begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} = 3(A^T) \ .$$

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7.2.6 Symmetric and Skew-symmetric Matrices

Definition 7.17

A square matrix A is said to be **symmetric** if $A^T = A$.

That is, $A=[a_{ij}]_{n\times n}$ is a symmetric matrix, then $a_{ij} = a_{ji}$ for all i and j.

For instance, $A=\begin{bmatrix} 3 & -6 & 9 \\ -6 & 8 & 5 \\ 9 & 5 & 2 \end{bmatrix}$ is a symmetric matrix since $A^T = A$.

Observe that transpose of A^T is the matrix A itself. That is $\left(A^T\right)^T = A$.

Definition 7.18

A square matrix A is said to be **skew-symmetric** if $A^T = -A$.

If $A=[a_{ij}]_{n\times n}$ is a skew-symmetric matrix, then $a_{ij} = -a_{ji}$ for all i and j.

Now, if we put $i = j$, then $2a_{ii} = 0$ or $a_{ii} = 0$ for all i. This means that all the diagonal elements of a skew-symmetric matrix are zero.

For instance, $A=\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ is a skew-symmetric matrix since $A^T = -A$.

It is interesting to note that any square matrix can be written as the sum of symmetric and skew-symmetric matrices.

Theorem 7.1

For any square matrix A with real number entries, $A+A^T$ is a symmetric matrix and $A-A^T$ is a skew-symmetric matrix.

Proof

Let $B = A + A^T$.

$$B^T = (A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = B.$$

This implies $A+A^T$ is a symmetric matrix.

Next, we let $C = A - A^T$. Then we see that

$$C^T = (A+(-A^T))^T = A^T + (-A^T)^T = A^T - (A^T)^T = A^T - A = -(A-A^T) = -C$$

This implies $A-A^T$ is a skew-symmetric matrix. ■

Theorem 7.2

Any square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Proof

Let A be a square matrix. Then, we can write

$$A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T).$$

From Theorem 7.1, it follows that $(A+A^T)$ and $(A-A^T)$ are symmetric and skew-symmetric matrices respectively. Since $(kA)^T = kA^T$, it follows that $\frac{1}{2}(A+A^T)$ and $\frac{1}{2}(A-A^T)$ are symmetric and skew-symmetric matrices, respectively. Now, the desired result follows. ■

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Note 7.4

A matrix which is both symmetric and skew-symmetric is a zero matrix.

Example 7.13

Express the matrix $A=\begin{bmatrix}1 & 3 & 5\\ -6 & 8 & 3\\ -4 & 6 & 5\end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrices.

Solution

$$A=\begin{bmatrix}1 & 3 & 5\\ -6 & 8 & 3\\ -4 & 6 & 5\end{bmatrix} \Rightarrow A^T=\begin{bmatrix}1 & -6 & -4\\ 3 & 8 & 6\\ 5 & 3 & 5\end{bmatrix}$$

$$\text{Let } P=\frac{1}{2}(A+A^T) = \frac{1}{2}\begin{bmatrix}2 & -3 & 1\\ -3 & 16 & 9\\ 1 & 9 & 10\end{bmatrix}$$

$$\text{Now } P^T = \frac{1}{2}\begin{bmatrix}2 & -3 & 1\\ -3 & 16 & 9\\ 1 & 9 & 10\end{bmatrix}=P$$

Thus, $P = \frac{1}{2}(A+A^T)$ is a symmetric matrix.

$$\text{Let } Q = \frac{1}{2}(A-A^T)$$

$$= \frac{1}{2}\begin{bmatrix}0 & 9 & 9\\ -9 & 0 & -3\\ -9 & 3 & 0\end{bmatrix}$$

$$\text{Then } Q^T = \frac{1}{2}\begin{bmatrix}0 & -9 & -9\\ 9 & 0 & 3\\ 9 & -3 & 0\end{bmatrix}=-Q$$

Thus $Q = \frac{1}{2}(A-A^T)$ is a skew-symmetric matrix.

$$A = P+Q=\frac{1}{2}\begin{bmatrix}2 & -3 & 1\\ -3 & 16 & 9\\ 1 & 9 & 10\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0 & 9 & 9\\ -9 & 0 & -3\\ -9 & 3 & 0\end{bmatrix}$$

Thus A is expressed as the sum of symmetric and skew-symmetric matrices.

EXERCISE 7.1

(1) Construct an $m \times n$ matrix $A=[a_{ij}]$, where a_{ij} is given by

(i) $a_{ij}=\frac{(i-2j)^2}{2}$ with $m=2, n=3$ (ii) $a_{ij}=\frac{|3i-4j|}{4}$ with $m=3, n=4$

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(2) Find the values of p, q, r, and s if

$$\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & \frac{3}{2} & 9 \\ -2 & 8 & -\pi \end{bmatrix}.$$

(3) Determine the value of $x + y$ if $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$.

(4) Determine the matrices A and B if they satisfy

$$2A - B + \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} = 0 \text{ and } A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$$

(5) If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then compute A^4.

(6) Consider the matrix $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

(i) Show that $A_\alpha \, A_\beta = A_{(\alpha+\beta)}$.

(ii) Find all possible real values of α satisfying the condition $A_\alpha + A_\alpha^T = I$.

(7) If $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$ and such that $(A-2I)(A-3I) = O$, find the value of x.

(8) If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, show that A^2 is a unit matrix.

(9) If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $A^3 - 6A^2 + 7A + kI = O$, find the value of k.

(10) Give your own examples of matrices satisfying the following conditions in each case:

(i) A and B such that $AB \neq BA$.

(ii) A and B such that $AB = O = BA$, $A \neq O$ and $B \neq O$.

(iii) A and B such that $AB = O$ and $BA \neq O$.

(11) Show that $f(x)\,f(y) = f(x+y)$, where $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(12) If A is a square matrix such that $A^2 = A$, find the value of $7A - (I + A)^3$.

(13) Verify the property $A(B + C) = AB + AC$, when the matrices A, B, and C are given by

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

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(14) Find the matrix A which satisfies the matrix relation $A\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}=\begin{bmatrix}-7 & -8 & -9\\ 2 & 4 & 6\end{bmatrix}$.

(15) If $A^T=\begin{bmatrix}4 & 5\\ -1 & 0\\ 2 & 3\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1 & 1\\ 7 & 5 & -2\end{bmatrix}$, verify the following

(i) $(A+B)^T=A^T+B^T=B^T+A^T$ (ii) $(A-B)^T=A^T-B^T$ (iii) $(B^T)^T=B$.

(16) If A is a 3×4 matrix and B is a matrix such that both A^TB and BA^T are defined, what is the order of the matrix B?

(17) Express the following matrices as the sum of a symmetric matrix and a skew-symmetric matrix:

(i) $\begin{bmatrix}4 & -2\\ 3 & -5\end{bmatrix}$ and (ii) $\begin{bmatrix}3 & 3 & -1\\ -2 & -2 & 1\\ -4 & -5 & 2\end{bmatrix}$.

(18) Find the matrix A such that $\begin{bmatrix}2 & -1\\ 1 & 0\\ -3 & 4\end{bmatrix}A^T=\begin{bmatrix}-1 & -8 & -10\\ 1 & 2 & -5\\ 9 & 22 & 15\end{bmatrix}$.

(19) If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ x & 2 & y\end{bmatrix}$ is a matrix such that $AA^T=9I$, find the values of x and y.

(20) (i) For what value of x, the matrix $A=\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & x^3\\ 2 & -3 & 0\end{bmatrix}$ is skew-symmetric.

(ii) If $\begin{bmatrix}0 & p & 3\\ 2 & q^2 & -1\\ r & 1 & 0\end{bmatrix}$ is skew-symmetric, find the values of p, q, and r.

(21) Construct the matrix $A=[a_{ij}]_{3\times 3}$, where $a_{ij}=i-j$. State whether A is symmetric or skew-symmetric.

(22) Let A and B be two symmetric matrices. Prove that $AB = BA$ if and only if AB is a symmetric matrix.

(23) If A and B are symmetric matrices of same order, prove that

(i) $AB + BA$ is a symmetric matrix.

(ii) $AB - BA$ is a skew-symmetric matrix.

(24) A shopkeeper in a Nuts and Spices shop makes gift packs of cashew nuts, raisins and almonds.

Pack I contains 100 gm of cashew nuts, 100 gm of raisins and 50 gm of almonds.

Pack-II contains 200 gm of cashew nuts, 100 gm of raisins and 100 gm of almonds.

Pack-III contains 250 gm of cashew nuts, 250 gm of raisins and 150 gm of almonds.

The cost of 50 gm of cashew nuts is ₹ 50, 50 gm of raisins is ₹10, and 50 gm of almonds is ₹ 60. What is the cost of each gift pack?

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7.3 Determinants

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number called **determinant** of the matrix A.

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

Note 7.5

(i) Determinants can be defined only for square matrices.

(ii) For a square matrix A, $|A|$ is read as **determinant of A.**

(iii) Matrix is only a representation whereas determinant is a value of a matrix.

7.3.1 Determinants of Matrices of different order

Determinant of a matrix of order 1

Let $A = [a]$ be the matrix of order 1, then the determinant of A is defined as 'a'.

Determinant of a matrix of order 2

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2. Then the determinant of A is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\ a_{22} - a_{21}\ a_{12}.$$

Example 7.14

Evaluate : (i) $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$.

Solution

(i) $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = (2 \times 2) - (-1 \times 4) = 4 + 4 = 8.$

(ii) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = (\cos\theta\cos\theta) - (-\sin\theta\sin\theta) = \cos^2\theta + \sin^2\theta = 1.$

Determinant of a Matrix of order 3

We consider the determinant of a 3×3 matrix with entries as real numbers or real valued functions defined on $\mathbb{R}$ and study its properties and discuss various methods of evaluation of certain determinants.

Definition 7.19

Let $A = [a_{ij}]_{3\times3}$ be a given square matrix of order 3. The **minor** of an arbitrary element a_{ij} is the determinant obtained by deleting the ith row and j^{th} column in which the element a_{ij} stands. The minor of a_{ij} is usually denoted by M_{ij}.

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Definition 7.20

The **cofactor** is a signed minor. The cofactor of a_{ij} is usually denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

For instance, consider the 3 × 3 matrix defined by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then the minors and cofactors of the elements a_{11}, a_{12}, a_{13} are given as follows :

(i) Minor of a_{11} is M_{11} $= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

Cofactor of a_{11} is A_{11} $= (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

(ii) Minor of a_{12} is M_{12} $= \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$

Cofactor a_{12} is A_{12} $= (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{31}a_{23})$

(iii) Minor of a_{13} is M_{13} $= \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$

Cofactor of a_{13} is A_{13} $= (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$.

Result 7.1 (Laplace Expansion)

For a given matrix $A = [a_{ij}]_{3\times3}$, the sum of the product of elements of the first row with their corresponding cofactors is the **determinant** of A.

That is, $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.

This can also be written using minors. That is, $|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$.

The determinant can be computed by expanding along any row or column and it is important to note that the value in all cases remains the same. For example,

expansion along R_1 is $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.

along R_2 is $|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$.

along C_1 is $|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$.

Example 7.15

Compute all minors, cofactors of A and hence compute $|A|$ if $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ -3 & 5 & 2 \end{bmatrix}$. Also check that $|A|$ remains unaltered by expanding along any row or any column.

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Solution

Minors :

$$M_{11} = \begin{vmatrix} -5 & 6 \\ 5 & 2 \end{vmatrix} = -10 - 30 = -40$$

$$M_{12} = \begin{vmatrix} 4 & 6 \\ -3 & 2 \end{vmatrix} = 8 + 18 = 26$$

$$M_{13} = \begin{vmatrix} 4 & -5 \\ -3 & 5 \end{vmatrix} = 20 - 15 = 5$$

$$M_{21} = \begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} = 6 + 10 = 16$$

$$M_{22} = \begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} = 2 - 6 = -4$$

$$M_{23} = \begin{vmatrix} 1 & 3 \\ -3 & 5 \end{vmatrix} = 5 + 9 = 14$$

$$M_{31} = \begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix} = 18 - 10 = 8$$

$$M_{32} = \begin{vmatrix} 1 & -2 \\ 4 & 6 \end{vmatrix} = 6 + 8 = 14$$

$$M_{33} = \begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} = -5 - 12 = -17$$

Cofactors :

$$A_{11} = (-1)^{1+1}(-40) = -40$$

$$A_{12} = (-1)^{1+2}(+26) = -26$$

$$A_{13} = (-1)^{1+3}(5) = 5$$

$$A_{21} = (-1)^{2+1}(16) = -16$$

$$A_{22} = (-1)^{2+2}(-4) = -4$$

$$A_{23} = (-1)^{2+3}(14) = -14$$

$$A_{31} = (-1)^{3+1}(8) = 8$$

$$A_{32} = (-1)^{3+2}(14) = -14$$

$$A_{33} = (-1)^{3+3}(-17) = -17$$

Expanding along R_1 yields

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \ .$$

$$|A| = 1(-40) + (3)(-26) + (-2)(5) = -128 \ . \qquad \text{... (3)}$$

Expanding along C_1 yields

$$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \ .$$

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$$= 1(-40) + 4(-16) + -3(8) = -128 \qquad \text{... (4)}$$

From (3) and (4), we have

|A| obtained by expanding along R_1 is equal to expanding along C_1 .

Evaluation of determinant of order 3 by using Sarrus Rule (named after the French Mathematician Pierre Frédéic Sarrus)

Let $A = [a_{ij}]_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

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Write the entries of Matrix A as follows :

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

Then $|A|$ is computed as follows :

$$|A| = \left[a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}\right] - \left[a_{33}a_{21}a_{12} + a_{32}a_{23}a_{11} + a_{31}a_{22}a_{13}\right]$$

Example 7.16

Find |A| if $A = \begin{bmatrix} 0 & \sin\alpha & \cos\alpha \\ \sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{bmatrix}$.

Solution

$$\begin{vmatrix} 0 & \sin\alpha & \cos\alpha \\ \sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix} = 0M_{11} - \sin\alpha M_{12} + \cos\alpha M_{13}$$

$$= 0 - \sin\alpha(0 - \cos\alpha\sin\beta) + \cos\alpha(-\sin\alpha\sin\beta - 0) = 0.$$

Example 7.17

Compute |A| using Sarrus rule if $A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{bmatrix}$.

Solution

$$\begin{matrix} 3 & 4 & 1 & 3 & 4 \\ 0 & -1 & 2 & 0 & -1 \\ 5 & -2 & 6 & 5 & -2 \end{matrix}$$

$$|A| = \left[3(-1)(6) + 4(2)(5) + 1(0)(-2)\right] - \left[5(-1)(1) + (-2)(2)3 + 6(0)(4)\right]$$

$$= [-18 + 40 + 0] - [-5 - 12 + 0] = 22 + 17 = 39.$$

Note 7.6

For easier calculations, we expand the determinant along a row or column which contains maximum number of zeros.

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Determinant of square matrix of order n, $n \geq 4$

The concept of determinant can be extended to the case of square matrix of order $n, n \geq 4$. Let $A = [a_{ij}]_{n\times n}, n \geq 4$.

If we delete the ith row and j^{th} column from the matrix of $A = [a_{ij}]_{n\times m}$, we obtain a determinant of order $(n-1)$, which is called the **minor of the element** a_{ij}. We denote this minor by M_{ij}. The cofactor of the element a_{ij} is defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

Result 7.2

For a given square matrix $A = [a_{ij}]_{n\times n}$ of order n, the sum of the products of elements of the first row with their corresponding cofactors is the determinant of A. That is,

$|A| = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} = \sum_{j=1}^{n} a_{1j}A_{1j}$ which, by the definition of cofactors and minors, is same as

$$|A| = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j},$$

where A_{1j} denotes the cofactor of a_{1j} and M_{1j} denotes the minor of a_{1j}, $j = 1, 2, \ldots, n$.

Note 7.7

(i) If $A = [a_{ij}]_{n\times n}$ then determinant of A can also be denoted as det(A) or *det A* or Δ.

(ii) It can be computed by using any row or column.

7.3.2 Properties of Determinants

We can use one or more of the following properties of the determinants to simplify the evaluation of determinants.

Property 1

The determinant of a matrix remains unaltered if its rows are changed into columns and columns into rows. That is, $|A| = |A^T|$.

Since the row-wise expansion is same as the column-wise expansion, the result holds good.

Property 2

If any two rows / columns of a determinant are interchanged, then the determinant changes in sign but its absolute value remains unaltered.

Verification

$$\text{Let } |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$\text{Let } |A_1| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix} \quad (\text{since } R_2 \leftrightarrow R_3)$$

$$= a_1(b_3c_2 - b_2c_3) - b_1(a_3c_2 - a_2c_3) + c_1(a_3b_2 - a_2b_3)$$

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$$= -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$

$$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$$

$$= -|A|$$

Therefore, $|A_1| = -|A|$. Thus the property is verified.

Property 3

If there are n interchanges of rows (columns) of a matrix A then the determinant of the resulting matrix is $(-1)^n |A|$.

Property 4

If two rows (columns) of a matrix are identical, then its determinant is zero.

Verification

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix}$, with 2nd and 3rd rows are identical.

Interchanging second and third rows, we get $-|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix} = |A|$.

$$\Rightarrow 2|A| = 0 \Rightarrow |A| = 0.$$

Property 5

If a row (column) of a matrix A is a scalar multiple of another row (or column) of A, then its determinant is zero.

Note 7.8

(i) If all entries of a row or a column are zero, then the determinant is zero.

(ii) The determinant of a triangular matrix is obtained by the product of the principal diagonal elements.

Property 6

If each element in a row (or column) of a matrix is multiplied by a scalar k, then the determinant is multiplied by the same scalar k.

Verification

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Let $|A_1| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= ka_1(b_2c_3 - b_3c_2) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2) = k|A|$$

$$= k[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] = k|A|$$

$$\Rightarrow |A_1| = k|A|.$$

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Note 7.9

If A is a square matrix of order n, then

(i) $|AB| = |A||B|$

(ii) If $AB = O$ then either $|A| = 0$ or $|B| = 0$.

(iii) $|A^n| = (|A|)^n$

Property 7

If each element of a row (or column) of a determinant is expressed as sum of two or more terms then the whole determinant is expressed as sum of two or more determinants.

That is, $$\begin{vmatrix} a_1+m_1 & b_1 & c_1 \\ a_2+m_2 & b_2 & c_2 \\ a_3+m_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}.$$

Verification

By taking first column expansion it can be verified easily.

$$\text{LHS} = (a_1+m_1)(b_2c_3-b_3c_2)-(a_2+m_2)(b_1c_3-b_3c_1)+(a_3+m_3)(b_1c_2-b_2c_1)$$

$$= a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)+m_1(b_2c_3-b_3c_2)-m_2(b_1c_3-b_3c_1)$$
$$+m_3(b_1c_2-b_2c_1)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix} = \text{RHS}.$$

Property 8

If, to each element of any row (column) of a determinant the equi-multiples of the corresponding entries of one or more rows (columns) are added or subtracted, then the value of the determinant remains unchanged.

Verification

Let $$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $$|A_1| = \begin{vmatrix} a_1+pa_2+qa_3 & b_1+pb_2+qb_3 & c_1+pc_2+qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pa_2 & pb_2 & pc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} qa_3 & qb_3 & qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \left\{ \begin{matrix} \text{using} \\ \text{Property 7} \end{matrix} \right.$$

$$|A_1| = |A| + p\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + q\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

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$$|A_1| = |A| + p(0) + q(0) = |A| \quad \text{(using Property 4)}$$

Therefore $|A_1| = |A|$

This property is independent of any fixed row or column.

Example 7.18

If a, b, c and x are positive real numbers, then show that $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix}$ is zero.

Solution

Applying $C_1 \to C_1 - C_2$, we get $\begin{vmatrix} 4 & (a^x-a^{-x})^2 & 1 \\ 4 & (b^x-b^{-x})^2 & 1 \\ 4 & (c^x-c^{-x})^2 & 1 \end{vmatrix} = 0$, since C_1 and C_3 are proportional.

Example 7.19

Without expanding the determinants, show that $|B| = 2|A|$.

Where $B = \begin{bmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

Solution

We have $$|B| = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} (R_1 \to R_1 + R_2 + R_3)$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} (R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1)$$

$$= 2\begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} (R_1 \to R_1 + R_2 + R_3)$$

$$= 2(-1)^2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= 2|A|.$$

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Example 7.20

Evaluate $\begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix} = \begin{vmatrix} 2014 & 2017-2014 & 0 \\ 2020 & 2023-2020 & 1 \\ 2023 & 2026-2023 & 0 \end{vmatrix} = \begin{vmatrix} 2014 & 3 & 0 \\ 2020 & 3 & 1 \\ 2023 & 3 & 0 \end{vmatrix} = 3\begin{vmatrix} 2014 & 1 & 0 \\ 2020 & 1 & 1 \\ 2023 & 1 & 0 \end{vmatrix}$$

$$= -3(2014-2023) = -3(-9) = 27.$$

Example 7.21

Find the value of x if $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$.

Solution

Since all the entries below the principal diagonal are zero, the value of the determinant is $(x-1)(x-2)(x-3) = 0$ which gives $x = 1, 2, 3$.

Example 7.22

Prove that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$.

Solution

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$\text{LHS} = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} = (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix}$$

$$= (y-x)(z-x)[(z+x)-(y+x)].$$

$$= (y-x)(z-x)(z-y).$$

$$= (x-y)(y-z)(z-x) = \text{RHS}.$$

EXERCISE 7.2

(1) Without expanding the determinant, prove that $\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix} = 0$.

(2) Show that $\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix} = 0$.

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(3) Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$.

(4) Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

(5) Prove that $\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix} = 0.$

(6) Show that $\begin{vmatrix} x+2a & y+2b & z+2c \\ x & y & z \\ a & b & c \end{vmatrix} = 0.$

(7) Write the general form of a 3×3 skew-symmetric matrix and prove that its determinant is 0.

(8) If $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$,

prove that a, b, c are in *G.P.* or α is a root of $ax^2 + 2bx + c = 0$.

(9) Prove that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$.

(10) If a, b, c are p^{th}, q^{th} and r^{th} terms of an *A.P*, find the value of $\begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$.

(11) Show that $\begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$ is divisible by x^4.

(12) If a, b, c are all positive, and are p^{th}, q^{th} and r^{th} terms of a *G.P.*, show that $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0.$

(13) Find the value of $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ if $x, y, z \neq 1$.

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(14) If $A = \begin{bmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{bmatrix}$, prove that $\sum_{k=1}^{n} \det(A^k) = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$.

(15) Without expanding, evaluate the following determinants :

(i) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ (ii) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

(16) If A is a square matrix and $|A| = 2$, find the value of $|AA^T|$.

(17) If A and B are square matrices of order 3 such that $|A| = -1$ and $|B| = 3$, find the value of $|3AB|$.

(18) If $\lambda = -2$, determine the value of $\begin{vmatrix} 0 & 2\lambda & 1 \\ \lambda^2 & 0 & 3\lambda^2+1 \\ -1 & 6\lambda-1 & 0 \end{vmatrix}$.

(19) Determine the roots of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$.

(20) Verify that $\det(AB) = (\det A)(\det B)$ for $A = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{bmatrix}$.

(21) Using cofactors of elements of second row, evaluate $|A|$, where $A = \begin{bmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

7.3.3 Application of Factor Theorem to Determinants.

Theorem 7.3 (Factor Theorem)

If each element of a matrix A is a polynomial in x and if $|A|$ vanishes for $x = a$, then $(x - a)$ is a factor of $|A|$.

Note 7.10

(i) This theorem is very much useful when we have to obtain the value of the determinant in 'factors' form.

(ii) If we substitute b for a in the determinant $|A|$, any two of its rows or columns become identical, then $|A| = 0$, and hence by factor theorem $(a - b)$ is a factor of $|A|$.

(iii) If r rows (columns) are identical in a determinant of order n ($n \geq r$), when we put $x = a$, then $(x - a)^{r-1}$ is a factor of $|A|$.

(iv) A square matrix (or its determinant) is said to be in cyclic symmetric form if each row is obtained from the first row by changing the variables cyclically.

(v) If the determinant is in cyclic symmetric form and if m is the difference between the degree of the product of the factors (obtained by substitution) and the degree of the product of the leading diagonal elements and if

(1) m is zero, then the required factor is a constant k

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(2) m is 1, then the required factor is $k(a + b + c)$ and

(3) m is 2, then the required factor is $k(a^2 + b^2 + c^2) + l\,(ab + bc + ca)$.

Example 7.23

Using Factor Theorem, prove that $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = (x-1)^2(x+9)$.

Solution

$$\text{Let } |A| = \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix}.$$

$$\text{Putting } x = 1, \text{ we get } |A| = \begin{vmatrix} 2 & 3 & 5 \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{vmatrix} = 0$$

Since all the three rows are identical, $(x - 1)^2$ is a factor of $|A|$

$$\text{Putting } x = -9 \text{ in } |A|, \text{we get } |A| = \begin{vmatrix} -8 & 3 & 5 \\ 2 & -7 & 5 \\ 2 & 3 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 5 \\ 0 & -7 & 5 \\ 0 & 3 & -5 \end{vmatrix} = 0$$

Therefore $(x+9)$ is a factor of $|A|$ [since $C_1 \to C_1 + C_2 + C_3$].

The product $(x - 1)^2 (x + 9)$ is a factor of $|A|$. Now the determinant is a cubic polynomial in x.

Therefore the remaining factor must be a constant 'k'.

$$\text{Therefore } \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = k(x-1)^2(x+9).$$

Equating x^3 term on both sides, we get $k = 1$. Thus $|A| = (x - 1)^2 (x + 9)$.

Example 7.24

Prove that $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)\,(y-z)\,(z-x)\,(xy+yz+zx)$.

Solution

$$\text{Let} \qquad |A| = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}.$$

$$\text{Putting } x = y \textit{ gives} \qquad |A| = \begin{vmatrix} 1 & y^2 & y^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = 0 \quad (\text{since } R_1 \equiv R_2).$$

Therefore $(x - y)$ is a factor.

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The given determinant is in cyclic symmetric form in x, y and z. Therefore $(y - z)$ and $(z - x)$ are also factors.

The degree of the product of the factors $(x-y)(y-z)(z-x)$ is 3 and the degree of the product of the leading diagonal elements $1\times y^2\times z^3$ is 5.

Therefore the other factor is $k(x^2+y^2+z^2)+\ell(xy+yz+zx)$.

Thus $$\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = [k(x^2+y^2+z^2)+\ell(xy+yz+zx)]\times(x-y)(y-z)(z-x)\ .$$

Putting $x = 0$, $y = 1$ and $z = 2$, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 4 & 8 \end{vmatrix} = \left[k(0+1+4)+\ell(0+2+0)\right](-1)(1-2)(2-0)$$

$$\Rightarrow \quad (8-4) = [(5k+2\ell)](-1)(-1)(2)$$

$$4 = 10k+4\ell \Rightarrow 5k+2\ell = 2. \qquad \text{... (1)}$$

Putting $x = 0, y = -1$ and $z = 1$, We get

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = [k(2)+\ell(-1)](1)(-2)(1)$$

$$\Rightarrow \quad [(2k-\ell)(-2)] = 2$$

$$2k-\ell = -1. \qquad \text{... (2)}$$

Solving (1) and (2), we get $k = 0, \ell = 1$.

Therefore $$\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx).$$

Example 7.25

Prove that $$|A| = \begin{vmatrix} (q+r)^2 & p^2 & p^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = 2pqr\ (p+q+r)^3.$$

Solution :

Taking $p = 0$, we get $$|A| = \begin{vmatrix} (q+r)^2 & 0 & 0 \\ q^2 & r^2 & q^2 \\ r^2 & r^2 & q^2 \end{vmatrix} = 0.$$

Therefore, $(p - 0)$ is a factor. That is, p is a factor.

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Since $|A|$ is in cyclic symmetric form in p, q, r and hence q and r also factors.

Putting $p + q + r = 0 \Rightarrow q + r = -p$; $r + p = -q$; and $p + q = -r$.

$$|A| = \begin{vmatrix} p^2 & p^2 & p^2 \\ q^2 & q^2 & q^2 \\ r^2 & r^2 & r^2 \end{vmatrix} = 0 \text{ since 3 columns are identical.}$$

Therefore, $(p + q + r)^2$ is a factor of $|A|$.

The degree of the obtained factor $pqr\,(p + q + r)^2$ is 5. The degree of $|A|$ is 6.

Therefore, required factor is $k\,(p + q + r)$.

$$\begin{vmatrix} (q+r)^2 & p^2 & p^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = k(p+q+r)\,(p+q+r)^2 \times pqr$$

Taking $p = 1,\ q = 1,\ c = 1$, we get

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = k(1+1+1)^3\ (1)\ (1)\ (1).$$

$4(16 - 1) - 1(4 - 1) + 1(1 - 4) = 27k$

$60 - 3 - 3 = 27\,k \Rightarrow k = 2.$

$|A| = 2pqr\,(p + q + r)^3.$

Example 7.26

In a triangle ABC, if $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A(1+\sin A) & \sin B(1+\sin B) & \sin C(1+\sin C) \end{vmatrix} = 0,$

prove that ΔABC is an isosceles triangle.

Solution :

By putting $\sin A = \sin B$, we get

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin A & 1+\sin C \\ \sin A(1+\sin A) & \sin A(1+\sin A) & \sin C(1+\sin C) \end{vmatrix} = 0$$

That is, by putting $\sin A = \sin B$ we see that, the given equation is satisfied.

Similarly by putting $\sin B = \sin C$ and $\sin C = \sin A$, the given equation is satisfied.

Thus, we have $A = B$ or $B = C$ or $C = A$.

In all cases atleast two angles are equal. Thus the triangle is isosceles.

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EXERCISE 7.3

Solve the following problems by using Factor Theorem :

(1) Show that $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2(x+2a)$.

(2) Show that $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc$.

(3) Solve $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0.$

(4) Show that $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)\ (a-b)\ (b-c)\ (c-a)$.

(5) Solve $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$.

(6) Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$.

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7.3.4 Product of Determinants

While multiplying two matrices "row-by-column" rule alone can be followed. The process of interchanging the rows and columns will not affect the value of the determinant (by Property 1). Therefore we can also adopt the following procedures for multiplication of two determinants.

(i) Row by column multiplication rule
(ii) Row by row multiplication rule
(iii) Column by column multiplication rule
(iv) Column by row multiplication rule

Note 7.11

(i) If A and B are square matrices of the same order n, then $|AB| = |A|\,|B|$ holds.
(ii) In matrices, although $AB \neq BA$ in general, we do have $|AB| = |BA|$ always.

Example 7.27

Verify that $|AB| = |A|\ |B|$ if $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Solution

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

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$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow |AB| = 1. \qquad \text{... (1)}$$

$$|A| = \cos^2\theta + \sin^2\theta = 1.$$

$$|B| = \cos^2\theta + \sin^2\theta = 1.$$

$$|A|\,|B| = 1. \qquad \text{... (2)}$$

From (1) and (2), $|AB| = |A|\,|B|$.

Example 7.28

Show that $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$.

Solution

$$\text{LHS} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}.$$

$$= \begin{vmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ab & c^2+0+a^2 & bc+0+0 \\ 0+ac+0 & bc+0+0 & b^2+a^2+0 \end{vmatrix}.$$

$$= \begin{vmatrix} c^2+b^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & b^2+a^2 \end{vmatrix} = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \text{RHS}.$$

Example 7.29

Show that $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$.

Solution

$$\text{RHS} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1) \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \quad [\text{In the } 2^{\text{nd}} \text{ determinant } R_2 \leftrightarrow R_3]$$

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$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}.$$

Taking row by column method, we get

$$= \begin{vmatrix} -a^2+bc+cb & -ab+ab+c^2 & -ac+b^2+ac \\ -ab+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & -c^2+ab+ab \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \text{RHS.}$$

Example 7.30

Prove that $\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}^2 = \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix}$.

Solution

$$\text{LHS} = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}^2 = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}.$$

$$= \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times (-1)\,(-1) \begin{vmatrix} 1 & x & x \\ -x & -1 & -x \\ -x & -x & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & x \\ -x & -1 & -x \\ -x & -x & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-x^2-x^2 & x-x-x^2 & x-x^2-x \\ x-x-x^2 & x^2-1-x^2 & x^2-x-x \\ x-x^2-x & x^2-x-x & x^2-x^2-1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix}.$$

$$= \text{R.H.S.}$$

7.3.5 Relation between a Determinant and its Cofactor Determinant

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

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Let A_1, B_1, C_1 .... be the cofactors of a_1, b_1, c_1 ... in $|A|$.

Hence, the cofactor determinant is $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

$|A| = a_1A_1 + b_1B_1 + c_1C_1$

Similarly, $|A| = a_2A_2 + b_2B_2 + c_2C_2$ and $|A| = a_3A_3 + b_3B_3 + c_3C_3$

Note that the sum of the product of elements of any row (or column) with their corresponding cofactors is the value of the determinant.

$$\text{Now } a_1A_2 + b_1B_2 + c_1C_2 = -a_1\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_1\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_1(b_1c_3 - b_3c_1) + b_1(a_1c_3 - a_3c_1) - c_1(a_1b_3 - a_3b_1)$$

$$= a_1b_1c_3 + a_1b_3c_1 + a_1b_1c_3 - a_3b_1c_1 - a_1b_3c_1 + a_3b_1c_1 = 0$$

Similarly we get

$$a_1A_3 + b_1B_3 + c_1C_3 = 0 \;;\; a_2A_1 + b_2B_1 + c_2C_1 = 0 \;;$$

$$a_2A_3 + b_2B_3 + c_2C_3 = 0 \;;\; a_3A_1 + b_3B_1 + c_3C_1 = 0 \text{ and } a_3A_2 + b_3B_2 + c_3C_2 = 0.$$

Note 7.12

If elements of a row (or column) are multiplied with corresponding cofactors of any other row (or column) then their sum is zero.

Example 7.31

If A_i, B_i, C_i are the cofactors of a_i, b_i, c_i, respectively, $i = 1$ to 3 in

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ show that } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2 .$$

Solution

Consider the product $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} = |A|^3$$

That is, $|A| \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^3$.

$$\Rightarrow \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2 .$$

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7.3.6 Area of a Triangle

We know that the **area of a triangle** whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is equal to the absolute value of

$$\frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3).$$

This expression can be written in the form of a determinant as the absolute value of

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Example 7.32

If the area of the triangle with vertices (– 3, 0), (3, 0) and (0, k) is 9 square units, find the values of k .

Solution

$$\text{Area of the triangle} = \text{absolute value of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

$$9 = \left|\frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}\right| = \left|\frac{1}{2}(-k)(-3-3)\right|$$

$$\Rightarrow \quad 9 = 3|k| \text{ and hence, } k = \pm 3.$$

Note 7.13

The area of the triangle formed by three points is zero if and only if the three points are collinear. Also, we remind the reader that the determinant could be negative whereas area is always non-negative.

Example 7.33

Find the area of the triangle whose vertices are (– 2, – 3), (3, 2), and (– 1, – 8).

Solution

$$\text{Area of the triangle} = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|.$$

$$\left|\frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}\right| = \left|\frac{1}{2}(-20+12-22)\right| = |-15| = 15$$

and therefore required area is 15 sq.units.

Example 7.34

Show that the points (a, $b + c$), (b, $c + a$), and (c, $a + b$) are collinear.

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Solution

To prove the given points are collinear, it suffices to prove $|A| = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} = 0$.

Applying $C_1 \to C_1 + C_2$, we deduce that

$$|A| = \begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix} = (a+b+c)\times 0 = 0$$

which shows that the given points are collinear.

7.2.11 Singular and non-singular Matrices

Definition 7.21

A square matrix A is said to be **singular** if $|A| = 0$. A square matrix A is said to be **non-singular** if $|A| \neq 0$.

For instance, the matrix $A = \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ -4 & 1 & 1 \end{bmatrix}$ is a singular matrix, since

$$|A| = 3(1-1) - 8(-4+4) + 1(-4+4) = 0.$$

If $B = \begin{bmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{bmatrix}$ then $|B| = 2(0-20) - (-3)(-42-4) + 5(30-0) = -28 \neq 0$.

Thus B is a non-singular matrix.

Note 7.14

If A and B are non-singular matrices of the same order then AB and BA are also non-singular matrices because $|AB| = |A|\,|B| = |BA|$.

EXERCISE 7.4

(1) Find the area of the triangle whose vertices are (0, 0), (1, 2) and (4, 3).

(2) If $(k, 2)$, (2, 4) and (3, 2) are vertices of the triangle of area 4 square units then determine the value of k.

(3) Identify the singular and non-singular matrices:

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & a-b & k \\ b-a & 0 & 5 \\ -k & -5 & 0 \end{bmatrix}$

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(4) Determine the values of a and b so that the following matrices are singular:

(i) $A=\begin{bmatrix} 7 & 3 \\ -2 & a \end{bmatrix}$ (ii) $B=\begin{bmatrix} b-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$

(5) If $\cos 2\theta = 0$, determine $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$.

(6) Find the value of the product ; $\begin{vmatrix} \log_3 64 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix}$.

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EXERCISE 7.5

Choose the correct or the most suitable answer from the given four alternatives.

(1) If $a_{ij} = \frac{1}{2}(3i-2j)$ and $A=[a_{ij}]_{2\times 2}$ is

(1) $\begin{bmatrix} \frac{1}{2} & 2 \\ -\frac{1}{2} & 1 \end{bmatrix}$ (2) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 2 & 1 \end{bmatrix}$ (3) $\begin{bmatrix} 2 & 2 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ (4) $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & 2 \end{bmatrix}$

(2) What must be the matrix X, if $2X + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$?

(1) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}$ (3) $\begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix}$ (4) $\begin{bmatrix} 2 & -6 \\ 4 & -2 \end{bmatrix}$

(3) Which one of the following is not true about the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$?

(1) a scalar matrix (2) a diagonal matrix
(3) an upper triangular matrix (4) a lower triangular matrix

(4) If A and B are two matrices such that $A + B$ and AB are both defined, then
(1) A and B are two matrices not necessarily of same order
(2) A and B are square matrices of same order
(3) Number of columns of A is equal to the number of rows of B
(4) $A = B$.

(5) If $A = \begin{bmatrix} \lambda & 1 \\ -1 & -\lambda \end{bmatrix}$, then for what value of $\lambda, A^2 = O$?

(1) 0 (2) ± 1 (3) -1 (4) 1

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(6) If $A=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$, $B=\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}$ and $(A+B)^2 = A^2+B^2$, then the values of a and b are

(1) $a=4, b=1$ (2) $a=1, b=4$ (3) $a=0, b=4$ (4) $a=2, b=4$

(7) If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & 2 & b\end{bmatrix}$ is a matrix satisfying the equation $AA^T=9I$, where I is 3 × 3 identity matrix, then the ordered pair (a, b) is equal to

(1) (2, – 1) (2) (– 2, 1) (3) (2, 1) (4) (– 2, – 1)

(8) If A is a square matrix, then which of the following is not symmetric?

(1) $A+A^T$ (2) AA^T (3) A^TA (4) $A-A^T$

(9) If A and B are symmetric matrices of order n, where $(A \neq B)$, then

(1) $A + B$ is skew-symmetric (2) $A + B$ is symmetric

(3) $A + B$ is a diagonal matrix (4) $A + B$ is a zero matrix

(10) If $A=\begin{bmatrix}a & x\\ y & a\end{bmatrix}$ and if $xy=1$, then det $(A\,A^T)$ is equal to

(1) $(a-1)^2$ (2) $(a^2+1)^2$ (3) a^2-1 (4) $(a^2-1)^2$

(11) The value of x, for which the matrix $A=\begin{bmatrix}e^{x-2} & e^{7+x}\\ e^{2+x} & e^{2x+3}\end{bmatrix}$ is singular

(1) 9 (2) 8 (3) 7 (4) 6

(12) If the points $(x,-2)$, $(5,2)$, $(8,8)$ are collinear, then x is equal to

(1) – 3 (2) $\frac{1}{3}$ (3) 1 (4) 3

(13) If $\begin{vmatrix}2a & x_1 & y_1\\ 2b & x_2 & y_2\\ 2c & x_3 & y_3\end{vmatrix}=\frac{abc}{2}\neq 0$, then the area of the triangle whose vertices are $\left(\frac{x_1}{a},\frac{y_1}{a}\right)$, $\left(\frac{x_2}{b},\frac{y_2}{b}\right)$, $\left(\frac{x_3}{c},\frac{y_3}{c}\right)$ is

(1) $\frac{1}{4}$ (2) $\frac{1}{4}abc$ (3) $\frac{1}{8}$ (4) $\frac{1}{8}abc$

(14) If the square of the matrix $\begin{bmatrix}\alpha & \beta\\ \gamma & -\alpha\end{bmatrix}$ is the unit matrix of order 2, then α, β and γ should satisfy the relation.

(1) $1+\alpha^2+\beta\gamma=0$ (2) $1-\alpha^2-\beta\gamma=0$

(3) $1-\alpha^2+\beta\gamma=0$ (4) $1+\alpha^2-\beta\gamma=0$

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(15) If $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$, then $\begin{vmatrix} ka & kb & kc \\ kx & ky & kz \\ kp & kq & kr \end{vmatrix}$ is

(1) Δ (2) $k\Delta$ (3) $3k\Delta$ (4) $k^3\Delta$

(16) A root of the equation $\begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$ is

(1) 6 (2) 3 (3) 0 (4) – 6

(17) The value of the determinant of $A = \begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is

(1) $-2abc$ (2) abc (3) 0 (4) $a^2+b^2+c^2$

(18) If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in geometric progression with the same common ratio, then the points (x_1, y_1), (x_2, y_2), (x_3, y_3) are

(1) vertices of an equilateral triangle

(2) vertices of a right angled triangle

(3) vertices of a right angled isosceles triangle

(4) collinear

(19) If $\lfloor . \rfloor$ denotes the greatest integer less than or equal to the real number under consideration and $-1 \le x < 0,\ 0 \le y < 1,\ 1 \le z < 2$, then the value of the determinant $\begin{vmatrix} \lfloor x \rfloor + 1 & \lfloor y \rfloor & \lfloor z \rfloor \\ \lfloor x \rfloor & \lfloor y \rfloor + 1 & \lfloor z \rfloor \\ \lfloor x \rfloor & \lfloor y \rfloor & \lfloor z \rfloor + 1 \end{vmatrix}$ is

(1) $\lfloor z \rfloor$ (2) $\lfloor y \rfloor$ (3) $\lfloor x \rfloor$ (4) $\lfloor x \rfloor + 1$

(20) If $a \ne b, b, c$ satisfy $\begin{vmatrix} a & 2b & 2c \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0$, then $abc =$

(1) $a+b+c$ (2) 0 (3) b^3 (4) $ab+bc$

(21) If $A = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 1 & 0 \\ -2 & 4 & 2 \end{vmatrix}$ and $B = \begin{vmatrix} -2 & 4 & 2 \\ 6 & 2 & 0 \\ -2 & 4 & 8 \end{vmatrix}$, then B is given by

(1) $B = 4A$ (2) $B = -4A$ (3) $B = -A$ (4) $B = 6A$

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(22) If A is skew-symmetric of order n and C is a column matrix of order $n \times 1$, then $C^T AC$ is

(1) an identity matrix of order n (2) an identity matrix of order 1

(3) a zero matrix of order 1 (4) an identity matrix of order 2

(23) The matrix A satisfying the equation $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ is

(1) $\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & -4 \\ 1 & 0 \end{bmatrix}$ (3) $\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$

(24) If $A + I = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$, then $(A+I)(A-I)$ is equal to

(1) $\begin{bmatrix} -5 & -4 \\ 8 & -9 \end{bmatrix}$ (2) $\begin{bmatrix} -5 & 4 \\ -8 & 9 \end{bmatrix}$ (3) $\begin{bmatrix} 5 & 4 \\ 8 & 9 \end{bmatrix}$ (4) $\begin{bmatrix} -5 & -4 \\ -8 & -9 \end{bmatrix}$

(25) Let A and B be two symmetric matrices of same order. Then which one of the following statement is not true?

(1) $A + B$ is a symmetric matrix (2) AB is a symmetric matrix

(3) $AB = (BA)^T$ (4) $A^T B = AB^T$

SUMMARY

In this chapter we have acquired the knowledge of

- A matrix is a rectangular array of real numbers or real functions on $\mathbb{R}$ or complex numbers.
- A matrix having m rows and n columns, then the order of the matrix is $m \times n$.
- A matrix $A = [a_{ij}]_{m\times n}$ is said to be a
  square matrix if $m = n$
  row matrix if $m = 1$
  column matrix if $n = 1$
  zero matrix if $a_{ij} = 0 \,\forall\, i$ and j
  diagonal matrix if $m = n$ and $a_{ij} = 0 \,\forall\, i \neq j$
  scalar matrix if $m = n$ and $a_{ij} = 0 \,\forall\, i \neq j$ and $a_{ii} = \lambda$ for all i
  unit matrix or identity matrix if $m = n$ and $a_{ij} = 0$ for all $i \neq j$ and $a_{ii} = 1 \;\forall\; i$
  upper triangular matrix if $m = n$ and $a_{ij} = 0 \;\forall\; i > j$
  lower triangular matrix if $m = n$ and $a_{ij} = 0 \;\forall\; i < j$.
- Matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, are said to be equal if $a_{ij} = b_{ij} \;\forall\; i$ and j
- If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, then $A + B = [c_{ij}]_{m\times n}$, where $c_{ij} = a_{ij} + b_{ij}$
- If $A = [a_{ij}]_{m\times n}$ and λ is a scalar, then $\lambda A = [\lambda a_{ij}]_{m\times n}$

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- $-A = (-1)A$
- $A + B = B + A$
- $A - B = A + (-1)B$
- $(A + B) + C = A + (B + C)$ where A, B and C have the same order.
- $A(BC) = (AB)C$ (ii) $A(B+C) = AB+AC$ (iii) $(A+B)C = AC+BC$
- The transpose of A, denoted by A^T is obtained by interchanging rows and columns of A.
  (i) $(A^T)^T = A$, (ii) $(kA)^T = kA^T$, (iii) $(A+B)^T = A^T + B^T$, (iv) $(AB)^T = B^T A^T$
- A square matrix A is called
  (i) symmetric if $A^T = A$ and (ii) skew-symmetric if $A^T = -A$
- Any square matrix can be expressed as sum of a symmetric and skew-symmetric matrices.
- The diagonal entries of a skew-symmetric must be zero.
- For any square matrix A with real entries, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric and further $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.
- Determinant is defined only for square matrices.
- $|A^T| = |A|$.
- $|AB| = |A|\,|B|$ where A and B are square matrices of same order.
- If $A = [a_{ij}]_{m\times n}$, then $|kA| = k^n |A|$, where k is a scalar.
- A determinant of a square matrix A is the sum of products of elements of any row (or column) with its corresponding cofactors; for instance, $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.
- If the elements of a row or column is multiplied by the cofactors of another row or column, then their sum is zero; for example, $a_{11}A_{13} + a_{12}A_{23} + a_{13}A_{33} = 0$.
- The determinant value remains unchanged if all its rows are interchanged by its columns.
- If all the elements of a row or a column are zero, then the determinant is zero.
- If any two rows or columns are interchanged, then the determinant changes its sign.
- If any two rows or columns are identical or proportional, then the determinant is zero.
- If each element of a row or a column is multiplied by constant k, then determinant gets multiplied by k.
- If each element in any row (column) is the sum of r terms, then the determinant can be expressed as the sum of r determinants.
- A determinant remains unaltered under a row (R_i) operation of the form $R_i + \alpha R_j + \beta R_k\,(j, k \neq i)$ or a Column (C_i) operation of the form $C_i + \alpha C_j + \beta C_k\,(j, k \neq i)$ where α, β are scalars.
- **Factor theorem :** If each element of $|A|$ is a polynomial in x and if $|A|$ vanishes for $x = a$, then $x - a$ is a factor of $|A|$.
- Area of the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by the absolute value of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.
  If the area is zero, then the three points are collinear.
- A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

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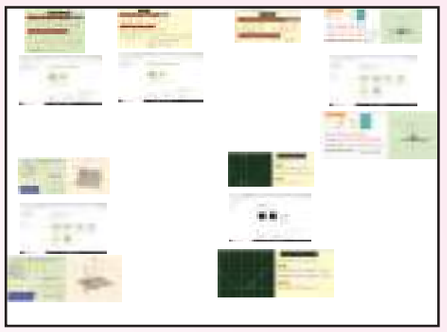
ICT CORNER 7(a)

Matrices and Determinants

Expected Outcome



Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code.

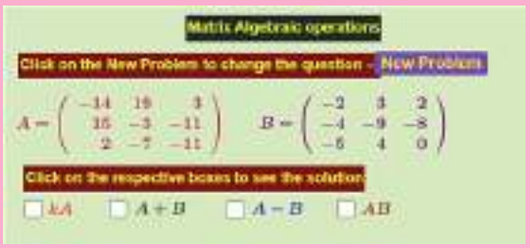
GeoGebra Workbook called "Matrices and Determinants" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Matrices-Algebraic operations"work out the operations given and Select the check boxes to verify corresponding answers.

Click on "New Problem" to get new question.

Step1



Step2

Browse in the link:

Matrices and Determinants: https://ggbm.at/cpknpvvh

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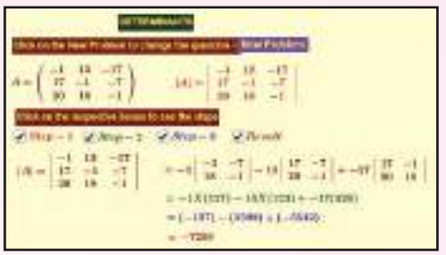
ICT CORNER 7(b)

Matrices and Determinants

Expected Outcome



Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code.

GeoGebra Workbook called "Matrices and Determinants" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Determinants" Evaluate the determinant for the matrix given and Select the check boxes to verify steps.

Click on "New Problem" to get new question.

Step1

Step2

Browse in the link:

Matrices and Determinants: https://ggbm.at/cpknpvvh

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Chapter 8 Vector Algebra

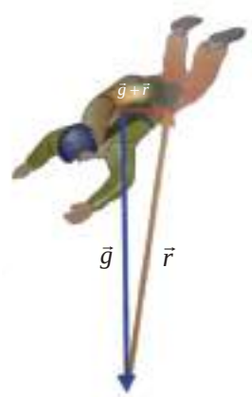
BA7QAF

"On earth there is nothing great but man;
In man there is nothing great but mind"

- Hamilton

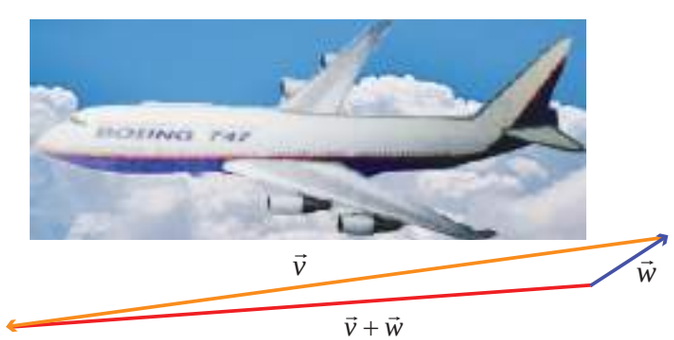
8.1 Introduction

A **pilot constructing a flight plan** has to be concerned about the plane's course, heading, air speed, and ground speed. In order for the plane to proceed directly toward its destination, it must head into the wind at an angle such that the wind is exactly counteracted. If available, a navigation computer will do the calculation quickly and accurately. If, however, a navigation computer is not accessible, the pilot may have to depend on pencil-and-paper work supplemented by a calculator with a knowledge of vectors. An understanding of vectors and their operations is therefore vitally important.



At a certain point during a jump, there are two principal forces acting on a **skydiver**. One force $(\vec{g})$ gravity exerting straight down and another air resistance $(\vec{r})$ exerting up as well as to some direction. What is the net force acting on the skydiver? The answer is $\vec{g}+\vec{r}$. (how?)

Let $\vec{v}$ be the velocity vector of an aircraft. Suppose that the wind velocity is given by the vector $\vec{w}$, what is the effective velocity of aircraft? The answer is $\vec{v}+\vec{w}$. In what direction should the aircraft head in order to fly due west?



A **global positioning system** (GPS) is a system designed to help to navigate on the earth, in the air and on water. Vectors are also used in GPS.

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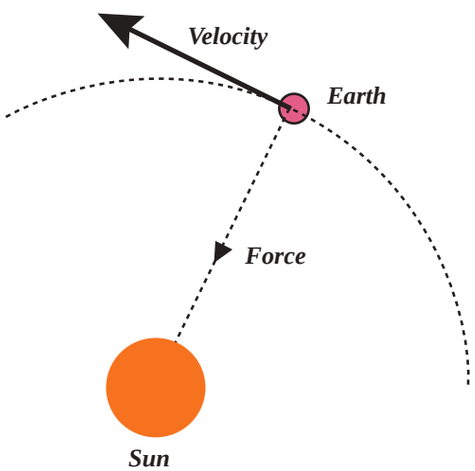
The development of the concept of vectors was influenced by the works of the German mathematician **H.G. Grassmann** (1809 - 1877) and the Irish mathematician **W.R. Hamilton** (1805 – 1865). While Hamilton occupied high positions, Grassman was a secondary school teacher.

Hamilton
(1805 - 1865)

The best features of Quaternion Calculus and Cartesian Geometry were united, largely through the efforts of the American Mathematician **J.B. Gibbs** (1839 - 1903) and **Q.Heaviside** (1850 - 1925) of England and new subject called Vector Algebra was created. The development of the algebra of vectors and of vector analysis as we know it today was first revealed in sets of remarkable notes made by Gibbs for his students at Yale University. **Clifford** (1845 – 1879), in his **Elements of Dynamics** (1878), broke down the product of two quaternions into two very different vector products, which he called the scalar product and the vector product. The term vectors was due to Hamilton and it was derived from the Latin word **'to carry'**.

The theory of vector was also based on Grassman's theory of extension.



It was soon realised that vectors would be the ideal tools for the fruitful study of many ideas in geometry and physics. **Vectors are now the modern language** of a great deal of physics and applied mathematics and they continue to hold their own intrinsic mathematical interest.

Learning Objectives

On completion of this chapter, the students are expected to

- realise vectors as a tool to study the various geometric and physics problems.
- distinguish the scalars from vectors.
- understand different types of vectors and algebra of vectors.
- understand the geometrical interpretations and resolutions of 2D and 3D vectors.
- appreciate the usage of matrix theory in vector algebra.
- visualise scalar product and vector product yielding scalars and vectors respectively as a unique feature.

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8.2 Scalars and Vectors

Definition 8.1

A scalar is a quantity that is determined by its magnitude.

For instance, distance, length, speed, temperature, voltage, mass, pressure, and work are scalars.

Definition 8.2

A **vector** is a quantity that is determined by both its magnitude and its direction and hence it is a directed line segment.

For instance, force, displacement, and velocity (which gives the speed and direction of the motion) are vectors.

We denote vectors by lower case letters with arrow. A two dimensional vector is a directed line segment in $\mathbb{R}^2$ and a three dimensional vector is a directed line segment in $\mathbb{R}^3$.

8.3 Representation of a vector and types of vectors

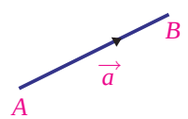

Fig 8.1

A vector has a tail and a tip. Consider the diagram as in Fig. 8.1.

Definition 8.3

The tail point A is called the **initial point** and the tip point B is called the **terminal point** of the vector $\vec{a}$. The initial point of a vector is also taken as origin of the vector.

The initial point A of the vector $\vec{a}$ is the original position of a point and the terminal point B is its position after the translation.

The length or magnitude of the vector $\vec{a}$ is the length of the line segment AB and is denoted by $|\vec{a}|$.

The undirected line AB is called the **support of the vector** $\vec{a}$.

To distinguish between an ordinary line segment without a direction and a line segment representing a vector, we make an arrow mark for the vector as $\overrightarrow{AB}$ and $\vec{a}$. So AB denotes the line segment.

Definition 8.4

If we have a liberty to choose the origin of the vector at any point then it is said to be **free vector,** whereas if it is restricted to a certain specified point then the vector is said to be **localized vector.**

Upto vector product we will be dealing with free vectors only. Localised vectors are involved in finding equations of straight lines.

Definition 8.5

Co-initial vectors are having the same initial point. On the other hand, the **co-terminous vectors** are having the same terminal point.

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Definition 8.6

Two or more vectors are said to be **collinear** or **parallel** if they have same line of action or have the lines of action parallel to one another.

Two or more vectors are said to be **coplanar** if they lie on the same plane or parallel to the same plane.

Definition 8.7

Two vectors are said to be **equal** if they have same direction and same magnitude.

Let us note that it is not necessary to have the same initial point and same terminal point for two equal vectors. For instance, in Fig. 8.2, the vectors $\vec{b}$ and $\vec{c}$ are equal since they have same direction and same length, whereas $\vec{a}$ and $\vec{b}$ are not equal because of opposite direction even though they are having same length. The vectors $\vec{c}$ and $\vec{d}$ are not equal even though they are having same direction but not having same length.

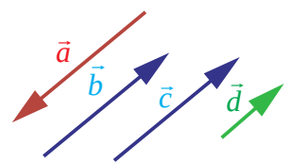

Fig. 8.2

Definition 8.8

Zero vector is a vector which has zero magnitude and an arbitrary direction and it is denoted by $\vec{0}$.

That is, a vector whose initial and terminal points are coincident is called a zero vector.

We observe that the initial and terminal points of a zero vector are the same. The zero vector is also called **null vector** or **void vector**.

A vector of magnitude 1 is called a **unit vector**. The unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$ (read as '*a* cap' or '*a* hat'). Clearly $|\hat{a}| = 1$.

We observe that there are infinitely many directions and hence there are infinitely many unit vectors. In fact, for each direction there is one unit vector in that direction.

Any non-zero vector $\vec{a}$ can be written as the scalar multiple of a unit vector in the direction of $\vec{a}$. This scalar is nothing but the magnitude of the vector.

Thus for any vector $\vec{a} = |\vec{a}|\,\hat{a}$, where $\hat{a}$ is the unit vector along the direction of $\vec{a}$.

Clearly $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ for any non-zero vector.

Definition 8.9

Two vectors are said to be **like vectors** if they have the same direction. Two vectors are said to be **unlike vectors** if they have opposite directions.

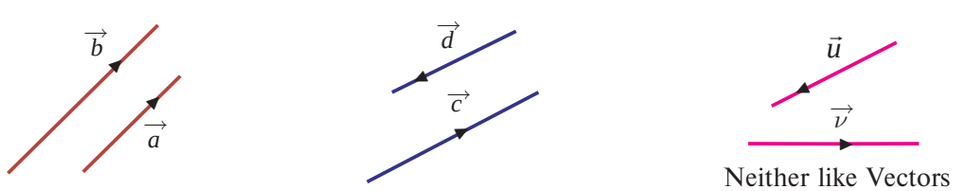

Fig. 8.3

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We observe that if two vectors are like vectors or unlike vectors, then the undirected lines (support) of the vectors are parallel to each other. There are pair of vectors which are neither like nor unlike vectors.

8.4 Algebra of Vectors

We have studied basic algebraic operations on real numbers and on matrices. Similarly we studied some operations on vectors. Now let us see how to add two vectors, subtract a vector from another vector and multiply a vector by a scalar.

8.4.1 Addition of Vectors

Let us define the sum of two vectors in two ways and see that they are the same. Let us assume that an object of unit mass is placed at the origin (0,0) in $\mathbb{R}^2$. We assume that the size of the object is just a point. Let us assume that two forces $\vec{a}$ and $\vec{b}$ of unit magnitude act on the object in the positive directions of x-axis and y-axis respectively (Fig.8.4). It is easy to guess that the object will move in the direction 45° to the x-axis as indicated in Fig.8.5. The forces $\vec{a}$ and $\vec{b}$ are equal to the vectors $\vec{a}$ and $\vec{b}$ as indicated in Fig. 8.6. We may think that the forces push the object in Fig. 8.4 and pull the object in Fig.8.6.

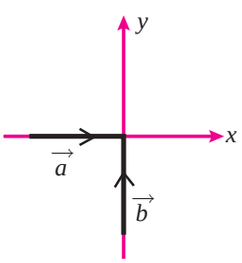

Fig. 8.4

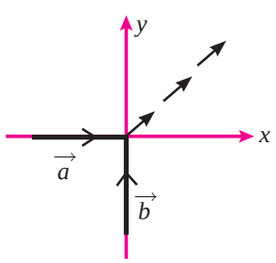

Fig. 8.5

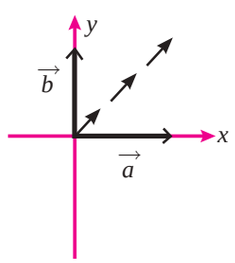

Fig. 8.6

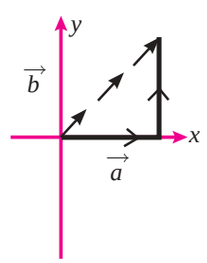

Fig. 8.7

The next question before us is 'How long will it go?'. Let us assume that the forces act one after the other. The force $\vec{a}$ will move the object one unit along the x-axis. So the object will move from (0, 0) to (1, 0). Now the force $\vec{b}$ will move the object vertically from (1, 0) to (1, 1). So finally the object will be at (1, 1) (Fig. 8.7). Thus the sum of the two vectors may be defined as the line segment joining (0, 0) and (1, 1) in the direction '(0, 0) to (1, 1)'.

Now, as in the same situation discussed above, let us assume that the force $\vec{a}$ has magnitude 2 instead of 1 (Fig. 8.8). It will not be difficult to guess that the object will move in a direction much closure to the x-axis as indicated in Fig. 8.9. Also we may guess that the object will go to the point (2,1). Thus the sum of the two vectors may be defined as the line segment joining (0,0) and (2,1) in the direction "(0,0) to (2,1)".

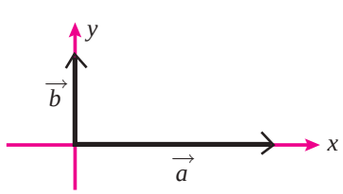

Fig. 8.8

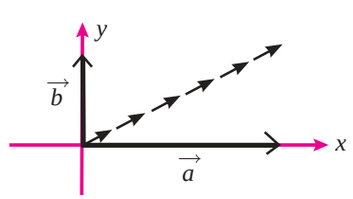

Fig. 8.9

In the two situations discussed above the directions of the forces are perpendicular to each other. This need not be the case in general. Even then we can add the forces by considering one after the other. For example let $\vec{a}$ and $\vec{b}$ be two forces in a plane as shown in Fig. 8.10.

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Fig. 8.10

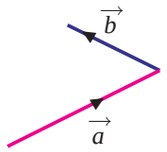

Fig. 8.11

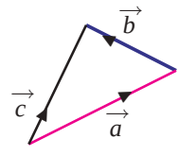

Fig. 8.12

Bringing the initial point of $\vec{b}$ to the terminal point of $\vec{a}$ (Fig. 8.11), we can get the resultant of these two forces (see Fig. 8.12). This motivates us to define the sum of two vectors.

Triangle law of addition

Let $\vec{a}$ and $\vec{b}$ be two vectors. Let A_1 and B_1 be the initial points of $\vec{a}$ and $\vec{b}$, and A_2 and B_2 be the terminal points of $\vec{a}$ and $\vec{b}$ respectively.

Draw A_3B_3 parallel to B_1B_2 so that $A_3B_3 = B_1B_2$. Then the vector $\overrightarrow{A_1B_3}$ is defined as the sum of the vectors $\vec{a}$ and $\vec{b}$, and it is denoted as $\vec{a}+\vec{b}$. This can be restated as,

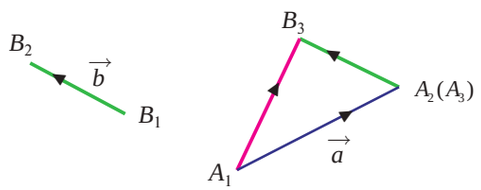

Fig. 8.13

Definition 8.10 (Triangle law of addition)

If two vectors are represented in magnitude and direction by the two sides of a triangle taken in order, then their sum is represented by the third side taken in the reverse order.

Result 8.1

If $\vec{a}, \vec{b}$ and $\vec{c}$ are the sides of a triangle taken in order then $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

Proof

Let $\overrightarrow{AB} = \vec{a}, \overrightarrow{BC} = \vec{b}$, and $\overrightarrow{CA} = \vec{c}$.

Now $\vec{a}+\vec{b}+\vec{c} = \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = \overrightarrow{AC}+\overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$.

Thus the result is proved.

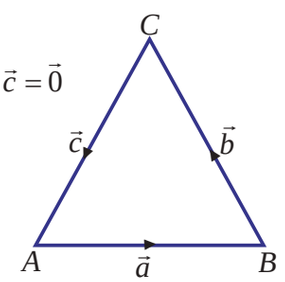

Fig. 8.14

Parallelogram law of vector addition

Let $\vec{a}$ and $\vec{b}$ be two vectors. Assuming that the initial points of the two vectors are the same, let us find the sum according to Definition 8.7. Let A and B be the terminal points of $\vec{a}$ and $\vec{b}$ respectively (Fig. 8.15). To find $\vec{a}+\vec{b}$, we draw AC parallel to OB so that $OB = AC$ and declare that $\overrightarrow{OC}$ is the sum (Fig. 8.16). We observe that OA and BC are parallel (Fig. 8.17).

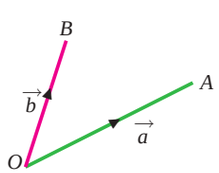

Fig. 8.15

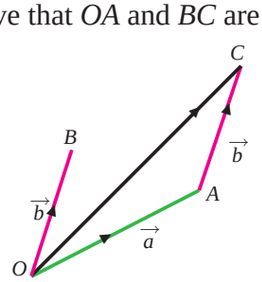

Fig. 8.16

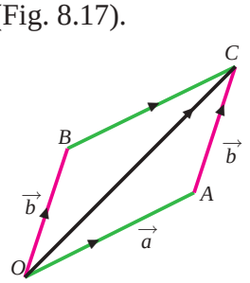

Fig. 8.17

So to find the sum of two vectors with the same initial point, draw the parallelogram with the given vectors as adjacent sides and declare the diagonal as the sum. Even the vectors do not have the

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same initial point, we can move one of the vectors suitably and make them to have same initial point. This leads us to the following Definition 8.11.

Let $\vec{a}$ and $\vec{b}$ be two vectors with the same initial point O. Let A and B be the terminal points of $\vec{a}$ and $\vec{b}$ respectively.

Complete the parallelogram $OACB$. Then the vector $\overrightarrow{OC}$ is defined as the sum of the vectors $\vec{a}$ and $\vec{b}$. Thus

Definition 8.11 (**Parallelogram law of addition**)

In a parallelogram $OABC$ if $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represents two adjacent sides, then the diagonal $\overrightarrow{OC}$ represents their sum (see Fig. 8.17).

Though we have two definitions for addition of vectors, they are one and the same. Definition 8.10 is defined using the triangle law for addition of vectors and Definition 8.11 is defined using the parallelogram law for addition of vectors:

In a triangle ABC if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ represent two sides, then the third side $\overrightarrow{AC}$ represents their sum.

8.4.2 Difference between two Vectors

Now let us see how to subtract one vector from another vector.

Definition 8.12

Let $\vec{a}$ be a vector. Then the **reverse** of $\vec{a}$, denoted by $-\vec{a}$, is defined as the vector having the magnitude of $\vec{a}$ and the direction opposite to the direction of $\vec{a}$.

Notice that if $\overrightarrow{AB} = \vec{a}$, then $\overrightarrow{BA} = -\vec{a}$.

Geometrical interpretation of difference between two vectors

Let $\vec{a}$ be a vector with initial point P and terminal point Q. Let $\vec{b}$ be the vector with initial point Q and terminal point P. The magnitude of both of the vectors is the length of the line segment joining P and Q. So they have the same magnitude. But clearly they have opposite directions. So $\vec{b}$ is equal to $-\vec{a}$.

If $\vec{a}$ and $\vec{b}$ are two vectors, then the vector $\vec{a} - \vec{b}$ is defined as the sum of the vectors $\vec{a}$ and $-\vec{b}$. That is $\vec{a} + (-\vec{b})$.

We can view the vector $\vec{a} - \vec{b}$ geometrically. Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the vectors $\vec{a}$ and $\vec{b}$ respectively (Fig. 8.18) . Draw AC parallel to OB with $AC = OB$. Then $\overrightarrow{AC}$ is equal to $\vec{b}$. Extend the line CA to D so that $CA = AD$. Then $\overrightarrow{AD}$ is equal to $-\vec{b}$. Thus $\vec{a} + (-\vec{b}) = \overrightarrow{OD}$. Hence $\vec{a} - \vec{b} = \overrightarrow{OD}$ (Fig. 8.19).

Let us complete the parallelogram $OACB$. We observe that BA and OD are parallel and they have equal length. Thus the two vectors $\overrightarrow{BA}$ and $\overrightarrow{OD}$ are equal. So we may consider $\overrightarrow{BA}$ as $\vec{a} - \vec{b}$. This shows that if the sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$ of the parallelogram $OACB$ represent the vectors $\vec{a}$ and $\vec{b}$ respectively, then the diagonal $\overrightarrow{BA}$ will represent the vector $\vec{a} - \vec{b}$. (Fig. 8.20). We note that we have already seen that the diagonal $\overrightarrow{OC}$ represents the vector $\vec{a} + \vec{b}$.

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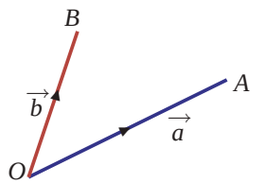

Fig. 8.18

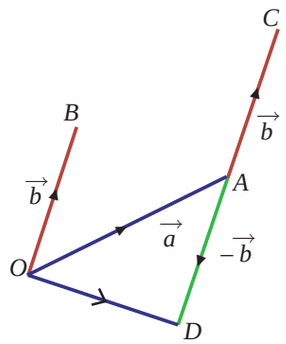

Fig. 8.19

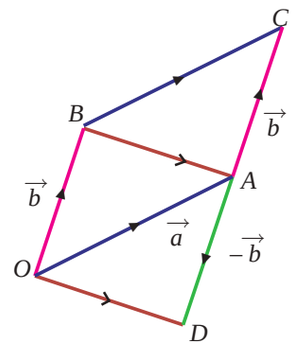

Fig. 8.20

Thus, **if** $\vec{a}$ and $\vec{b}$ **represent two adjacent sides of a parallelogram then the diagonals represent** $\vec{a}+\vec{b}$ **and** $\vec{a}-\vec{b}$ **.**

8.4.3 Scalar multiplication of a vector

Now let us see how to multiply a vector by a scalar.

Let $\vec{a}$ be a vector and m be a scalar. Then the vector $m\vec{a}$ is called the **scalar multiple of a vector** $\vec{a}$ by the scalar m.

Let us note that when m is zero, the magnitude of $m\vec{a}$ becomes 0 and hence $m\vec{a}$ becomes the zero vector. If m is positive, then both $\vec{a}$ and $m\vec{a}$ have the same direction and when m is negative, then $\vec{a}$ and $m\vec{a}$ have opposite directions. Thus $\vec{a}$ and $m\vec{a}$ are like vectors if m is positive and unlike vectors if m is negative. The magnitude of $m\vec{a}$ is $|m\vec{a}| = |m||\vec{a}|$.

Definition 8.13

Two vectors $\vec{a}$ and $\vec{b}$ are said to be **parallel** if $\vec{a} = \lambda\vec{b}$, where λ is a scalar. If $\lambda > 0$, they are in the same direction. If $\lambda < 0$ then they are in the opposite direction to each other.

8.4.4 Some properties and results

For any two vectors $\vec{a}$ and $\vec{b}$ and scalars m and n, we have

(i) $m(n\vec{a}) = mn(\vec{a}) = n(m\vec{a})$

(ii) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

(iii) $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

BAGLC3

Result 8.2

Vector addition is associative.

For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$,

$$(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c}).$$

Result 8.3

For any vector $\vec{a}$, $\vec{a}+\vec{0} = \vec{0}+\vec{a} = \vec{a}$.

Result 8.4

For any vector $\vec{a}$, $\vec{a}+(-\vec{a}) = (-\vec{a})+\vec{a} = \vec{0}$.

This result states that the additive inverse exists for every vector.

Result 8.5

Vector addition is commutative.

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Proof

Let $\vec{a}$ and $\vec{b}$ be two vectors. Let $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$.

Complete the parallelogram $OACB$ with $\vec{a}$ and $\vec{b}$ as adjacent sides. The vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$ have same direction and equal magnitude; so $\overrightarrow{OB} = \overrightarrow{AC}$. Thus

$$\vec{a} + \vec{b} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}.$$

As ,
$$\overrightarrow{OA} = \overrightarrow{BC},$$
$$\vec{b} + \vec{a} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC}.$$

Thus
$$\vec{a} + \vec{b} = \vec{b} + \vec{a}.$$

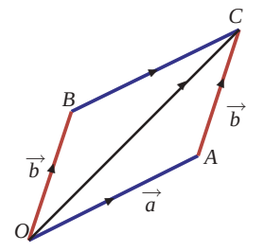

Fig. 8.21

Polygon law of addition

Let $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$, and $\overrightarrow{DE}$ be any five vectors as shown in the Fig. 8.22.

We observe from the figure that each vector is drawn from the terminal point of its previous one. By the triangle law,

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \;;\; \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC}$$

$$\overrightarrow{OC} + \overrightarrow{CD} = \overrightarrow{OD} \;;\; \overrightarrow{OD} + \overrightarrow{DE} = \overrightarrow{OE}$$

Thus $\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{OE}$.

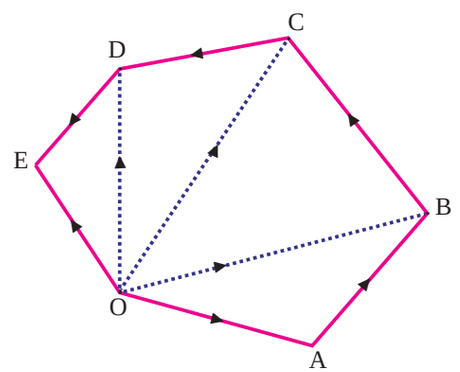

Fig. 8.22

Thus the line joining the initial point of first vector to the terminal point at the last vector is the sum of all the vectors. This is called the **polygon law of addition** of vectors.

Example 8.1

Represent graphically the displacement of

(i) 30 km 60° west of north

(ii) 60 km 50° south of east.

Solution

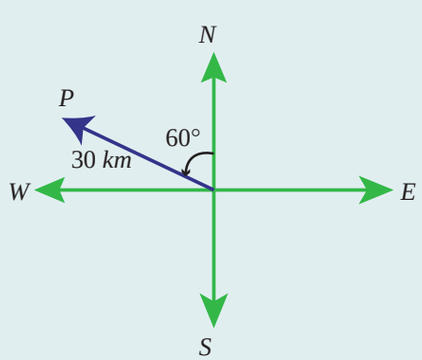

Fig. 8.23

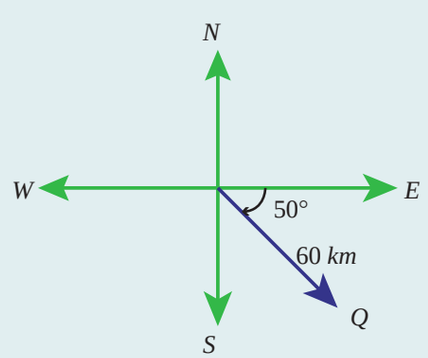

Fig. 8.24

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Example 8.2

If $\vec{a}$ and $\vec{b}$ are vectors represented by two adjacent sides of a regular hexagon, then find the vectors represented by other sides.

Solution

Let A,B,C,D,E,F be the vertices of a regular hexagon.

Let $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{BC}$

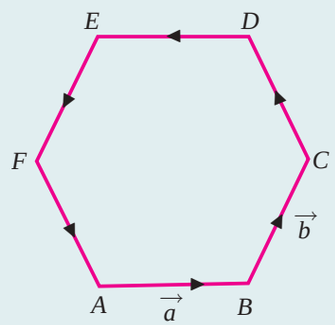

Fig. 8.25

We use the following facts about regular hexagon.

(i) The lines AB, CF and ED are parallel and the lines BC, AD and EF are parallel.

(ii) The length of CF is twice the length of AB and the length of AD is twice the length of BC.

Since the lines AB and DE are parallel, equal in length and opposite in direction we have

$$\overrightarrow{DE} = -\vec{a}\ .$$

Since the lines AB and CF are parallel and opposite in direction we have

$$\overrightarrow{CF} = -2\vec{a}.$$

Similarly $\overrightarrow{EF} = -\vec{b}$ and $\overrightarrow{AD} = 2\vec{b}$.

Since $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ we have

$$\overrightarrow{AC} = \vec{a} + \vec{b}.$$

Since $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$ we have

$$\vec{a} + \vec{b} + \overrightarrow{CD} = 2\vec{b}.$$

Thus $\overrightarrow{CD} = 2\vec{b} - (\vec{a} + \vec{b}) = \vec{b} - \vec{a}.$

As $\overrightarrow{FA} = -\overrightarrow{CD}$, we have

$$\overrightarrow{FA} = \vec{a} - \vec{b}.$$

Hence, for given sides $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$, we have obtained all other sides of the hexagon as $\overrightarrow{CD} = \vec{b} - \vec{a}, \overrightarrow{DE} = -\vec{a}, \overrightarrow{EF} = -\vec{b}$, and $\overrightarrow{FA} = \vec{a} - \vec{b}$.

8.5 Position vectors

Definition 8.14

Let O be the origin and P be any point (in the plane or space). Then the vector $\overrightarrow{OP}$ is called the **position vector** of the point P with respect to the origin O (point of reference).

The relation between the vectors and position vectors are given in the following result.

Result 8.6

Let O be the origin, A and B be two points. Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ where, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are position vectors of A and B respectively.

Proof

We know that, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$. Thus $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

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Theorem 8.1 (Section Formula - Internal Division)

Let O be the origin. Let A and B be two points. Let P be the point which divides the line segment AB internally in the ratio $m : n$. If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B, then the position vector $\overrightarrow{OP}$ of P is given by

$$\overrightarrow{OP} = \frac{n\vec{a} + m\vec{b}}{n+m}.$$

Proof

Since O is the origin, $\vec{a}$ and $\vec{b}$ are the position vectors of A and B, we have

$$\overrightarrow{OA} = \vec{a} \text{ and } \overrightarrow{OB} = \vec{b}.$$

Let $\overrightarrow{OP} = \vec{r}$.

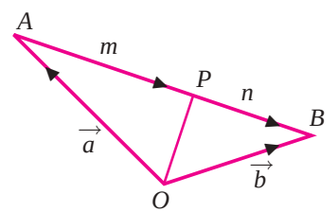

Fig. 8.26

Since P divides the line segment AB internally in the ratio $m : n$, we have,

$$\frac{|\overrightarrow{AP}|}{|\overrightarrow{PB}|} = \frac{m}{n}$$

and hence $\quad n|\overrightarrow{AP}| = m|\overrightarrow{PB}|.$

But the vectors $\overrightarrow{AP}$ and $\overrightarrow{PB}$ have the same direction. Thus

$$n\overrightarrow{AP} = m\overrightarrow{PB}. \tag{8.1}$$

But $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - \vec{a}$ and $\overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP} = \vec{b} - \vec{r}$

Substituting this in (8.1), we get

$$n(\vec{r} - \vec{a}) = m(\vec{b} - \vec{r})$$

and hence

$$(n+m)\vec{r} = n\vec{a} + m\vec{b}.$$

Thus $\quad \overrightarrow{OP} = \dfrac{n\vec{a} + m\vec{b}}{n+m}.$ ■

Theorem 8.2 Section Formula - External Division (Without proof)

Let O be the origin. Let A and B be two points. Let P be the point which divides the line segment AB externally in the ratio $m : n$. If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B, then the position vector $\overrightarrow{OP}$ of P is given by

$$\overrightarrow{OP} = \frac{n\vec{a} - m\vec{b}}{n-m}.$$

Note 8.1

By taking $m = n = 1$ in Theorem 8.1, we see that the position vector of the midpoint of the line joining the points A and B is $\dfrac{\vec{a}+\vec{b}}{2}$, where $\vec{a}$ and $\vec{b}$ are the position vectors of the points A and B respectively.

From the above theorem we can get a condition for three points to be collinear.

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Result 8.7

Three distinct points A, B and C with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear if and only if there exist real numbers x,y,z, none of them is zero, such that

$$x + y + z = 0 \text{ and } x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}.$$

Example 8.3

Let A and B be two points with position vectors $2\vec{a} + 4\vec{b}$ and $2\vec{a} - 8\vec{b}$. Find the position vectors of the points which divide the line segment joining A and B in the ratio 1:3 internally and externally.

Solution

Let O be the origin. It is given that

$$\overrightarrow{OA} = 2\vec{a} + 4\vec{b} \text{ and } \overrightarrow{OB} = 2\vec{a} - 8\vec{b}.$$

Let C and D be the points which divide the segment AB in the ratio 1 : 3 internally and externally respectively. Then

$$\overrightarrow{OC} = \frac{3\overrightarrow{OA} + \overrightarrow{OB}}{3+1} = \frac{3(2\vec{a} + 4\vec{b}) + (2\vec{a} - 8\vec{b})}{4} = 2\vec{a} + \vec{b}.$$

$$\overrightarrow{OD} = \frac{3\overrightarrow{OA} - \overrightarrow{OB}}{3-1} = \frac{3(2\vec{a} + 4\vec{b}) - (2\vec{a} - 8\vec{b})}{2} = 2\vec{a} + 10\vec{b}.$$

Let us recall the definition that the line joining a vertex of a triangle with the midpoint of its opposite side is called a median. The centroid divides the median from vertex to the midpoint of the opposite side internally in the ratio 2:1.

Theorem 8.3

The medians of a triangle are concurrent.

Proof

Let ABC be a triangle and let D, E, F be the mid points of its sides BC, CA and AB respectively. We have to prove that the medians AD, BE, CF are concurrent.

Let O be the origin and $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of A, B, and C respectively.

The position vectors of D, E, and F are respectively

$$\frac{\vec{b}+\vec{c}}{2}, \frac{\vec{c}+\vec{a}}{2}, \frac{\vec{a}+\vec{b}}{2}.$$

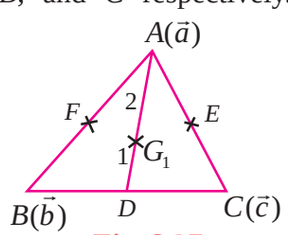

Fig. 8.27

Let G_1 be the point on AD dividing it internally in the ratio 2 : 1

Therefore, position vector of $G_1 = \dfrac{1\overrightarrow{OA} + 2\overrightarrow{OD}}{1+2}$

$$\overrightarrow{OG_1} = \frac{1\vec{a} + 2\left(\dfrac{\vec{b}+\vec{c}}{2}\right)}{3} = \frac{\vec{a}+\vec{b}+\vec{c}}{3} \qquad (1)$$

Let G_2 be the point on BE dividing it internally in the ratio 2 : 1

Therefore, $\overrightarrow{OG_2} = \dfrac{1\overrightarrow{OB} + 2\overrightarrow{OE}}{1+2}$

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$$\overrightarrow{OG_2} = \frac{1\vec{b} + 2\left(\frac{\vec{c}+\vec{a}}{2}\right)}{3} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}. \qquad (2)$$

Similarly if G_3 divides CF in the ratio 2 : 1 then

$$\overrightarrow{OG_3} = \frac{\vec{a}+\vec{b}+\vec{c}}{3} \qquad (3)$$

From (1), (2), and (3) we find that the position vectors of the three points G_1, G_2, G_3 are one and the same. Hence they are not different points. Let the common point be G.

Therefore the three medians are concurrent and the point of concurrence is G.

Theorem 8.4

A quadrilateral is a parallelogram if and only if its diagonals bisect each other.

Proof

Let A, B, C, D be the vertices of a quadrilateral with diagonals AC and BD. Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be the position vectors of A, B, C, and D respectively with respect to O.

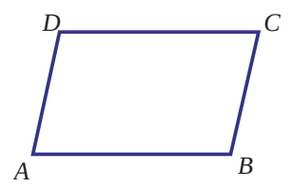

Fig. 8.28

Let the quadrilateral $ABCD$ be a parallelogram. Then

$$\overrightarrow{AB} = \overrightarrow{DC} \Rightarrow \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD} \Rightarrow \vec{b} - \vec{a} = \vec{c} - \vec{d} \Rightarrow \vec{b} + \vec{d} = \vec{a} + \vec{c}$$

and hence $$\frac{\vec{b}+\vec{d}}{2} = \frac{\vec{a}+\vec{c}}{2}.$$

This shows that the position vectors of the midpoint of the line segments AC and BD are the same. In other words, the diagonals bisect each other.

Conversely let us assume that the diagonal bisects each other. Thus the position vectors of the midpoint of the line segments AC and BD are the same. Thus

$$\frac{\vec{a}+\vec{c}}{2} = \frac{\vec{b}+\vec{d}}{2} \Rightarrow \vec{a}+\vec{c} = \vec{b}+\vec{d} \Rightarrow \vec{c}-\vec{d} = \vec{b}-\vec{a}.$$

This implies that $\overrightarrow{OC} - \overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OA}$ and hence $\overrightarrow{DC} = \overrightarrow{AB}$. This shows that the lines AB and DC are parallel. From $\vec{a}+\vec{c} = \vec{b}+\vec{d}$ we see that $\vec{a}-\vec{d} = \vec{b}-\vec{c}$ which shows that the lines AD and BC are parallel. Hence $ABCD$ is a parallelogram.

EXERCISE 8.1

(1) Represent graphically the displacement of

(i) $45cm$ $30°$ north of east. (ii) $80km$, $60°$ south of west

(2) Prove that the relation R defined on the set V of all vectors by '$\vec{a}$ R $\vec{b}$ if $\vec{a} = \vec{b}$' is an equivalence relation on V.

(3) Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A and B. Prove that the position vectors of the points which trisects the line segment AB are $\frac{\vec{a}+2\vec{b}}{3}$ and $\frac{\vec{b}+2\vec{a}}{3}$.

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(4) If D and E are the midpoints of the sides AB and AC of a triangle ABC, prove that $\overrightarrow{BE}+\overrightarrow{DC}=\frac{3}{2}\overrightarrow{BC}$.

(5) Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side whose length is half of the length of the third side.

(6) Prove that the line segments joining the midpoints of the adjacent sides of a quadrilateral form a parallelogram.

(7) If $\vec{a}$ and $\vec{b}$ represent a side and a diagonal of a parallelogram, find the other sides and the other diagonal.

(8) If $\overrightarrow{PO}+\overrightarrow{OQ}=\overrightarrow{QO}+\overrightarrow{OR}$, prove that the points P, Q, R are collinear.

(9) If D is the midpoint of the side BC of a triangle ABC, prove that $\overrightarrow{AB}+\overrightarrow{AC}=2\overrightarrow{AD}$.

(10) If G is the centroid of a triangle ABC, prove that $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\vec{0}$.

(11) Let A, B, and C be the vertices of a triangle. Let D,E, and F be the midpoints of the sides BC, CA, and AB respectively. Show that $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\vec{0}$.

(12) If $ABCD$ is a quadrilateral and E and F are the midpoints of AC and BD respectively, then prove that $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD}=4\overrightarrow{EF}$.

8.6 Resolution of Vectors

Resolution of a vector can be done for any finite dimension. But we will discuss only in two and three dimensions. Let us start with two dimension.

8.6.1 Resolution of a vector in two dimension

Theorem 8.5

Let $\hat{i}$ and $\hat{j}$ be the unit vectors along the positive x-axis and the y-axis having initial point at the origin O. Now $\overrightarrow{OP}$ is the position vector of any point P in the plane. Then $\overrightarrow{OP}$ can be uniquely written as

$\overrightarrow{OP} = x\hat{i}+y\hat{j}$ for some real numbers x and y. Further $|\overrightarrow{OP}|=\sqrt{x^2+y^2}$.

Proof

Let (x, y) be the coordinates of the point P. Let L and M be the foots of the perpendiculars drawn from P to the x and y axes. Then $\overrightarrow{OP}=\overrightarrow{OL}+\overrightarrow{LP}=\overrightarrow{OL}+\overrightarrow{OM}$.

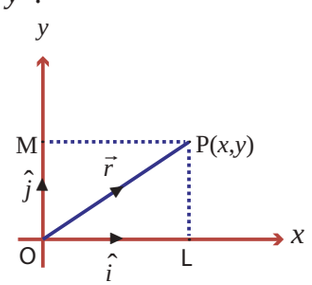

Fig. 8.29

Since $\hat{i}$ and $\hat{j}$ are unit vectors, we have $\overrightarrow{OL}=x\hat{i}$ and $\overrightarrow{OM}=y\hat{j}$.

Thus $\overrightarrow{OP} = x\hat{i}+y\hat{j}$

If $\overrightarrow{OP}=\vec{r}$ then $\vec{r}=x\hat{i}+y\hat{j}$.

To prove the uniqueness, let $x_1\hat{i}+y_1\hat{j}$ and $x_2\hat{i}+y_2\hat{j}$ be two representations of the same point P. Then

$$x_1\hat{i}+y_1\hat{j} = x_2\hat{i}+y_2\hat{j}.$$

This implies that $(x_1-x_2)\hat{i}-(y_2-y_1)\hat{j} = \vec{0} \Rightarrow x_1-x_2=0, y_2-y_1=0.$

In other words $x_1=x_2$ and $y_1=y_2$ and hence the uniqueness follows.

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In the triangle OLP, $OP^2 = OL^2 + LP^2$; hence $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$.

That is, $|\vec{r}| = r = \sqrt{x^2 + y^2}$.

Observe that if $\hat{i}$ and $\hat{j}$ are the unit vectors in the postive directions of x and y axes, then the position vector of the point (6,4) can be written as $6\hat{i} + 4\hat{j}$ and this is the only way of writing it.

Result 8.8

If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors in a plane, then any vector in the plane can be written as the linear combination of $\vec{a}$ and $\vec{b}$ in a unique way. That is, any vector in the plane is of the form $l\vec{a} + m\vec{b}$ for some scalars l and m.

Proof

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\vec{r}$ be any vector coplanar with $\vec{a}$ and $\vec{b}$.

Draw PL parallel to OB. Clearly $\overrightarrow{LP} = m\vec{b}$ and $\overrightarrow{OL} = l\vec{a}$ for some l and m.

Now $\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP}$.

That is, $\vec{r} = l\vec{a} + m\vec{b}$.

Therefore if $\vec{r}, \vec{a}, \vec{b}$ are coplanar then $\vec{r}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

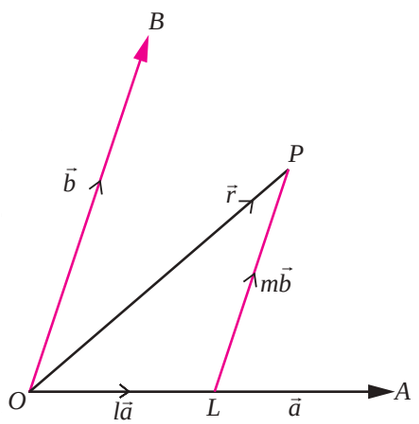

Fig. 8.30

Note 8.2

Further if three non collinear vectors are coplanar then any one of the vector can be written as a linear combination of other two. Note that the converse is also true.

Result 8.9

If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors in the space, then any vector in the space can be written as $l\vec{a} + m\vec{b} + n\vec{c}$ in a unique way for some scalars l, m and n.

Definition 8.15

Let $\hat{i}$ and $\hat{j}$ be the unit vectors in the positive directions of x and y axes respectively. Let $\vec{r}$ be any vector in the plane. Then $\vec{r} = x\hat{i} + y\hat{j}$ for some real numbers x and y. Here $x\hat{i}$ and $y\hat{j}$ are called the **rectangular components** of $\vec{r}$ along the x and y axes respectively in two dimension.

What we discussed so far can be discussed in the three dimensional space also.

8.6.2 Resolution of a vector in three dimension

Theorem 8.6

Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ be the unit vectors in the direction of postive x, y and z axes respectively having initial point at the origin O. Let $\overrightarrow{OP}$ be the position vector of any point P in the space. Then $\overrightarrow{OP}$ can be uniquely written as $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ for some real numbers x, y and z. Further $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.

Proof

Let (x, y, z) be the coordinates of the point P. Let Q be the foot of the perpendicular drawn from P to the xy-plane. Let R and S be the foots of the perpendiculars drawn from Q to the x and y axes respectively. Let $\overrightarrow{OP} = \vec{r}$.

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Then, $OR = x$, $OS = y$, and $QP = z$.

Thus, $\overrightarrow{OR} = x\hat{i}$, $\overrightarrow{RQ} = \overrightarrow{OS} = y\hat{j}$, and $\overrightarrow{QP} = z\hat{k}$

$$\overrightarrow{OP} = \vec{r} = \overrightarrow{OQ} + \overrightarrow{QP} = \overrightarrow{OR} + \overrightarrow{RQ} + \overrightarrow{QP} = x\hat{i} + y\hat{j} + z\hat{k}.$$

That is $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

This $\overrightarrow{OP}$ vector is called the position vector of P with respect to the origin O in three dimension.

In the triangle ORQ,

$$OQ^2 = OR^2 + RQ^2 \text{ (how?)}$$

and in the triangle OQP,

$$OP^2 = OQ^2 + QP^2.$$

Thus $OP^2 = OQ^2 + QP^2 = OR^2 + RQ^2 + QP^2 = x^2 + y^2 + z^2$

and hence $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$, that is $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$

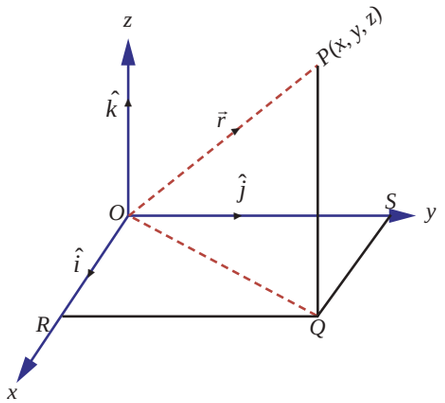

Fig. 8.31

Components of vector joining two points

Let us find the components of the vector joining the point (x_1, y_1) to (x_2, y_2).

Let A and B be the points (x_1, y_1) and (x_2, y_2). Let P be the point $(x_2 - x_1, y_2 - y_1)$. Then $\overrightarrow{AB} = \overrightarrow{OP}$. The components of $\overrightarrow{OP}$ are $(x_2 - x_1)\hat{i}$ and $(y_2 - y_1)\hat{j}$. Hence the components of $\overrightarrow{AB}$ in the directions of x and y axes are $(x_2 - x_1)\hat{i}$ and $(y_2 - y_1)\hat{j}$.

Similarly if A and B are the points (x_1, y_1, z_1) and (x_2, y_2, z_2), then the components of $\overrightarrow{AB}$ in the directions of x, y and z axes are $(x_2 - x_1)\hat{i}$, $(y_2 - y_1)\hat{j}$, and $(z_2 - z_1)\hat{k}$.

8.6.3 Matrix representation of a vector

A vector with three components can be visualised as either a row or column matrix as $[x, y, z]$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ respectively.

Thus any vector $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ can be obtained from $[a_1 \;\; a_2 \;\; a_3]\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{A}$.

Hence addition of vectors and multiplication of a vector by a scalar can be defined as follows.

If $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{A} + \vec{B} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$ resulting in $\vec{A} + \vec{B} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$.

Also $k\vec{A} = k\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} ka_1 \\ ka_2 \\ ka_3 \end{bmatrix}$ yielding

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$k\vec{A} = ka_1\hat{i} + ka_2\hat{j} + ka_3\hat{k}$

For $k \in \mathbb{R}, k > 1$ yields magnification, $0 < k < 1$ yields contraction of a vector and $k = 0$ yields a zero vector $\overrightarrow{OA} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$.

Result 8.10

Using the commutative, associative properties of vector addition and the distributive property of the scalar multiplication we can prove the following.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and if m is a scalar, then

(i) $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

(iii) $m\vec{a} = ma_1\hat{i} + ma_2\hat{j} + ma_3\hat{k}$ and

(iv) $\vec{a} = \vec{b}$ if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

Example 8.4

Find a unit vector along the direction of the vector $5\hat{i} - 3\hat{j} + 4\hat{k}$.

Solution

We know that a unit vector along the direction of the vector $\vec{a}$ is given by $\frac{\vec{a}}{|\vec{a}|}$. So a unit vector along the direction of $5\hat{i} - 3\hat{j} + 4\hat{k}$ is given by

$$\frac{5\hat{i} - 3\hat{j} + 4\hat{k}}{|5\hat{i} - 3\hat{j} + 4\hat{k}|} = \frac{5\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{5^2 + 3^2 + 4^2}} = \frac{5\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{50}}.$$

Note 8.3

Now we have another unit vector parallel to $5\hat{i} - 3\hat{j} + 4\hat{k}$ in the opposite direction. That is,

$$-\frac{5\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{50}}.$$

8.7 Direction Cosines and Direction Ratios

Let P be a point in the space with coordinates (x, y, z) and of distance r from the origin. Let R, S and T be the foots of the perpendiculars drawn from P to the x, y and z axes respectively. Then

$$\angle PRO = \angle PSO = \angle PTO = 90°.$$
$$OR = x, \quad OS = y, \quad OT = z \text{ and } OP = r.$$

(It may be difficult to visualize that $\angle PRO = \angle PSO = \angle PTO = 90°$ in the figure; as they are foot of the perpendiculars to the axes from P; in a three dimensional model we can easily visualize the fact.)

Let α, β, γ be the angles made by the vector $\overrightarrow{OP}$ with the positive x, y and z axes respectively. That is,

$$\angle POR = \alpha, \quad \angle POS = \beta, \quad \text{and} \quad \angle POT = \gamma.$$

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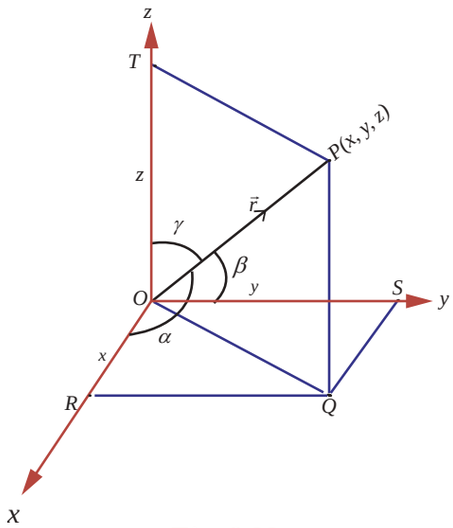

Fig. 8.32

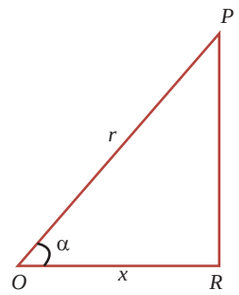

Fig. 8.33

In $\Delta OPR, \angle PRO = 90°, \angle POR = \alpha, OR = x,$ and $OP = r.$ Therefore

$$\cos\alpha = \frac{OR}{OP} = \frac{x}{r}.$$

In a similar way we can find that $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$.

Here the angles α, β, γ are called direction angles of the vector $\overrightarrow{OP} = \vec{r}$ and $\cos\alpha, \cos\beta, \cos\gamma$ are called **direction cosines** of the vector $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$. Thus $\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$, where $r = \sqrt{x^2 + y^2 + z^2}$, are the direction cosines of the vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Any three numbers which are proportional to the direction cosines of vector are called the **direction ratios** of the vector. Hence the direction ratios of a vector is not unique. For a given vector, we have, infinitely many set of direction ratios.

Observations

(i) For a given non-zero vector, one can find the direction ratios as well as the direction cosines.
(ii) For a given set of direction ratios, one cannot find the corresponding vector.
(iii) For a given set of direction cosines, one cannot find the corresponding vector.
(iv) For a given vector, the triplet of direction cosines is also a triplet of direction ratios.
(v) To find the vector, the magnitude as well as either the set of direction cosines or a set of direction ratios are essential.

Note 8.4

Here we consider a vector $\overrightarrow{OP}$ whose initial point is at the origin. If the vector whose initial point is not the origin, then, in order to find its direction cosines, we draw a vector with initial point at the origin and parallel to the given vector of same magnitude by translation. By the principle of two equal vectors having the same set of direction cosines, we can find direction cosines of any vector.

Result 8.11

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point and let α, β, γ be the direction angles of $\vec{r}$. Then

(i) the sum of the squares of the direction cosines of $\vec{r}$ is 1.

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(ii) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

(iii) the direction cosines of $\vec{r}$ are $\dfrac{x}{\sqrt{x^2+y^2+z^2}}, \dfrac{y}{\sqrt{x^2+y^2+z^2}}, \dfrac{z}{\sqrt{x^2+y^2+z^2}}$.

(iv) l, m, n are the direction cosines of a vector if and only if $l^2+m^2+n^2=1$.

(v) any unit vector can be written as $\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$

Proof

(i) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} + \dfrac{z^2}{r^2} = \dfrac{x^2+y^2+z^2}{r^2} = \dfrac{r^2}{r^2} = 1$

The proofs of (ii), (iii), (iv), and (v) are left as exercise.

Example 8.5

Find a direction ratio and direction cosines of the following vectors.

(i) $3\hat{i} + 4\hat{j} - 6\hat{k}$, (ii) $3\hat{i} - 4\hat{k}$.

Solution

(i) The direction ratios of $3\hat{i} + 4\hat{j} - 6\hat{k}$ are 3, 4, – 6.

The direction cosines are $\dfrac{x}{r}, \dfrac{y}{r}, \dfrac{z}{r}$, where $r = \sqrt{x^2+y^2+z^2}$.

Therefore, the direction cosines are $\dfrac{3}{\sqrt{61}}, \dfrac{4}{\sqrt{61}}, \dfrac{-6}{\sqrt{61}}$

(ii) The direction ratios of $3\hat{i} - 4\hat{k}$ are 3, 0, – 4.

The direction cosines are $\dfrac{3}{5}, 0, \dfrac{-4}{5}$.

Example 8.6

(i) Find the direction cosines of a vector whose direction ratios are 2, 3, – 6.

(ii) Can a vector have direction angles $30°, 45°, 60°$?

(iii) Find the direction cosines of $\overrightarrow{AB}$, where A is (2, 3, 1) and B is (3, – 1, 2).

(iv) Find the direction cosines of the line joining (2, 3, 1) and (3, – 1, 2).

(v) The direction ratios of a vector are 2, 3, 6 and it's magnitude is 5. Find the vector.

Solution

(i) The direction cosines are $\dfrac{x}{\sqrt{x^2+y^2+z^2}}, \dfrac{y}{\sqrt{x^2+y^2+z^2}}, \dfrac{z}{\sqrt{x^2+y^2+z^2}}$.

That is, $\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{-6}{7}$.

(ii) The condition is $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Here $\alpha = 30°, \beta = 45°, \gamma = 60°$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{3}{4} + \dfrac{1}{2} + \dfrac{1}{4} \neq 1$.

Therefore they are not direction angles of any vector.

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(iii) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \hat{i} - 4\hat{j} + \hat{k}$

Direction cosines are $\frac{1}{\sqrt{18}}, \frac{-4}{\sqrt{18}}, \frac{1}{\sqrt{18}}$.

(iv) Let A and B be the points (2, 3, 1) and (3,–1, 2). The direction cosines of $\overrightarrow{AB}$ are $\frac{1}{\sqrt{18}}, \frac{-4}{\sqrt{18}}, \frac{1}{\sqrt{18}}$.

But any point can be taken as first point. Hence we have another set of direction cosines with opposite direction. Thus, we have another set of direction ratios $\frac{-1}{\sqrt{18}}, \frac{4}{\sqrt{18}}, \frac{-1}{\sqrt{18}}$.

(v) The direction cosines are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$.

The unit vector is $\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$.

The required vector is $\frac{5}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$.

Example 8.7

Show that the points whose position vectors are $2\hat{i} + 3\hat{j} - 5\hat{k}$, $3\hat{i} + \hat{j} - 2\hat{k}$ and , $6\hat{i} - 5\hat{j} + 7\hat{k}$ are collinear.

Solution

Let O be the origin and let $\overrightarrow{OA}, \overrightarrow{OB}$, and $\overrightarrow{OC}$ be the vectors $2\hat{i} + 3\hat{j} - 5\hat{k}$, $3\hat{i} + \hat{j} - 2\hat{k}$ and $6\hat{i} - 5\hat{j} + 7\hat{k}$ respectively. Then

$$\overrightarrow{AB} = \hat{i} - 2\hat{j} + 3\hat{k} \text{ and } \overrightarrow{AC} = 4\hat{i} - 8\hat{j} + 12\hat{k}.$$

Thus $\overrightarrow{AC} = 4\overrightarrow{AB}$ and hence $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel. They have a common point namely A. Thus, the three points are collinear.

Alternative method

Let O be the point of reference.

Let $\overrightarrow{OA} = 2\hat{i} + 3\hat{j} - 5\hat{k}, \overrightarrow{OB} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\overrightarrow{OC} = 6\hat{i} - 5\hat{j} + 7\hat{k}$

$\overrightarrow{AB} = \hat{i} - 2\hat{j} + 3\hat{k}$; $\overrightarrow{BC} = 3\hat{i} - 6\hat{j} + 9\hat{k}$; $\overrightarrow{CA} = -4\hat{i} + 8\hat{j} - 12\hat{k}$

$|\overrightarrow{AB}| = \sqrt{14}$; $|\overrightarrow{BC}| = \sqrt{126} = 3\sqrt{14}$; $|\overrightarrow{CA}| = \sqrt{224} = 4\sqrt{14}$.

Thus, $AC = AB + BC$.

Hence A, B, C are lying on the same line. That is, they are collinear.

Example 8.8

Find a point whose position vector has magnitude 5 and parallel to the vector $4\hat{i} - 3\hat{j} + 10\hat{k}$.

Solution :

Let $\vec{a}$ be the vector $4\hat{i} - 3\hat{j} + 10\hat{k}$.

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The unit vector $\hat{a}$ along the direction of $\vec{a}$ is $\frac{\vec{a}}{|\vec{a}|}$ which is equal to $\frac{4\hat{i}-3\hat{j}+10\hat{k}}{5\sqrt{5}}$. The vector whose magnitude is 5 and parallel to $4\hat{i}-3\hat{j}+10\hat{k}$, is $5\left(\frac{4\hat{i}-3\hat{j}+10\hat{k}}{5\sqrt{5}}\right)=\frac{4\hat{i}}{\sqrt{5}}-\frac{3}{\sqrt{5}}\hat{j}+2\sqrt{5}\,\hat{k}$.

So a required point is $\left(\frac{4}{\sqrt{5}},-\frac{3}{\sqrt{5}},2\sqrt{5}\right)$.

Example 8:9

Prove that the points whose position vectors $2\hat{i}+4\hat{j}+3\hat{k}$, $4\hat{i}+\hat{j}+9\hat{k}$ and $10\hat{i}-\hat{j}+6\hat{k}$ form a right angled triangle.

Solution

Let A, B, C be the given points and O be the point of reference or origin.

Then $\overrightarrow{OA} = 2\hat{i}+4\hat{j}+3\hat{k}$, $\overrightarrow{OB}=4\hat{i}+\hat{j}+9\hat{k}$ and $\overrightarrow{OC}=10\hat{i}-\hat{j}+6\hat{k}$

$$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA}=(4\hat{i}+\hat{j}+9\hat{k})-(2\hat{i}+4\hat{j}+3\hat{k})=2\hat{i}-3\hat{j}+6\hat{k}.$$

$$AB = |\overrightarrow{AB}| = \sqrt{2^2+(-3)^2+6^2} = \sqrt{4+9+36}=7$$

$$\overrightarrow{BC} = \overrightarrow{OC}-\overrightarrow{OB}=(10\hat{i}-\hat{j}+6\hat{k})-(4\hat{i}+\hat{j}+9\hat{k})=6\hat{i}-2\hat{j}-3\hat{k}.$$

$$BC = |\overrightarrow{BC}| = \sqrt{6^2+(-2)^2+(-3)^2} = \sqrt{36+4+9}=7$$

$$\overrightarrow{CA} = \overrightarrow{OA}-\overrightarrow{OC}=(2\hat{i}+4\hat{j}+3\hat{k})-(10\hat{i}-\hat{j}+6\hat{k})=-8\hat{i}+5\hat{j}-3\hat{k}.$$

$$CA = |\overrightarrow{CA}| = \sqrt{(-8)^2+5^2+(-3)^2} = \sqrt{64+25+9}=\sqrt{98}$$

$$BC^2 = 49,\ CA^2 = 98,\ AB^2 = 49.$$

Clearly $CA^2 = BC^2 + AB^2$.

Therefore, the given points form a right angled triangle.

Example 8.10

Show that the vectors $5\hat{i}+6\hat{j}+7\hat{k}$, $7\hat{i}-8\hat{j}+9\hat{k}$, $3\hat{i}+20\hat{j}+5\hat{k}$ are coplanar.

Solution

Let $5\hat{i}+6\hat{j}+7\hat{k} = s(7\hat{i}-8\hat{j}+9\hat{k})+t(3\hat{i}+20\hat{j}+5\hat{k})$

Equating the components, we have

$$7s + 3t = 5$$
$$-8s + 20t = 6$$
$$9s + 5t = 7$$

Solving first two equations, we get, $s = t = \frac{1}{2}$, which satisfies the third equation.

Thus one vector is a linear combination of other two vectors.

Hence the given vectors are coplanar.

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EXERCISE 8.2

(1) Verify whether the following ratios are direction cosines of some vector or not.

(i) $\frac{1}{5}, \frac{3}{5}, \frac{4}{5}$ (ii) $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$ (iii) $\frac{4}{3}, 0, \frac{3}{4}$

(2) Find the direction cosines of a vector whose direction ratios are

(i) 1 , 2 , 3 (ii) 3 , – 1 , 3 (iii) 0 , 0 , 7

(3) Find the direction cosines and direction ratios for the following vectors.

(i) $3\hat{i} - 4\hat{j} + 8\hat{k}$ (ii) $3\hat{i} + \hat{j} + \hat{k}$ (iii) $\hat{j}$

(iv) $5\hat{i} - 3\hat{j} - 48\hat{k}$ (v) $3\hat{i} - 3\hat{k} + 4\hat{j}$ (vi) $\hat{i} - \hat{k}$

(4) A triangle is formed by joining the points (1, 0, 0), (0, 1, 0) and (0, 0, 1). Find the direction cosines of the medians.

(5) If $\frac{1}{2}, \frac{1}{\sqrt{2}}, a$ are the direction cosines of some vector, then find a.

(6) If (a , $a + b$, $a + b + c$) is one set of direction ratios of the line joining (1, 0, 0) and (0, 1, 0), then find a set of values of a, b, c.

(7) Show that the vectors

$2\hat{i} - \hat{j} + \hat{k},\ 3\hat{i} - 4\hat{j} - 4\hat{k},\ \hat{i} - 3\hat{j} - 5\hat{k}$ form a right angled triangle.

(8) Find the value of λ for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + \lambda\hat{j} + 3\hat{k}$ are parallel.

(9) Show that the following vectors are coplanar

(i) $\hat{i} - 2\hat{j} + 3\hat{k},\ -2\hat{i} + 3\hat{j} - 4\hat{k},\ -\hat{j} + 2\hat{k}$

(ii) $2\hat{i} + 3\hat{j} + \hat{k},\ \hat{i} - \hat{j},\ 7\hat{i} + 3\hat{j} + 2\hat{k}$.

(10) Show that the points whose position vectors $4\hat{i} + 5\hat{j} + \hat{k},\ -\hat{j} - \hat{k},\ 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $-4\hat{i} + 4\hat{j} + 4\hat{k}$ are coplanar.

(11) If $\vec{a} = 2\hat{i} + 3\hat{j} - 4\hat{k},\ \vec{b} = 3\hat{i} - 4\hat{j} - 5\hat{k}$, and $\vec{c} = -3\hat{i} + 2\hat{j} + 3\hat{k}$, find the magnitude and direction cosines of (i) $\vec{a} + \vec{b} + \vec{c}$ (ii) $3\vec{a} - 2\vec{b} + 5\vec{c}$.

(12) The position vectors of the vertices of a triangle are $\hat{i} + 2\hat{j} + 3\hat{k};\ 3\hat{i} - 4\hat{j} + 5\hat{k}$ and $-2\hat{i} + 3\hat{j} - 7\hat{k}$. Find the perimeter of the triangle.

(13) Find the unit vector parallel to

$3\vec{a} - 2\vec{b} + 4\vec{c}$ if $\vec{a} = 3\hat{i} - \hat{j} - 4\hat{k},\ \vec{b} = -2\hat{i} + 4\hat{j} - 3\hat{k}$, and $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$.

(14) The position vectors $\vec{a}, \vec{b}, \vec{c}$ of three points satisfy the relation $2\vec{a} - 7\vec{b} + 5\vec{c} = \vec{0}$. Are these points collinear?

(15) The position vectors of the points P, Q, R, S are $i + j + k,\ 2\hat{i} + 5\hat{j},\ 3\hat{i} + 2\hat{j} - 3\hat{k}$, and $\hat{i} - 6\hat{j} - \hat{k}$ respectively. Prove that the line PQ and RS are parallel.

(16) Find the value or values of m for which $m\ (\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

(17) Show that the points A (1, 1, 1), B(1, 2, 3) and C(2, – 1, 1) are vertices of an isosceles triangle.

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8.8 Product of Vectors

We have seen the notion of addition of two vectors, subtraction of one vector from another vector and the multiplication of a vector by a scalar. Now we study the notion of product of two vectors. There are two ways of multiplying two vectors.

(i) scalar product (dot product) and

(ii) vector product (cross product).

To define such products we need the angle between two vectors.

BAQHDQ

8.8.1 Angle between two vectors

Let $\vec{a}$ and $\vec{b}$ be any two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Angle between $\vec{a}$ and $\vec{b}$ is the angle between their directions when these directions are either both converge as in Fig. 8.36 or both diverge as in Fig. 8.34 from their point of intersection

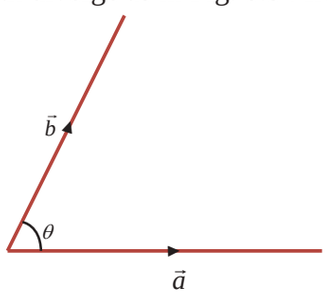

Fig. 8.34

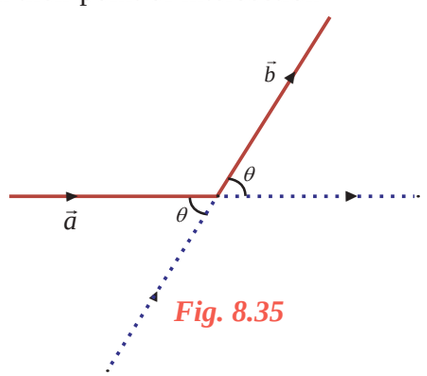

Fig. 8.35

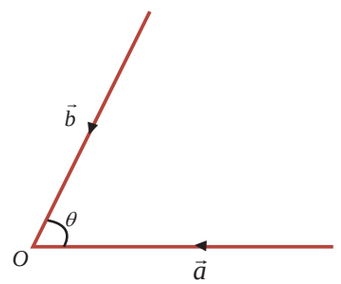

Fig. 8.36

Note that, if θ is the angle between two vectors then $0 \le \theta \le \pi$

When $\theta = 0$ or π, the vectors are parallel.

If two vectors neither converge nor diverge as in Fig. 8.35 then we can make them into either converge or diverge by extending the length of the vectors to find the angle between the two vectors.

8.8.2 Scalar product

Definition 8.16

Let $\vec{a}$ and $\vec{b}$ be any two non-zero vectors and θ be the included angle of the vectors as in Fig. 8.34.

Their **scalar product** or **dot product** is denoted by $\vec{a}.\vec{b}$ and is defined as a scalar $|\vec{a}||\vec{b}|\cos\theta$.

Thus $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

Since the resultant of $\vec{a}\cdot\vec{b}$ is a scalar, it is called scalar product. Further we use the symbol dot ('·') and hence another name dot product.

Geometrical meaning of scalar product (projection of one vector on another vector)

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and θ be the angle between $\vec{a}$ and $\vec{b}$.

Draw *BL* perpendicular to *OA*. From the right triangle *OLB*

$$\cos\theta = \frac{OL}{OB}$$

$$OL = OB\cos\theta = |\vec{b}|\cos\theta$$

But *OL* is the projection of $\vec{b}$ on $\vec{a}$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}|(OL)$$

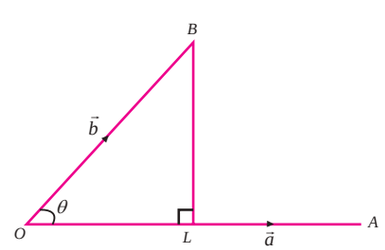

Fig. 8.37

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$\vec{a}\cdot\vec{b} = |\vec{a}|$ (projection of $\vec{b}$ on $\vec{a}$)

Thus, projection of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

In the same manner, projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{b}\cdot\vec{a}}{|\vec{b}|}$.

8.8.3 Properties of Scalar Product

(i) Scalar product of two vectors is commutative.

With usual definition, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{b}||\vec{a}|\cos\theta = \vec{b}\cdot\vec{a}$

That is, for any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

(ii) Nature of scalar product

We know that $0 \le \theta \le \pi$.

If $\theta = 0$ then $\vec{a}\cdot\vec{b} = ab$ [Two vectors are parallel in the same direction $\Rightarrow$ $\theta = 0$].

If $\theta = \pi$ then $\vec{a}\cdot\vec{b} = -ab$ [Two vectors are parallel in the opposite direction $\Rightarrow \theta = \pi$.].

If $\theta = \dfrac{\pi}{2}$ then $\vec{a}\cdot\vec{b} = 0$ [Two vectors are perpendicular $\Rightarrow \theta = \dfrac{\pi}{2}$].

If $0 < \theta < \dfrac{\pi}{2}$ then $\cos\theta$ is positive and hence $\vec{a}\cdot\vec{b}$ is positive.

If $\dfrac{\pi}{2} < \theta < \pi$ then $\cos\theta$ is negative and hence $\vec{a}\cdot\vec{b}$ is negative.

That is, $\vec{a}\cdot\vec{b}$ is $\begin{cases} > 0 \text{ for } 0 \le \theta < \pi/2 \\ 0 \quad \text{for } \theta = \pi/2 \\ < 0 \text{ for } \pi/2 < \theta \le \pi \end{cases}$

(iii) $\vec{a}\cdot\vec{b} = 0 \Rightarrow |\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\theta = \dfrac{\pi}{2}$

(iv) For any two non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}$ is perpendicular to $\vec{b}$.

(v) Different ways of representations of $\vec{a}\cdot\vec{a}$.

$\vec{a}\cdot\vec{a} = |\vec{a}|^2 = (\vec{a})^2 = \vec{a}^2 = a^2$.

These representations are essential while solving problems.

(vi) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ (how?).

(vii) For any two scalars λ and μ

$\lambda\vec{a}\cdot\mu\vec{b} = \lambda\mu(\vec{a}\cdot\vec{b}) = (\lambda\mu\vec{a})\cdot\vec{b} = \vec{a}\cdot(\lambda\mu\vec{b})$.

(viii) Scalar product is distributive over vector addition.

That is, for any three vectors $\vec{a}, \vec{b}, \vec{c}$

$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (Left distributivity)

$(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$ (Right distributivity)

Subsequently,

$\vec{a}\cdot(\vec{b}-\vec{c}) = \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c}$ and $(\vec{a}-\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{c}$

These can be extended to any number of vectors.

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(ix) Vector identities

$$(\vec{a}+\vec{b})^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$$

$$(\vec{a}-\vec{b})^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$$

Proof

By property (iii)

$$(\vec{a}+\vec{b})^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$$

Similarly one can prove other results. ■

(x) Working rule to find scalar product of two vectors

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})\\ &= a_1b_1(\hat{i}\cdot\hat{i}) + a_1b_2(\hat{i}\cdot\hat{j}) + a_1b_3(\hat{i}\cdot\hat{k})\\ &\quad + a_2b_1(\hat{j}\cdot\hat{i}) + a_2b_2(\hat{j}\cdot\hat{j}) + a_2b_3(\hat{j}\cdot\hat{k}) + a_3b_1(\hat{k}\cdot\hat{i}) + a_3b_2(\hat{k}\cdot\hat{j}) + a_3b_3(\hat{k}\cdot\hat{k})\\ &= a_1b_1 + a_2b_2 + a_3b_3.\end{aligned}$$

Therefore, the scalar product of two vectors is equal to the sum of the products of their corresponding rectangular components. ■

(xi) If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$ then $\theta = \cos^{-1}\left[\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right]$.

(xii) For any two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$.

We know that if $\vec{a}$ and $\vec{b}$ are the two sides of a triangle then the sum $\vec{a}+\vec{b}$ represents the third side of the triangle. Therefore, by triangular property, $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$ ■

(xiii) For any two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$.

If one of them is zero vector then the equality holds. So, let us assume that both are non-zero vectors. We have

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

That is, $\dfrac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|} = |\cos\theta| \le 1$

$$\Rightarrow |\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|.$$ ■

Example 8.11

Find $\vec{a}\cdot\vec{b}$ when

(i) $\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{k}$

(ii) $\vec{a}$ and $\vec{b}$ represent the points (2, 3, – 1) and (– 1, 2, 3).

Solution

(i) $\vec{a}\cdot\vec{b} = (\hat{i} - \hat{j} + 5\hat{k})\cdot(3\hat{i} - 2\hat{k}) = (1)(3) + (-1)(0) + (5)(-2) = 3 - 10 = -7$

(ii) $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = -\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{a}\cdot\vec{b} = (2)(-1) + (3)(2) + (-1)(3) = -2 + 6 - 3 = 1.$

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Example 8.12

Find $(\vec{a}+3\vec{b})\cdot(2\vec{a}-\vec{b})$ if $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=3\hat{i}+2\hat{j}-\hat{k}$

Solution

$$(\vec{a}+3\vec{b})\cdot(2\vec{a}-\vec{b}) = 2\vec{a}\cdot\vec{a}+5\vec{a}\cdot\vec{b}-3\vec{b}\cdot\vec{b} = 2(1+1+4)+5(3+2-2)-3(9+4+1) = -15\,.$$

Example 8.13

If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ be such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$ then find λ.

Solution

$$\begin{aligned}(\vec{a}+\lambda\vec{b})\cdot\vec{c}=0 &\Rightarrow \vec{a}\cdot\vec{c}+\lambda\vec{b}\cdot\vec{c}=0\\ &\Rightarrow (6+2)+\lambda(-3+2)=0\\ &\Rightarrow \lambda=8.\end{aligned}$$

Example 8.14

If $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$ prove that $\vec{a}$ and $\vec{b}$ are perpendicular.

Solution

$$\begin{aligned}|\vec{a}+\vec{b}| &= |\vec{a}-\vec{b}|\\ |\vec{a}+\vec{b}|^2 &= |\vec{a}-\vec{b}|^2\\ |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b} &= |\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}\\ \Rightarrow\quad 4\vec{a}\cdot\vec{b} &= 0\\ \vec{a}\cdot\vec{b} &= 0\end{aligned}$$

Hence $\vec{a}$ and $\vec{b}$ are perpendicular.

Example 8.15

For any vector $\vec{r}$ prove that $\vec{r}=(\vec{r}\cdot\hat{i})\hat{i}+(\vec{r}\cdot\hat{j})\hat{j}+(\vec{r}\cdot\hat{k})\hat{k}$.

Solution

Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$$\begin{aligned}\vec{r}\cdot\hat{i} &= \left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\hat{i}=x\\ \vec{r}\cdot\hat{j} &= \left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\hat{j}=y\\ \vec{r}\cdot\hat{k} &= \left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\hat{k}=z\end{aligned}$$

$$(\vec{r}\cdot\hat{i})\hat{i}+(\vec{r}\cdot\hat{j})\hat{j}+(\vec{r}\cdot\hat{k})\hat{k}=x\hat{i}+y\hat{j}+z\hat{k}=\vec{r}$$

Thus $\vec{r}=(\vec{r}\cdot\hat{i})\hat{i}+(\vec{r}\cdot\hat{j})\hat{j}+(\vec{r}\cdot\hat{k})\hat{k}$.

Example 8.16

Find the angle between the vectors $5\hat{i}+3\hat{j}+4\hat{k}$ and $6\hat{i}-8\hat{j}-\hat{k}$.

Solution

Let $\vec{a}=5\hat{i}+3\hat{j}+4\hat{k}$, and $\vec{b}=6\hat{i}-8\hat{j}-\hat{k}$.

Let θ be the angle between them.

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$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{30-24-4}{\sqrt{50}\,\sqrt{101}} = \frac{\sqrt{2}}{5\sqrt{101}} \quad \Rightarrow \theta = \cos^{-1}\left[\frac{\sqrt{2}}{5\sqrt{101}}\right].$$

Example 8.17

Find the projection of $\overrightarrow{AB}$ on $\overrightarrow{CD}$ where A, B, C, D are the points (4, – 3, 0), (7, – 5, – 1), (– 2, 1, 3), (0, 2, 5).

Solution

Let O be the origin.

Therefore, $\overrightarrow{OA} = 4\hat{i} - 3\hat{j}$; $\overrightarrow{OB} = 7\hat{i} - 5\hat{j} - \hat{k}$; $\overrightarrow{OC} = -2\hat{i} + \hat{j} + 3\hat{k}$; $\overrightarrow{OD} = 2\hat{j} + 5\hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\hat{i} - 2\hat{j} - \hat{k}$; $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = 2\hat{i} + \hat{j} + 2\hat{k}$

Projection of $\overrightarrow{AB}$ on $\overrightarrow{CD} = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{\left|\overrightarrow{CD}\right|} = \dfrac{6-2-2}{3} = \dfrac{2}{3}$.

Example 8.18

If $\vec{a}, \vec{b}$, and $\vec{c}$ are three unit vectors satisfying $\vec{a} - \sqrt{3}\,\vec{b} + \vec{c} = \vec{0}$ then find the angle between $\vec{a}$ and $\vec{c}$.

Solution

Let θ be the angle between $\vec{a}$ and $\vec{c}$.

$$\vec{a} - \sqrt{3}\,\vec{b} + \vec{c} = \vec{0}$$

$$\Rightarrow \quad |(\vec{a}+\vec{c})| = |\sqrt{3}\,\vec{b}|$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{c}|^2 + 2|\vec{a}||\vec{c}|\cos\theta = 3|\vec{b}|^2$$

$$\Rightarrow \quad 1 + 1 + (2)(1)(1)\cos\theta = 3(1)$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

Example 8.19

Show that the points (4, – 3, 1), (2, – 4, 5) and (1, – 1, 0) form a right angled triangle.

Solution

Trivially they form a triangle. It is enough to prove one angle is $\frac{\pi}{2}$. So find the sides of the triangle.

Let O be the point of reference and A,B,C be (4, – 3, 1), (2, – 4, 5) and (1, – 1, 0) respectively.

$\overrightarrow{OA} = 4\hat{i} - 3\hat{j} + \hat{k}$, $\overrightarrow{OB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$, $\overrightarrow{OC} = \hat{i} - \hat{j}$

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\hat{i} - \hat{j} + 4\hat{k}$

Similarly, $\overrightarrow{BC} = -\hat{i} + 3\hat{j} - 5\hat{k}$; $\overrightarrow{CA} = 3\hat{i} - 2\hat{j} + \hat{k}$

Clearly, $\overrightarrow{AB}\cdot\overrightarrow{CA} = 0$

Thus one angle is $\frac{\pi}{2}$. Hence they form a right angled triangle.

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Note 8.5

Suppose three sides are given in vector form, prove

(i) either sum of the vectors is $\vec{0}$ or sum of any two vectors is equal to the third vector, to form a triangle.

(ii) dot product between any two vectors is 0 to ensure one angle is $\frac{\pi}{2}$.

EXERCISE 8.3

(1) Find $\vec{a}.\vec{b}$ when

(i) $\vec{a}=\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b}=3\hat{i}-4\hat{j}-2\hat{k}$ (ii) $\vec{a}=2\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b}=6\hat{i}-3\hat{j}+2\hat{k}$.

(2) Find the value λ for which the vectors $\vec{a}$ and $\vec{b}$ are perpendicular, where

(i) $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ (ii) $\vec{a}=2\hat{i}+4\hat{j}-\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\lambda\hat{k}$.

(3) If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}|=10, |\vec{b}|=15$ and $\vec{a}\cdot\vec{b}=75\sqrt{2}$, find the angle between $\vec{a}$ and $\vec{b}$.

(4) Find the angle between the vectors

(i) $2\hat{i}+3\hat{j}-6\hat{k}$ and $6\hat{i}-3\hat{j}+2\hat{k}$ (ii) $\hat{i}-\hat{j}$ and $\hat{j}-\hat{k}$.

(5) If $\vec{a},\vec{b},\vec{c}$ are three vectors such that $\vec{a}+2\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=3, |\vec{b}|=4, |\vec{c}|=7$, find the angle between $\vec{a}$ and $\vec{b}$.

(6) Show that the vectors $\vec{a}=2\hat{i}+3\hat{j}+6\hat{k}$, $\vec{b}=6\hat{i}+2\hat{j}-3\hat{k}$, and $\vec{c}=3\hat{i}-6\hat{j}+2\hat{k}$ are mutually orthogonal.

(7) Show that the vectors $-\hat{i}-2\hat{j}-6\hat{k}, 2\hat{i}-\hat{j}+\hat{k}$, and $-\hat{i}+3\hat{j}+5\hat{k}$ form a right angled triangle.

(8) If $|\vec{a}|=5$, $|\vec{b}|=6$, $|\vec{c}|=7$ and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

(9) Show that the points (2, – 1, 3), (4, 3, 1) and (3, 1, 2) are collinear.

(10) If $\vec{a},\vec{b}$ are unit vectors and θ is the angle between them, show that

(i) $\sin\frac{\theta}{2}=\frac{1}{2}|\vec{a}-\vec{b}|$ (ii) $\cos\frac{\theta}{2}=\frac{1}{2}|\vec{a}+\vec{b}|$ (iii) $\tan\frac{\theta}{2}=\frac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|}$.

(11) Let $\vec{a},\vec{b},\vec{c}$ be three vectors such that $|\vec{a}|=3$, $|\vec{b}|=4$, $|\vec{c}|=5$ and each one of them being perpendicular to the sum of the other two, find $|\vec{a}+\vec{b}+\vec{c}|$.

(12) Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}+6\hat{j}+3\hat{k}$.

(13) Find λ, when the projection of $\vec{a}=\lambda\hat{i}+\hat{j}+4\hat{k}$ on $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$ is 4 units.

(14) Three vectors $\vec{a},\vec{b}$ and $\vec{c}$ are such that $|\vec{a}|=2$, $|\vec{b}|=3, |\vec{c}|=4$, and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Find $4\vec{a}\cdot\vec{b}+3\vec{b}\cdot\vec{c}+3\vec{c}\cdot\vec{a}$.

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8.8.4 Vector Product

To define vector product between two vectors, we need the concept of right handed and left handed system.

If we align the fingers of our right hand along the vector $\vec{a}$ and bend our fingers around in the direction of rotation from $\vec{a}$ towards $\vec{b}$ (through an angle less than 180°), our thumb will point in the direction of $\vec{a}\times\vec{b}$. Now, following the right-hand rule, $\vec{b}\times\vec{a}$ will point in the direction opposite to $\vec{a}\times\vec{b}$ (See Fig. 8.38).

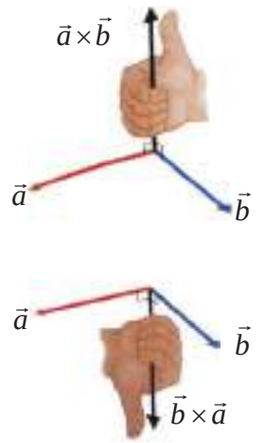
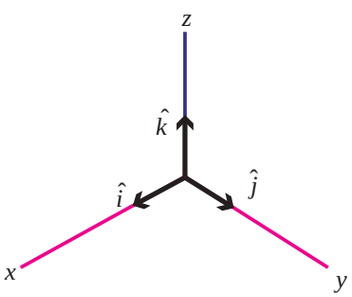


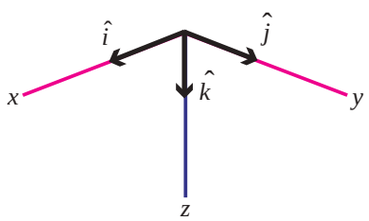
Fig. 8.38

We may also say that if $\vec{a}$ is rotated into the direction of $\vec{b}$ through the angle $\theta(<\pi)$, then $\vec{a}\times\vec{b}$ advances in the same direction as a right-handed screw would if turned in the same way.

A Cartesian coordinate system is called right-handed if the corresponding unit vectors $\hat{i},\hat{j},\hat{k}$ in the positive direction of the axes form a right-handed triple as in Fig. 8.39. The system is called left handed if the sense of $\hat{k}$ is reversed as in Fig 8.40.



Right handed
Fig. 8.39



Left handed
Fig. 8.40

Right handed screw
Fi. 8.41

Left handed screw
Fig. 8.42

Definition 8.17

Vector product of any two non-zero vectors $\vec{a}$ and $\vec{b}$ is written as $\vec{a}\times\vec{b}$ and is defined as

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n},$$

where θ is the angle between $\vec{a}$ and $\vec{b}, 0\le\theta\le\pi$.

Here $\vec{a},\vec{b},\hat{n}$ form a right handed system.

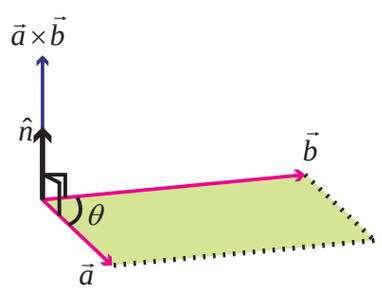


Fig. 8.43

The resultant is a vector with magnitude $|\vec{a}||\vec{b}|\sin\theta$ and has the direction $\hat{n}$.

Further $\vec{a}\times\vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$. That is, $\vec{a}\times\vec{b}$ is normal to the plane containing $\vec{a}$ and $\vec{b}$.

Note 8.6

(i) Note that the order of the vectors is very important to decide the direction of $\hat{n}$.

(ii) Since the resultant is a vector, this product is named as vector product. Again, we use the symbol cross '×' to define such a product and hence it has another name **cross product**.

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Geometrical interpretation of vector product

Construct a parallelogram $OACB$ with $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ as adjacent sides.

Let $\angle AOB = \theta$

From the diagram,

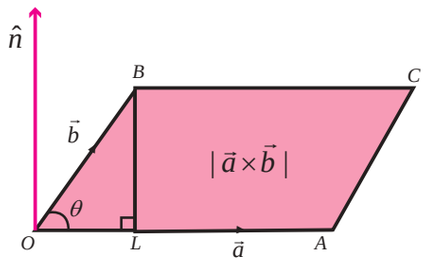

Fig. 8.44

$$\sin\theta = \frac{BL}{OB}$$

$$BL = (OB)\sin\theta = |\vec{b}|\sin\theta$$

Now $\quad |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta = |\vec{a}|(BL)$

$= $ (base) (height) = area of the parallelogram $OACB$.

Thus, $\vec{a}\times\vec{b}$ is a vector whose magnitude is the area of the parallelogram, having $\vec{a}$ and $\vec{b}$ as its adjacent sides and whose direction $\hat{n}$ is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{n}$ form a right handed system.

Thus, $|\vec{a}\times\vec{b}|$ = area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.

From the area of the parallelogram, we can deduct the area of the triangle OAC as half of the area of $OACB$.

Deduction

The area of any triangle whose two sides are $\vec{a}$ and $\vec{b} = \frac{1}{2}|\vec{a}\times\vec{b}|$.

8.8.5 Properties

(i) Vector product is non-commutative

By definition

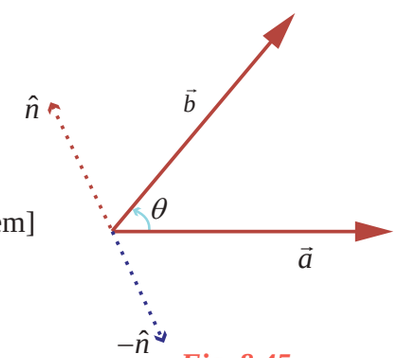

Fig. 8.45

$$\vec{b}\times\vec{a} = |\vec{b}||\vec{a}|\sin\theta(-\hat{n})$$

[since $\vec{b}, \vec{a}, -\vec{n}$ form a right handed system]

$$= -|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

$$= -(\vec{a}\times\vec{b})$$

Thus vector product is non-commutative.

(ii) If two vectors are collinear or parallel then $\vec{a}\times\vec{b} = \vec{0}$ (how?)

But $\vec{a}\times\vec{b} = \vec{0} \Rightarrow \quad \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ are parallel.

(iii) For any two non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel.

Deduction

$$\vec{a}\times\vec{a} = \vec{0}$$

(iv) With usual meaning of $\hat{i}, \hat{j}$ and $\hat{k}$ (they form a right handed system), the following results are obtained.

It is clear that,

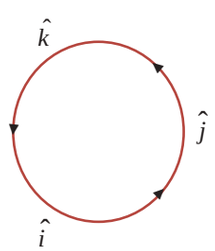

Fig. 8.46

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$$

$$\hat{i}\times\hat{j} = \hat{k}\ ;\ \hat{j}\times\hat{k} = \hat{i}\ ;\ \hat{k}\times\hat{i} = \hat{j}$$

$$\hat{j}\times\hat{i} = -\hat{k}\ ;\ \hat{k}\times\hat{j} = -\hat{i}\ ;\ \hat{i}\times\hat{k} = -\hat{j} \quad \text{(how?)}$$

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(v) If $\theta = \frac{\pi}{2}$ then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\hat{n}$.

(vi) For any scalars m and n,

$$m\vec{a} \times n\vec{b} = mn(\vec{a} \times \vec{b}) = (mn\vec{a}) \times \vec{b} = \vec{a} \times (mn\vec{b}) = n\vec{a} \times m\vec{b} .$$

(vii) Vector product is distributive over addition.

That is,
$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$$
$$(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}).$$

This property can be extended to subtraction and to any number of vectors.

That is,
$$\vec{a} \times (\vec{b} - \vec{c}) = (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c})$$
$$\vec{a} \times (\vec{b} + \vec{c} + \vec{d}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) + (\vec{a} \times \vec{d}).$$

(viii) Working rule to find the cross product

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

$$\begin{aligned}
\vec{a} \times \vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\
&= a_1b_1(\hat{i} \times \hat{i}) + a_1b_2(\hat{i} \times \hat{j}) + a_1b_3(\hat{i} \times \hat{k}) + a_2b_1(\hat{j} \times \hat{i}) \\
&\quad + a_2b_2(\hat{j} \times \hat{j}) + a_2b_3(\hat{j} \times \hat{k}) + a_3b_1(\hat{k} \times \hat{i}) + a_3b_2(\hat{k} \times \hat{j}) + a_3b_3(\hat{k} \times \hat{k}) \\
&= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}
\end{aligned}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} .$$

(ix) If θ is the angle between $\vec{a}$ and $\vec{b}$ then $\theta = \sin^{-1}\left[\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right]$.

(The proof of this result is left as an exercise)

Note 8.7

In this case θ is always acute. Thus, if we try to find the angle using vector product, we get only the acute angle. Hence in problems of finding the angle, the use of dot product is preferable since it specifies the position of the angle θ .

(x) The unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$ are $\pm\hat{n} = \pm\frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$ (how?)

Vectors of magnitude λ, perpendicular to both $\vec{a}$ and $\vec{b}$ are $\pm\lambda\hat{n} = \pm\lambda\frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

Example 8.20

Find $|\vec{a} \times \vec{b}|$, where $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$.

Solution

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \hat{i}(4-0) - \hat{j}(3-0) + \hat{k}(3-4) = 4\hat{i} - 3\hat{j} - \hat{k} .$$

$$|\vec{a} \times \vec{b}| = |4\vec{i} - 3\vec{j} - \vec{k}| = \sqrt{16+9+1} = \sqrt{26} .$$

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Example 8.21

If $\vec{a} = -3\hat{i} + 4\hat{j} - 7\hat{k}$ and $\vec{b} = 6\hat{i} + 2\hat{j} - 3\hat{k}$,

verify (i) $\vec{a}$ and $\vec{a} \times \vec{b}$ are perpendicular to each other.

(ii) $\vec{b}$ and $\vec{a} \times \vec{b}$ are perpendicular to each other.

Solution :

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 4 & -7 \\ 6 & 2 & -3 \end{vmatrix} = \hat{i}(-12+14) - \hat{j}(9+42) + \hat{k}(-6-24) = 2\hat{i} - 51\hat{j} - 30\hat{k}$$

(i) $$\vec{a} \cdot (\vec{a} \times \vec{b}) = (-3\hat{i} + 4\hat{j} - 7\hat{k}) \cdot (2\hat{i} - 51\hat{j} - 30\hat{k}) = -6 - 204 + 210 = 0 .$$

Therefore, $\vec{a}$ and $\vec{a} \times \vec{b}$ are perpendicular.

(ii) $$\vec{b} \cdot (\vec{a} \times \vec{b}) = (6\hat{i} + 2\hat{j} - 3\hat{k}) \cdot (2\hat{i} - 51\hat{j} - 30\hat{k}) = 12 - 102 + 90 = 0.$$

Therefore $\vec{b}$ and $\vec{a} \times \vec{b}$ are perpendicular.

Example 8.22

Find the vectors of magnitude 6 which are perpendicular to both vectors

$\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + j - 2\hat{k}$.

Solution

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = \hat{i}(2-3) - \hat{j}(-8+6) + \hat{k}(4-2) = -\hat{i} + 2\hat{j} + 2\hat{k}$$

Unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$ are $\pm \dfrac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|} = \pm \left[\dfrac{-\hat{i} + 2\hat{j} + 2\hat{k}}{3} \right]$

Vectors of magnitude 6 perpendicular to both $\vec{a}$ and $\vec{b}$ are $\pm 2(-\hat{i} + 2\hat{j} + 2\hat{k})$.

Example 8.23

Find the cosine and sine angle between the vectors $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 4\hat{i} - 2\hat{j} + 2\hat{k}$.

Solution

Let θ be the angle between $\vec{a}$ and $\vec{b}$

$$\vec{a} \cdot \vec{b} = (2\hat{i} + \hat{j} + 3\hat{k}) \cdot (4\hat{i} - 2\hat{j} + 2\hat{k}) = 8 - 2 + 6 = 12$$

$$|\vec{a}| = |2\hat{i} + \hat{j} + 3\hat{k}| = \sqrt{14} \quad ; \quad |\vec{b}| = |4\hat{i} - 2\hat{j} + 2\hat{k}| = \sqrt{24}$$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{12}{\sqrt{14}\sqrt{24}} = \sqrt{\frac{3}{7}}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 4 & -2 & 2 \end{vmatrix} = \hat{i}(2+6) - \hat{j}(4-12) + \hat{k}(-4-4) = 8\hat{i} + 8\hat{j} - 8\hat{k}$$

$$|\vec{a} \times \vec{b}| = |8\hat{i} + 8\hat{j} - 8\hat{k}| = 8\sqrt{3}$$

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{8\sqrt{3}}{\sqrt{14}\sqrt{24}} = \frac{4\sqrt{3}}{\sqrt{7}\sqrt{12}} = \frac{2}{\sqrt{7}}.$$

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Example 8.24

Find the area of the parallelogram whose adjacent sides are $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

Solution

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}(1+4) - \hat{j}(3-4) + \hat{k}(-3-1) = 5\hat{i} + \hat{j} - 4\hat{k}.$$

$$|\vec{a} \times \vec{b}| = |5\hat{i} + \hat{j} - 4\hat{k}| = \sqrt{42}$$

Area of the parallelogram is $\sqrt{42}$ sq.units.

Example 8.25

For any two vectors $\vec{a}$ and $\vec{b}$, prove that $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 \; |\vec{b}|^2$

Solution

We have, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 \; |\vec{b}|^2 \sin^2\theta + |\vec{a}|^2 \; |\vec{b}|^2 \cos^2\theta$$

$$= |\vec{a}|^2 \; |\vec{b}|^2 (\sin^2\theta + \cos^2\theta) = |\vec{a}|^2 \; |\vec{b}|^2.$$

Example 8.26

Find the area of a triangle having the points $A(1,0,0), B(0,1,0)$, and $C(0,0,1)$ as its vertices.

Solution :

Let us find two sides of the triangle.

$$\overrightarrow{AB} = -\hat{i} + \hat{j} \; ; \; \overrightarrow{AC} = -\hat{i} + \hat{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{3}$$

The area of the triangle ABC is $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{\sqrt{3}}{2}$.

Note 8.8

Instead of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, one can take any two sides.

EXERCISE 8.4

(1) Find the magnitude of $\vec{a} \times \vec{b}$ if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$.

(2) Show that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$.

(3) Find the vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane which contains $\hat{i} + 2\hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} + 4\hat{k}$.

(4) Find the unit vectors perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

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(5) Find the area of the parallelogram whose two adjacent sides are determined by the vectors $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$.

(6) Find the area of the triangle whose vertices are $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$.

(7) If $\vec{a},\vec{b},\vec{c}$ are position vectors of the vertices *A, B, C* of a triangle *ABC*, show that the area of the triangle *ABC* is $\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$. Also deduce the condition for collinearity of the points *A*, *B*, and *C*.

(8) For any vector $\vec{a}$ prove that $|\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=2|\vec{a}|^2$.

(9) Let $\vec{a},\vec{b},\vec{c}$ be unit vectors such that $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}=0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$. Prove that $\vec{a}=\pm\frac{2}{\sqrt{3}}(\vec{b}\times\vec{c})$.

(10) Find the angle between the vectors $2\hat{i}+\hat{j}-\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ using vector product.

BAZDFD

EXERCISE 8.5

Choose the correct or the most suitable answer from the given four alternatives

(1) The value of $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{DA}+\overrightarrow{CD}$ is

(1) $\overrightarrow{AD}$ (2) $\overrightarrow{CA}$ (3) $\vec{0}$ (4) $-\overrightarrow{AD}$

(2) If $\vec{a}+2\vec{b}$ and $3\vec{a}+m\vec{b}$ are parallel, then the value of *m* is

(1) 3 (2) $\frac{1}{3}$ (3) 6 (4) $\frac{1}{6}$

(3) The unit vector parallel to the resultant of the vectors $\hat{i}+\hat{j}-\hat{k}$ and $\hat{i}-2\hat{j}+\hat{k}$ is

(1) $\frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{5}}$ (2) $\frac{2\hat{i}+\hat{j}}{\sqrt{5}}$ (3) $\frac{2\hat{i}-\hat{j}+\hat{k}}{\sqrt{5}}$ (4) $\frac{2\hat{i}-\hat{j}}{\sqrt{5}}$

(4) A vector $\overrightarrow{OP}$ makes 60° and 45° with the positive direction of the *x* and *y* axes respectively. Then the angle between $\overrightarrow{OP}$ and the *z*-axis is

(1) 45° (2) 60° (3) 90° (4) 30°

(5) If $\overrightarrow{BA}=3\hat{i}+2\hat{j}+\hat{k}$ and the position vector of *B* is $\hat{i}+3\hat{j}-\hat{k}$, then the position vector *A* is

(1) $4\hat{i}+2\hat{j}+\hat{k}$ (2) $4\hat{i}+5\hat{j}$ (3) $4\hat{i}$ (4) $-4\hat{i}$

(6) A vector makes equal angle with the positive direction of the coordinate axes. Then each angle is equal to

(1) $\cos^{-1}\left(\frac{1}{3}\right)$ (2) $\cos^{-1}\left(\frac{2}{3}\right)$ (3) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (4) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$

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(7) The vectors $\vec{a}-\vec{b}$, $\vec{b}-\vec{c}$, $\vec{c}-\vec{a}$ are
(1) parallel to each other (2) unit vectors
(3) mutually perpendicular vectors (4) coplanar vectors.

(8) If $ABCD$ is a parallelogram, then $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD}$ is equal to
(1) $2(\overrightarrow{AB}+\overrightarrow{AD})$ (2) $4\overrightarrow{AC}$ (3) $4\overrightarrow{BD}$ (4) $\vec{0}$

(9) One of the diagonals of parallelogram $ABCD$ with $\vec{a}$ and $\vec{b}$ as adjacent sides is $\vec{a}+\vec{b}$. The other diagonal $\overrightarrow{BD}$ is
(1) $\vec{a}-\vec{b}$ (2) $\vec{b}-\vec{a}$ (3) $\vec{a}+\vec{b}$ (4) $\dfrac{\vec{a}+\vec{b}}{2}$

(10) If $\vec{a},\vec{b}$ are the position vectors A and B, then which one of the following points whose position vector lies on AB, is
(1) $\vec{a}+\vec{b}$ (2) $\dfrac{2\vec{a}-\vec{b}}{2}$ (3) $\dfrac{2\vec{a}+\vec{b}}{3}$ (4) $\dfrac{\vec{a}-\vec{b}}{3}$

(11) If $\vec{a}$, $\vec{b},\vec{c}$ are the position vectors of three collinear points, then which of the following is true?
(1) $\vec{a}=\vec{b}+\vec{c}$ (2) $2\vec{a}=\vec{b}+\vec{c}$ (3) $\vec{b}=\vec{c}+\vec{a}$ (4) $4\vec{a}+\vec{b}+\vec{c}=\vec{0}$

(12) If $\vec{r}=\dfrac{9\vec{a}+7\vec{b}}{16}$, then the point P whose position vector $\vec{r}$ divides the line joining the points with position vectors $\vec{a}$ and $\vec{b}$ in the ratio
(1) 7 : 9 internally (2) 9 : 7 internally
(3) 9 : 7 externally (4) 7 : 9 externally

(13) If $\lambda\hat{i}+2\lambda\hat{j}+2\lambda\hat{k}$ is a unit vector, then the value of λ is
(1) $\dfrac{1}{3}$ (2) $\dfrac{1}{4}$ (3) $\dfrac{1}{9}$ (4) $\dfrac{1}{2}$

(14) Two vertices of a triangle have position vectors $3\hat{i}+4\hat{j}-4\hat{k}$ and $2\hat{i}+3\hat{j}+4\hat{k}$. If the position vector of the centroid is $\hat{i}+2\hat{j}+3\hat{k}$, then the position vector of the third vertex is
(1) $-2\hat{i}-\hat{j}+9\hat{k}$ (2) $-2\hat{i}-\hat{j}-6\hat{k}$ (3) $2\hat{i}-\hat{j}+6\hat{k}$ (4) $-2\hat{i}+\hat{j}+6\hat{k}$

(15) If $|\vec{a}+\vec{b}|=60, |\vec{a}-\vec{b}|=40$ and $|\vec{b}|=46$, then $|\vec{a}|$ is
(1) 42 (2) 12 (3) 22 (4) 32

(16) If $\vec{a}$ and $\vec{b}$ having same magnitude and angle between them is $60°$ and their scalar product is $\dfrac{1}{2}$ then $|\vec{a}|$ is
(1) 2 (2) 3 (3) 7 (4) 1

(17) The value of $\theta\in\left(0,\dfrac{\pi}{2}\right)$ for which the vectors $\vec{a}=(\sin\theta)\hat{i}+(\cos\theta)\hat{j}$ and $\vec{b}=\hat{i}-\sqrt{3}\hat{j}+2\hat{k}$ are perpendicular, is equal to
(1) $\dfrac{\pi}{3}$ (2) $\dfrac{\pi}{6}$ (3) $\dfrac{\pi}{4}$ (4) $\dfrac{\pi}{2}$

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(18) If $|\vec{a}|=13, |\vec{b}|=5$ and $\vec{a}\cdot\vec{b}=60°$ then $|\vec{a}\times\vec{b}|$ is

(1) 15 (2) 35 (3) 45 (4) 25

(19) Vectors $\vec{a}$ and $\vec{b}$ are inclined at an angle $\theta=120°$. If $|\vec{a}|=1, |b|=2$, then $[(\vec{a}+3\vec{b})\times(3\vec{a}-\vec{b})]^2$ is equal to

(1) 225 (2) 275 (3) 325 (4) 300

(20) If $\vec{a}$ and $\vec{b}$ are two vectors of magnitude 2 and inclined at an angle $60°$, then the angle between $\vec{a}$ and $\vec{a}+\vec{b}$ is

(1) 30° (2) 60° (3) 45° (4) 90°

(21) If the projection of $5\hat{i}-\hat{j}-3\hat{k}$ on the vector $\hat{i}+3\hat{j}+\lambda\hat{k}$ is same as the projection of $\hat{i}+3\hat{j}+\lambda\hat{k}$ on $5\hat{i}-\hat{j}-3\hat{k}$, then λ is equal to

(1) ± 4 (2) ± 3 (3) ± 5 (4) ± 1

(22) If (1, 2, 4) and (2, – 3λ – 3) are the initial and terminal points of the vector $\hat{i}+5\hat{j}-7\hat{k}$, then the value of λ is equal to

(1) $\frac{7}{3}$ (2) $-\frac{7}{3}$ (3) $-\frac{5}{3}$ (4) $\frac{5}{3}$

(23) If the points whose position vectors $10\hat{i}+3\hat{j}$, $12\hat{i}-5\hat{j}$ and $a\hat{i}+11\hat{j}$ are collinear then a is equal to

(1) 6 (2) 3 (3) 5 (4) 8

(24) If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+x\hat{j}+\hat{k}$, $\vec{c}=\hat{i}-\hat{j}+4\hat{k}$ and $\vec{a}\cdot(\vec{b}\times\vec{c})=70$, then x is equal to

(1) 5 (2) 7 (3) 26 (4) 10

(25) If $\vec{a}=\hat{i}+2\hat{j}+2\hat{k}, |\vec{b}|=5$ and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then the area of the triangle formed by these two vectors as two sides, is

(1) $\frac{7}{4}$ (2) $\frac{15}{4}$ (3) $\frac{3}{4}$ (4) $\frac{17}{4}$

SUMMARY

In this chapter we have acquired the knowledge of the following :

- A **scalar** is a quantity that is determined by its magnitude.
- A **vector** is a quantity that is determined by both its magnitude and its direction
- If we have a liberty to choose the origins of the vector at any point then it is said to be a **free vector,** whereas if it is restricted to a certain specified point then the vector is said to be a **localized vector.**
- Two or more vectors are said to be **coplanar** if they lie on the same plane or parallel to the same plane.
- Two vectors are said to be **equal** if they have equal length and the same direction.
- A vector of magnitude 0 is called the **zero vector.**
- A vector of magnitude 1 is called a **unit vector**.

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- Let $\vec{a}$ be a vector and m be a scalar. Then the vector $m\vec{a}$ is called the **scalar multiple of a vector** $\vec{a}$ by the scalar m.
- Two vectors $\vec{a}$ and $\vec{b}$ are said to be **parallel** if $\vec{a} = \lambda\vec{b}$, where λ is a scalar.
- If $\vec{a}, \vec{b}$ and $\vec{c}$ are the sides of a triangle taken in order then $\vec{a} + \vec{b} + \vec{c} = \vec{0}$
- Vector addition is associative.
- For any vector $\vec{a}$, $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$.
- For any vector $\vec{a}$, $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$.
- Vector addition is commutative.
- "If two vectors are represented in magnitude and direction by the two sides of a triangle taken in the same order, then their sum is represented by the third side taken in the reverse order". This is known as the **triangle law of addition.**
- In a parallelogram $OABC$, if $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represents two adjacent sides, then the diagonal $\overrightarrow{OC}$ represents their sum. This is **parallelogram law of addition**.
- If α, β, γ are the direction angles then $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines.
- The direction ratios of the vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ are x, y, z.
- If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors in the space, then any vector in the space can be written as $l\vec{a} + m\vec{b} + n\vec{c}$ in a unique way.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point and let α, β, γ be the direction angles of $\vec{r}$. Then

(i) the sum of the squares of the direction cosines of $\vec{r}$ is 1.

(ii) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

(iii) the direction cosines of $\vec{r}$ are $\dfrac{x}{\sqrt{x^2+y^2+z^2}}, \dfrac{y}{\sqrt{x^2+y^2+z^2}}, \dfrac{z}{\sqrt{x^2+y^2+z^2}}$.

(iv) l, m, n are the direction cosines of a vector if and only if $l^2 + m^2 + n^2 = 1$.

(v) any unit vector can be written as $\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$.

- The **scalar product** of the vectors $\vec{a}$ and $\vec{b}$ is $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.
- **Vector product** of any two non-zero vectors $\vec{a}$ and $\vec{b}$ is written as $\vec{a}\times\vec{b}$ and is defined as $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}, 0 \le \theta \le \pi$. Here $\vec{a}, \vec{b}, \hat{n}$ form a right handed system.

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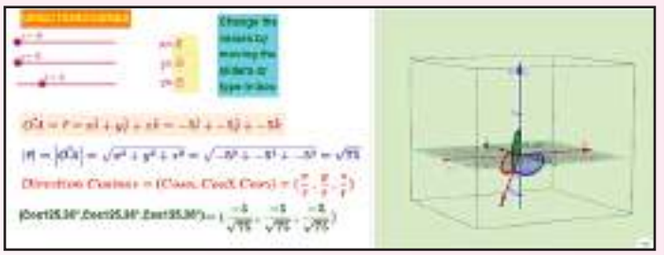
ICT CORNER 8(a)

Vector Algebra

Expected Outcome



Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code.

GeoGebra Workbook called "XI Standard Vector Algebra" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Direction Cosines". 3-D representation is found on Right side. You can rotate 3-D picture by right clicking on the mouse to see various positions

You can move the sliders or entering x, y and z values to change the vector.

Step1

Step2

Browse in the link:

XI Standard Vector Algebra: https://ggbm.at/cem3sdq5

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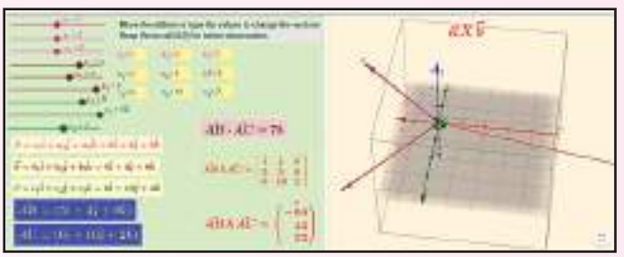
ICT CORNER 8(b)

Vector Algebra

Expected Outcome



Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code.

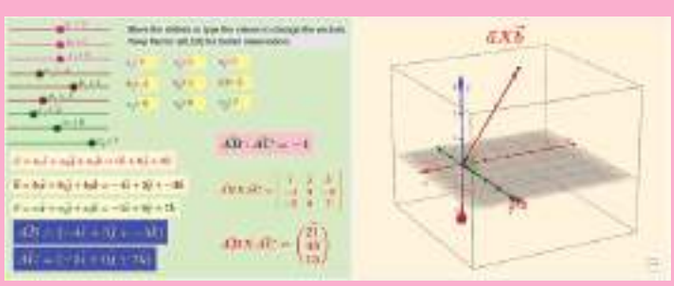
GeoGebra Workbook called "XI Standard Vector Algebra" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Product of Vectors". 3-D representation is found on Right side. You can rotate 3-D picture by right clicking on the mouse to see various positions

You can move the sliders or entering x, y and z values to change the vector.(for clear understanding do not change a_1,a_2, and a_3 values. For ABXAC, components are given vertically)

Step1



Step2

Browse in the link:

XI Standard Vector Algebra: https://ggbm.at/cem3sdq5

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Chapter 9 Differential Calculus - Limits and Continuity

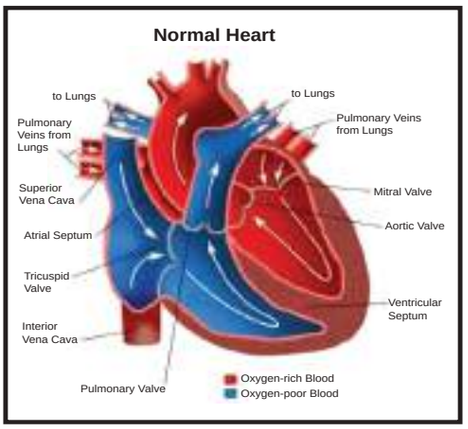
BB99H1

"Men pass away, but their deeds abide"

- Augustin-Louis Cauchy

9.1 Introduction

Calculus is about rates of change. Rates of change occur in all the sciences. A **mathematician** is interested in measuring the rate of change of the deviations of a straight line at a point on a curve, while a **physicist** is interested in the rate of change of displacement, and the velocity of a moving object. A **chemist** wants to know the rate of a chemical reaction that would result in the formation of one or more substances (called products) from one or more starting materials (called reactants).



A **biologist** would like to analyse the changes that take place in the number of individuals in an animal population or plant population at any time; he would also want to know the rate at which blood flows through a blood vessel, such as a vein or artery and the part of the vessel / artery in which this flow is lowest or highest.

An **economists** also studies marginal demand, marginal revenue, and marginal profit, which are drawn from rates of change (that is, derivatives) of this demand, revenue and profit functions.

A **geologist** is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks. An **engineer** wants to know the rate at which water flows into or out of a reservoir. An urban **geographer** is interested in the rate of change of population density in a city with the expansion of the city. A **meteorologist** is concerned with the rate of change of atmospheric pressure with respect to height.

In **psychology**, those interested in learning theory, study the so called learning curve, which graphs the performance of someone learning a skill as a function of the training time. Of particular interest is the rate at which performance improves as time passes.

When we enter a darkened room, our eyes adjust to the reduced level of light by increasing the size of our pupils, allowing more light to enter the eyes and making objects around us easier to see. By contrast, when we enter a brightly lit room, our pupils contract, reducing the amount of light entering the eyes, as too much light would overload our visual system. Researchers study such mechanisms based on limits.

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Velocity, density, current, power and temperature gradient in physics; rate of reaction and compressibility in chemistry, rate of growth and blood velocity in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology – these are all cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as derivatives) can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept, we can then apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science.

One of the greatest creations of the ancient past was Euclidean geometry. This monumental work was not matched in importance until the discovery of calculus almost two thousand years later.

Calculus was created independently in England by **Sir Isaac Newton** (1642 – 1727) and in Germany by **Gottfried Wilhelm Leibnitz** (1646 – 1716) in the last quarter of the seventeenth century.

Newton's interest in mathematics began with his study of two of the great books on mathematics at that time: **Euclid's Elements and Descartio La Geometric**. He also became aware of the work of the great scientists who preceded him, including Galileo and Fermat.

By the end of 1664, Newton seemed to have mastered all the mathematical knowledge of the time and had begun adding substantially to it. In 1665, he began his study of the rates of change or flexions, of quantities, such as distances or temperatures that varied continuously. The result of this study was what we today call differential calculus. All who study mathematics today stand on Isaac Newton's shoulders.

Many of Leibnitz's mathematical papers appeared in the journal 'Acta Eruditorum' which he cofounded in 1682. This journal contained his work on calculus and led to the bitter controversy with Newton over who first discovered calculus. Leibnitz was the first to publish the important results on calculus and was the first to use the notation that has now become standard.

Augustin – Louis Cauchy
(1789 – 1857)

Augustin-Louis Cauchy (1789 – 1857), born in Paris in 1789 is considered to be the most outstanding mathematical analyst of the first half of the nineteenth century. Cauchy made many contributions to calculus. In his 1829 text book '**Lecons le calcul differential**', he gave the first reasonably clear definition of a limit and was the first to define the derivative as the limit of the difference quotient,

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Karl Weierstrass (1815-1897), a German mathematician gave the precise definition ($\in - \delta$ definition) of the concepts of limit, continuity and differentiability.

What is Calculus?

Calculus is the mathematics of ratio of change of quantities. It is also the mathematics of tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures and a variety of other concepts that have enabled scientists, engineers and economists to model real-life situations.

Although pre calculus mathematics deals with velocities, accelerations, tangent lines, slopes and so on, there is a fundamental difference between pre calculus mathematics and calculus. Pre calculus mathematics is more static, whereas calculus is more dynamic. Here are some examples:

- An object travelling at a constant velocity can be analysed with pre calculus mathematics. To analyse the velocity of an accelerating object, you need calculus.
- The slope of a line can be analysed with pre calculus mathematics. To analyse the slope of a curve, you need calculus.

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- ★ A tangent line to a circle can be analysed with pre calculus mathematics. To analyse a line tangential to a general graph, you need calculus.
- ★ The area of a rectangle can be analysed with pre calculus mathematics. To analyse the area under a general curve, you need calculus.

Each of these situations involves the same general strategy, the reformulation of pre calculus mathematics through the use of a limit process. So, one way to answer the question 'What is calculus?' is to say that calculus is a 'limit machine' that involves three stages. The first stage is pre calculus mathematics such as the slope of a line or the area of a rectangle. The second stage is the limit process and the third stage is a new calculus formulation, such as a derivative or integral.

Pre calculus Mathematics	$\Rightarrow$	**Limit Process**	$\Rightarrow$	**Calculus**

It is cautioned that those who try to learn calculus as if it were simply a collection of new formulae rather than as a process, will miss a great deal of understanding, self-confidence and satisfaction.

Learning Objectives

On completion of this chapter, the students are expected to
- visualize the concept of limit / continuity through geometric process.
- relate the concept of limit / continuity with every day life activities.
- assimilate limit / continuity as the heart and spirit of calculus.
- understand limit / continuity as an operation (operator) to measure / quantify / mathmatize changes in physical world.
- concretize the concept of limit / continuity via illustrations of real life related situations.

9.2 Limits

9.2.1 The calculation of limits

The notion of a limit, which we will discuss extensively in this chapter, plays a central role in calculus and in much of modern mathematics. However, although mathematics dates back over three thousand years, limits were not really understood until the monumental work of the great French mathematician Augustin – Louis Cauchy and Karl Weierstrass in the nineteenth century, the age of rigour in mathematics.

In this section we define limit and show how limits can be calculated.

Illustration 9.1

We begin by looking at the function $y = f(x) = x^2 + 3$. Note that f is a function from $\square \to \square$.

Let us investigate the behaviour of this function near $x = 2$. We can use two sets of x values : one set that approaches 2 from the left (values less than 2) and one set that approaches 2 from the right (values greater than 2) as shown in the table.

	x approaches 2 from the left							x approaches 2 from the right					
x	1.7	1.9	1.95	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.05	2.1	2.3
$f(x)$	5.89	6.61	6.8025	6.9601	6.99601	6.99960001	7	7.0040001	7.004001	7.0401	7.2025	7.41	8.29

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It appears from the table that as x gets close to $x = 2$, $f(x) = x^2 + 3$ gets close to 7. This is not surprising since if we now calculate $f(x)$ at $x = 2$, we obtain $f(2) = 2^2 + 3 = 7$.

In order to guess at this limit, we didn't have to evaluate $x^2 + 3$ at $x = 2$.

That is, as x approaches 2 from either the left (values lower than 2) or right (values higher than 2) the functional values $f(x)$ are approaching 7 from either side; that is, when x is near 2, $f(x)$ is near 7. The above situation is described in a condensed form:

The value 7 is the left limit of $f(x)$ as x approaches 2 from the left as well as 7 is the right limit of $f(x)$ as x approaches 2 from the right and write :

$f(x) \to 7$ as $x \to 2^-$ and $f(x) \to 7$ as $x \to 2^+$

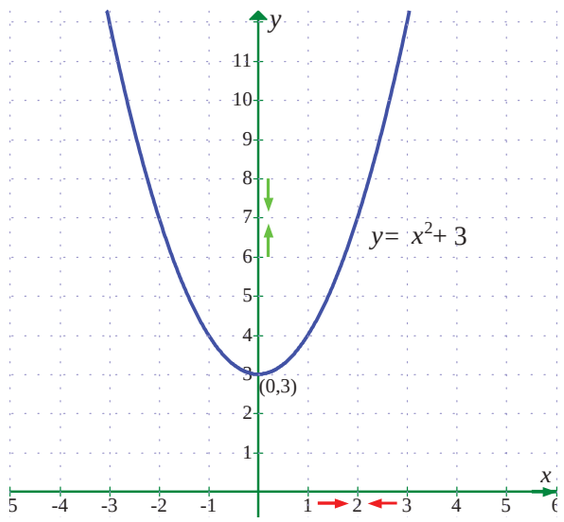

Fig. 9.1

or

$\lim\limits_{x \to 2^-} f(x) = 7$ and $\lim\limits_{x \to 2^+} f(x) = 7$.

Note also that $\lim\limits_{x \to 2^-} f(x) = 7 = \lim\limits_{x \to 2^+} f(x)$. The common value is written as $\lim\limits_{x \to 2} f(x) = 7$.

We also observe that the limit is a definite real number. Here, definiteness means that $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^+} f(x)$ are the same and

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x)$ is a unique real number.

The figure in Fig. 9.1 explains the geometrical significance of the above discussion of the behaviour of $f(x) = x^2 + 3$ as $x \to 2$.

Illustration 9.2

Next, let us look at the rational function $f(x) = \dfrac{16 - x^2}{4 + x}$.

The domain of this function is $\mathbb{R} \setminus \{-4\}$. Although $f(-4)$ is not defined, nonetheless, $f(x)$ can be calculated for any value of x near -4 because the symbol $\lim\limits_{x \to -4} \left(\dfrac{16 - x^2}{4 + x} \right)$ says that we consider values of x that are close to -4 but not equal to -4. The table below gives the values of $f(x)$ for values of x that approach -4.

$(x < -4)$ $(x \to -4^-)$	$f(x)$	$(x > -4)$ $(x \to -4^+)$	$f(x)$
-4.1	8.1	-3.9	7.9
-4.01	8.01	-3.99	7.99
-4.001	8.001	-3.999	7.999

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For $x \neq -4$, $f(x)$ can be simplified by cancellation :

$$f(x) = \frac{16 - x^2}{4 + x}$$

$$= \frac{(4+x)(4-x)}{(4+x)} = 4 - x.$$

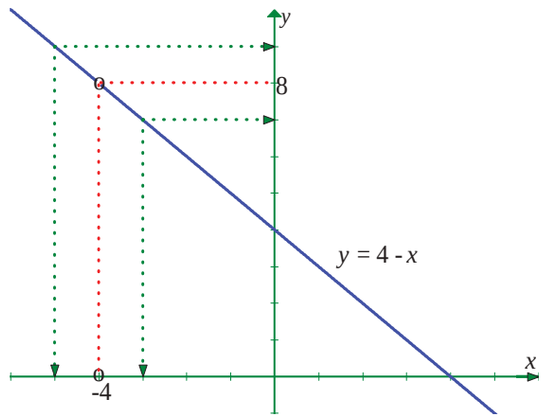

Fig. 9.2

As seen in Fig.9.2, the graph of $f(x)$ is essentially the graph of $y = 4 - x$ with the exception that the graph of f has a hole (puncture) at the point that corresponds to $x = -4$. As x gets closer and closer to -4, represented by the two arrow heads on the x-axis, the two arrow heads on the y-axis simultaneously get closer and closer to the number 8.

Here, note that

$$\lim_{x \to -4^-} f(x) = 8 = \lim_{x \to -4^+} f(x) \text{ and hence } \lim_{x \to -4} f(x) = \lim_{x \to -4} \frac{16 - x^2}{4 + x} = 8.$$

In Illustration 9.2, note that the function is not defined at $x = -4$ and yet $f(x)$ appears to be approaching a limit as x approaches -4. This often happens, and **it is important to realise that the existence or non-existence of $f(x)$ at $x = -4$ has no bearing on the existence of the limit of $f(x)$ as x approaches -4.**

Illustration 9.3

Now let us consider a function different from Illustrations 9.1 and 9.2.

Let $f(x) = \dfrac{|x|}{x}$.

$x = 0$ does not belong to the domain of this function, $\mathbb{R} \setminus \{0\}$. Look at the graph of this function. From the graph one can see that for positive values of x,

$$\frac{|x|}{x} = \frac{x}{x} = +1 \text{ and}$$

for negative x values, $\dfrac{|x|}{x} = \dfrac{-x}{x} = -1$.

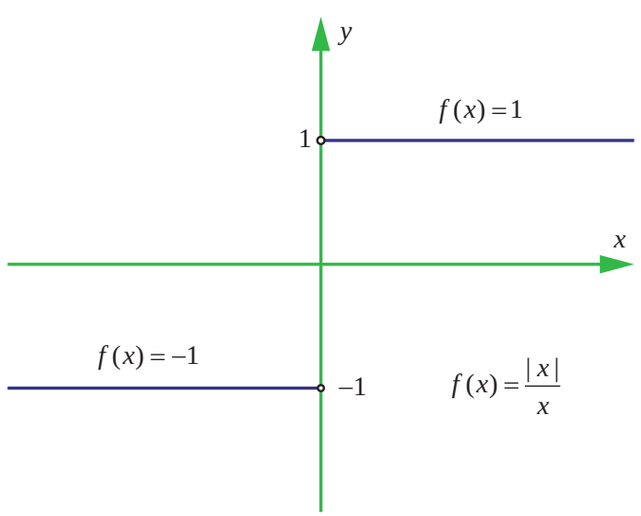

Fig. 9.3

This means that no matter how close x gets to 0 (in a small neighbourhood of 0), there will be both positive and negative x values that yield $f(x) = 1$ and $f(x) = -1$.

That is, $\lim_{x \to 0^-} f(x) = -1$ and $\lim_{x \to 0^+} f(x) = +1$.

This means that the limit does not exist. Of course, for any other value of x, there is a limit.

For example $\lim_{x \to 2^-} \dfrac{|x|}{x} = 1$ and $\lim_{x \to 2^+} \dfrac{|x|}{x} = 1$.

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Similarly, $\lim\limits_{x\to -3^-} \dfrac{|x|}{x} = \lim\limits_{x\to -3^-} \dfrac{-x}{x} = -1$

$$\lim_{x\to -3^+} \frac{|x|}{x} = \lim_{x\to -3^+} \frac{-x}{x} = -1.$$

In fact, for any real number $x_0 \neq 0$, $\lim\limits_{x\to x_0^-} \dfrac{|x|}{x} = -1 = \lim\limits_{x\to x_0^+} \dfrac{|x|}{x}$ if $x_0 < 0$ and

$$\lim_{x\to x_0^-} \frac{|x|}{x} = 1 = \lim_{x\to x_0^+} \frac{|x|}{x} \text{ if } x_0 > 0.$$

We call the attention of the reader to observe the differences reflected in Illustrations 9.1 to 9.3. In Illustration 9.1, the function $f(x) = x^2 + 3$ is defined at $x = 2$. i.e., 2 belongs to the domain of f namely $\mathbb{R} = (-\infty, \infty)$. In Illustration 9.2, the function is not defined at $x = -4$. In the former case we say the limit, $\lim\limits_{x\to 2} f(x)$ exists as x gets closer and closer to 2 to mean that $\lim\limits_{x\to 2^-} f(x)$ and $\lim\limits_{x\to 2^+} f(x)$ stand for a unique real number. In the later case, although it is not defined at $x = -4$, $\lim\limits_{x\to -4} f(x)$ exist as x gets closer and closer to -4. In Illustration 9.3, $\lim\limits_{x\to 0} \dfrac{|x|}{x}$ does not exist to mean that the one sided limits $\lim\limits_{x\to 0^-} \dfrac{|x|}{x}$ and $\lim\limits_{x\to 0^+} \dfrac{|x|}{x}$ are different as x gets sufficiently close to 0. In the light of these observations we have the **intuitive** notion of limit as in

Definition 9.1

Let I be an open interval containing $x_0 \in \mathbb{R}$. Let $f : I \to \mathbb{R}$. Then we say that the limit of $f(x)$ is L, as x approaches x_0 [symbollically written as $\lim\limits_{x\to x_0} f(x) = L$], if, whenever x becomes sufficiently close to x_0 from either side with $x \neq x_0$, $f(x)$ gets sufficiently close to L.

The following (Fig 9.4 and 9.5) graphs depict the above narrations.

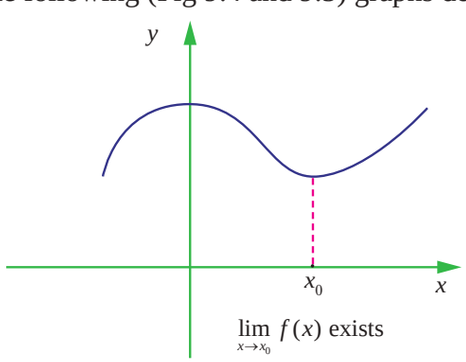

Fig. 9.4

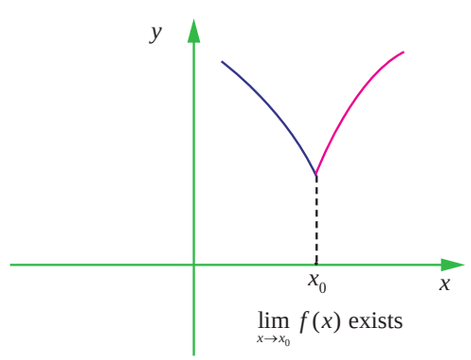

Fig. 9.5

9.2.2 One sided limits

Definition 9.2

We say that the **left-hand limit** of $f(x)$ as x approaches x_0 (or the limit of $f(x)$ as x approaches x_0 from the left) is equal to l_1 if we can make the values of $f(x)$ arbitrarily close to l_1 by taking x to be sufficiently close to x_0 and less than x_0. It is symbolically written as $f(x_0^-) = \lim\limits_{x\to x_0^-} f(x) = l_1$.

Similarly, we define the right hand limit.

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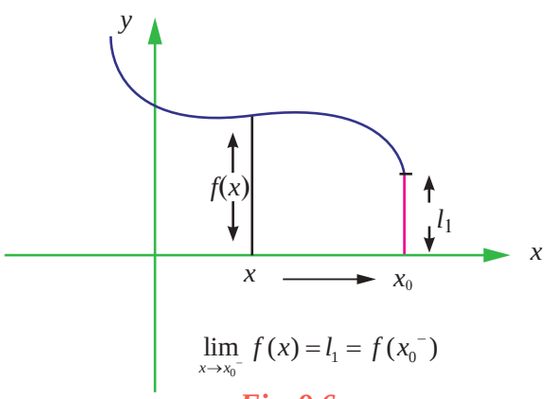
Definition 9.3

We say that the **right-hand limit** of $f(x)$ as x approaches x_0 (or the limit of $f(x)$ as x approaches x_0 from the right) is equal to l_2 if we can make the values of $f(x)$ arbitrarily close to l_2 by taking x to be sufficiently close to x_0 and greater than x_0. It is symbolically written as $f(x_0^+) = \lim\limits_{x \to x_0^+} f(x) = l_2$.

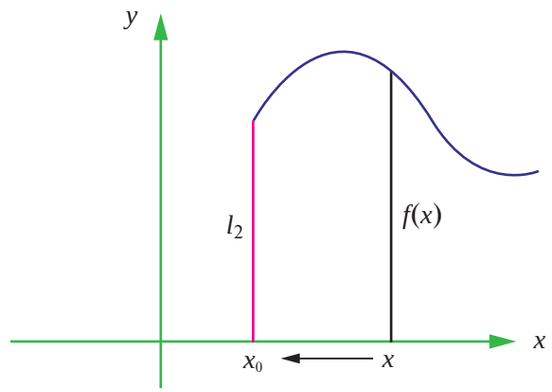
Thus the symbols "$x \to x_0^-$" and "$x \to x_0^+$" mean that we consider only $x < x_0$ and $x > x_0$ respectively.

These definitions are illustrated in the following Fig. 9.6 to 9.9.



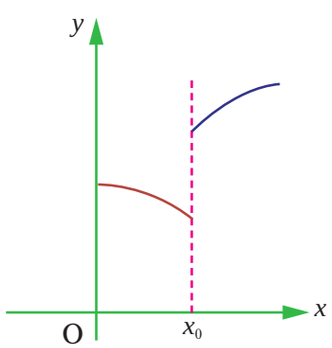
$\lim\limits_{x \to x_0^-} f(x) = l_1 = f(x_0^-)$

Fig. 9.6



$\lim\limits_{x \to x_0^+} f(x) = l_2 = f(x_0^+)$

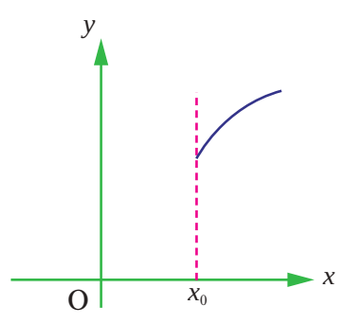
Fig. 9.7



$\lim\limits_{x \to x_0} f(x)$ does not exist

Fig. 9.8

(Different values are obtained as x_0 is approached from the left and from the right)



$\lim\limits_{x \to x_0} f(x)$ does not exist

Fig. 9.9

(Function not defined to the left of x_0)

From the above discussions we conclude that $\lim\limits_{x \to x_0} f(x) = L$ exists if the following hold :

(i) $\lim\limits_{x \to x_0^+} f(x)$ exists,

(ii) $\lim\limits_{x \to x_0^-} f(x)$ exists and

(iii) $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0^-} f(x) = L$.

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From the definitions of one sided limits and that of the limit of $f(x)$ as we have the following :

$\lim\limits_{x \to x_0} f(x) = L$ iff $\lim\limits_{x \to x_0^-} f(x) = L = \lim\limits_{x \to x_0^+} f(x)$.

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Thus, when we say $\lim_{x \to x_0} f(x)$ exists, it is understood that L is a unique real number. If any one of the above conditions fails then we say, the limit of $f(x)$ as x approaches x_0 does not exist.

We remark that the existence of one sided limits is weaker than the existence of limits.

Sometimes it is very useful to use the following in computing left and right limits. For $h > 0$,

$$f(x_0^-) = \lim_{h \to 0} f(x_0 - h) \text{ and } f(x_0^+) = \lim_{h \to 0} f(x_0 + h).$$

Note that $f(x_0^-)$ and $f(x_0^+)$ stand for the left and right limiting values. But $f(x_0)$ is the value of the function at $x = x_0$.

Example 9.1

Calculate $\lim_{x \to 0} |x|$.

Solution

Recall from the earlier chapter 1

that $|x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$

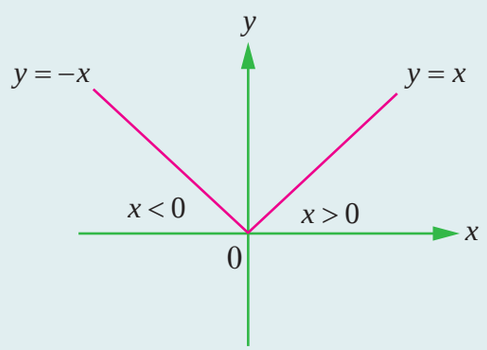

Fig. 9.10

If $x > 0$, then $|x| = x$, which tends to 0 as $x \to 0$ from the right of 0. That is, $\lim_{x \to 0^+} |x| = 0$

If $x < 0$, then $|x| = -x$ which again tends to 0 as $x \to 0$ from the left of 0. That is, $\lim_{x \to 0^-} |x| = 0$.

Thus, $\lim_{x \to 0^-} |x| = 0 = \lim_{x \to 0^+} |x|$.

Hence $\lim_{x \to 0} |x| = 0$.

Example 9.2

Consider the function $f(x) = \sqrt{x}$, $x \geq 0$.

Does $\lim_{x \to 0} f(x)$ exist?

Solution

No. $f(x) = \sqrt{x}$ is not even defined for $x < 0$.

Therefore as $x \to 0^-$, $\lim_{x \to 0^-} \sqrt{x}$ does not exist.

However, $\lim_{x \to 0^+} \sqrt{x} = 0$. Therefore $\lim_{x \to 0} \sqrt{x}$ does not exist.

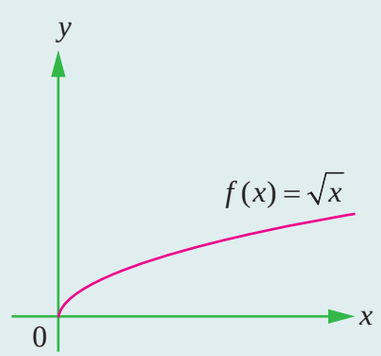

Fig. 9.11

Does $\lim_{x \to 0^-} \log x$ exist?

Look at the graph of $\log x$ for the answer.

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Example 9.3

Evaluate $\lim\limits_{x\to 2^-}\lfloor x\rfloor$ and $\lim\limits_{x\to 2^+}\lfloor x\rfloor$.

Solution

The greatest integer function $f(x)=\lfloor x\rfloor$ is defined as the greatest integer lesser than or equal to x.

From the graph (Fig. 1.25) of this function it is clear that $\lim\limits_{x\to 2^-}\lfloor x\rfloor=1$ and $\lim\limits_{x\to 2^+}\lfloor x\rfloor=2$.

Moreover, for any integer n, $\lim\limits_{x\to n^-}\lfloor x\rfloor=n-1$ and $\lim\limits_{x\to n^+}\lfloor x\rfloor=n$.

Does $f(x)=\lim\limits_{x\to n}\lceil x\rceil$ exist? Look at the graph of $\lceil x\rceil$ (Fig.1.26) for the answer.

Example 9.4

Let $f(x)=\begin{cases} x+1, & x>0 \\ x-1, & x<0 \end{cases}$.

Verify the existence of limit as $x\to 0$.

Solution

The function is graphed in Fig.9.12.

Clearly $\lim\limits_{x\to 0^-} f(x)=-1$ and $\lim\limits_{x\to 0^+} f(x)=1$. Since these limits are different, $\lim\limits_{x\to 0} f(x)$ does not exist.

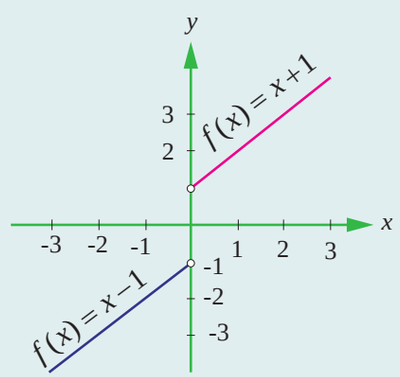

Fig. 9.12

Example 9.5

Check if $\lim\limits_{x\to -5} f(x)$ exists or not, where $f(x)=\begin{cases} \dfrac{|x+5|}{x+5}, & \text{for } x\neq -5 \\ 0, & \text{for } x=-5 \end{cases}$

Solution

(i) $f(-5^-)$.

For $x<-5$, $|x+5|=-(x+5)$

Thus $f(-5^-)=\lim\limits_{x\to -5^-}\dfrac{-(x+5)}{(x+5)}=-1$

(ii) $f(-5^+)$.

For $x>-5$, $|x+5|=(x+5)$

Thus $f(-5^+)=\lim\limits_{x\to -5^+}\dfrac{(x+5)}{(x+5)}=1$

Note that $f(-5^-)\neq f(-5^+)$. Hence the limit does not exist.

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Example 9.6

Test the existence of the limit, $\lim\limits_{x\to 1}\dfrac{4|x-1|+x-1}{|x-1|}$, $x \neq 1$.

Solution

For $x > 1$, $|x-1| = x-1$ and $f(1^+) = \lim\limits_{x\to 1^+}\dfrac{4(x-1)+x-1}{x-1} = \lim\limits_{x\to 1^+}\dfrac{5(x-1)}{(x-1)} = 5$.

For $x < 1$, $|x-1| = -(x-1)$, and $f(1^-) = \lim\limits_{x\to 1^-}\dfrac{-4(x-1)+(x-1)}{-(x-1)} = \lim\limits_{x\to 1^-}\dfrac{3(x-1)}{(x-1)} = 3$.

Thus $f(1^-) \neq f(1^+)$ and hence the limit does not exist.

EXERCISE 9.1

In problems 1-6, using the table estimate the value of the limit.

(1) $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-x-2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.344820	0.33444	0.33344	0.333222	0.33222	0.332258

(2) $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.25641	0.25062	0.250062	0.24993	0.24937	0.24390

(3) $\lim\limits_{x\to 0}\dfrac{\sqrt{x+3}-\sqrt{3}}{x}$

x	– 0.1	– 0.01	– 0.001	0.001	0.01	0.1
f(x)	0.2911	0.2891	0.2886	0.2886	0.2885	0.28631

(4) $\lim\limits_{x\to -3}\dfrac{\sqrt{1-x}-2}{x+3}$

x	– 3.1	– 3.01	– 3.00	– 2.999	– 2.99	– 2.9
$f(x)$	– 0.24845	– 0.24984	– 0.24998	– 0.25001	– 0.25015	– 0.25158

(5) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$

x	– 0.1	– 0.01	– 0.001	0.001	0.01	0.1
$f(x)$	0.99833	0.99998	0.99999	0.99999	0.99998	0.99833

(6) $\lim\limits_{x\to 0}\dfrac{\cos x-1}{x}$

x	– 0.1	– 0.01	– 0.001	0.0001	0.01	0.1
$f(x)$	0.04995	0.0049999	0.0004999	–0.0004999	–0.004999	–0.04995

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In exercise problems 7 – 15, use the graph to find the limits (if it exists). If the limit does not exist, explain why?

(7) $\lim\limits_{x\to 3}(4-x)$.

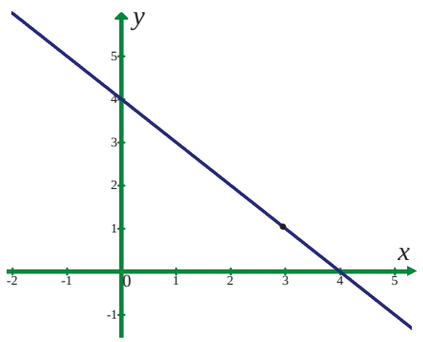

Fig. 9.13

(8) $\lim\limits_{x\to 1}(x^2+2)$.

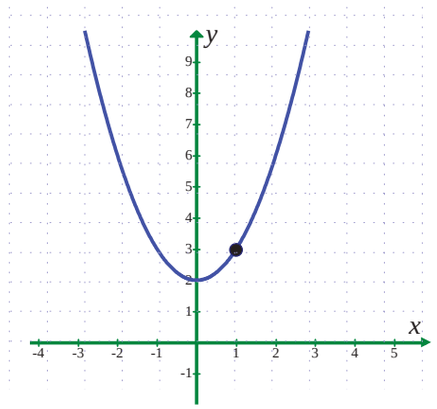

Fig. 9.14

(9) $\lim\limits_{x\to 2} f(x)$

where $f(x) = \begin{cases} 4-x, & x \neq 2 \\ 0, & x = 2 \end{cases}$.

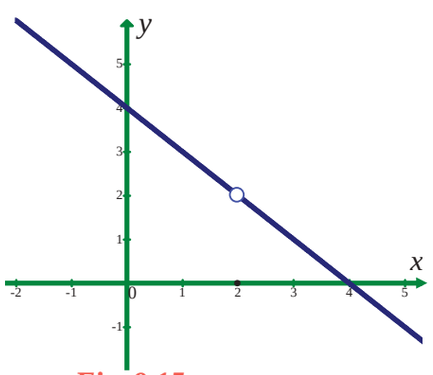

Fig. 9.15

(10) $\lim\limits_{x\to 1} f(x)$

where $f(x) = \begin{cases} x^2+2, & x \neq 1 \\ 1, & x = 1 \end{cases}$.

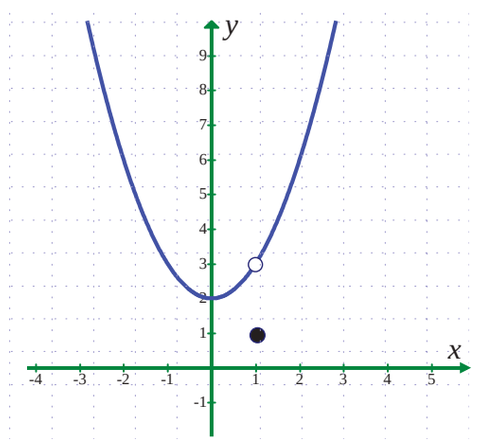

Fig. 9.16

(11) $\lim\limits_{x\to 3}\dfrac{1}{x-3}$

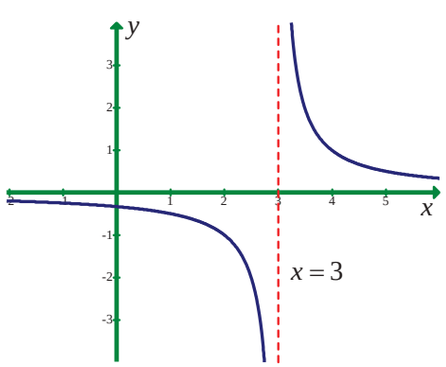

Fig. 9.17

(12) $\lim\limits_{x\to 5}\dfrac{|x-5|}{x-5}$

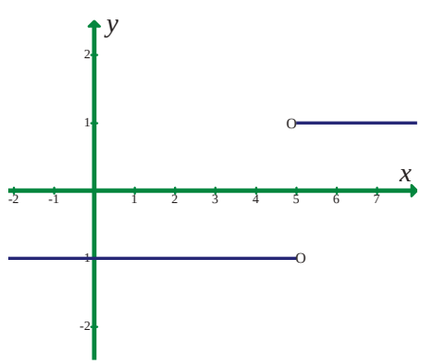

Fig. 9.18

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(13) $\lim\limits_{x \to 1} \sin \pi x$

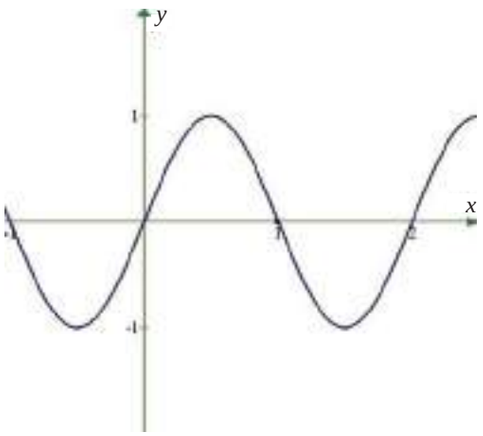

Fig. 9.19

(14) $\lim\limits_{x \to 0} \sec x$

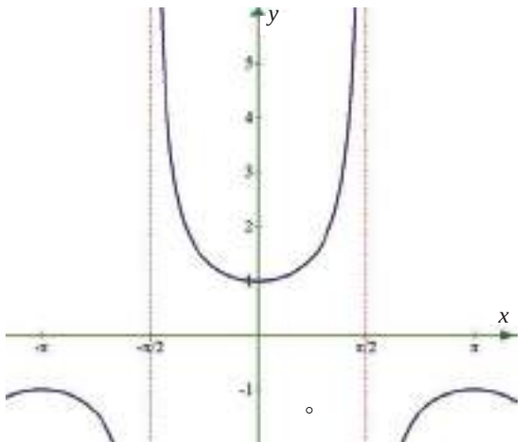

Fig. 9.20

(15) $\lim\limits_{x \to \frac{\pi}{2}} \tan x$

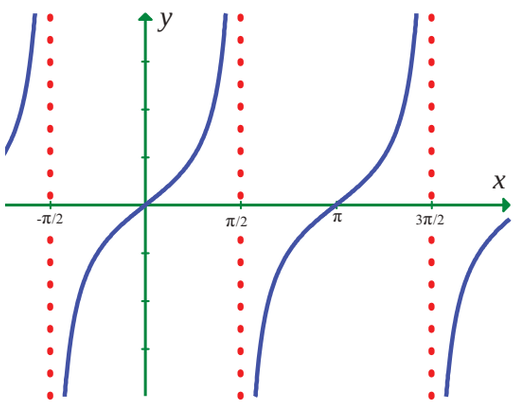

Fig. 9.21

Sketch the graph of f, then identify the values of x_0 for which $\lim\limits_{x \to x_0} f(x)$ exists.

(16) $f(x) = \begin{cases} x^2, & x \le 2 \\ 8 - 2x, & 2 < x < 4 \\ 4, & x \ge 4 \end{cases}$.

(17) $f(x) = \begin{cases} \sin x, & x < 0 \\ 1 - \cos x, & 0 \le x \le \pi \\ \cos x, & x > \pi \end{cases}$

(18) Sketch the graph of a function f that satisfies the given values :

(i) $f(0)$ is undefined

$\lim\limits_{x \to 0} f(x) = 4$

$f(2) = 6$

$\lim\limits_{x \to 2} f(x) = 3$

(ii) $f(-2) = 0$

$f(2) = 0$

$\lim\limits_{x \to -2} f(x) = 0$

$\lim\limits_{x \to 2} f(x)$ does not exist.

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(19) Write a brief description of the meaning of the notation $\lim_{x\to 8} f(x) = 25$.

(20) If $f(2) = 4$, can you conclude anything about the limit of $f(x)$ as x approaches 2?

(21) If the limit of $f(x)$ as x approaches 2 is 4, can you conclude anything about $f(2)$? Explain reasoning.

(22) Evaluate : $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$ if it exists by finding $f(3^-)$ and $f(3^+)$.

(23) Verify the existence of $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} \dfrac{|x-1|}{x-1}, & \text{for } x \neq 1 \\ 0, & \text{for } x = 1 \end{cases}$.

9.2.3 Theorems on limits

The intention of the informal discussion in the earlier section was to have an intuitive grasp of existence or non existence of the limit. However, it is neither desirable nor practical in every instance, to reach a conclusion about the existence of a limit based on a graph or table of functional values. We must be able to evaluate a limit, or discern its non existence, in a somewhat mechanical fashion. The theorems that we shall consider in this section establish such a means. The proofs of these theorems are more of technical and are beyond the scope of this textbook.

In Illustration 9.1, we concluded that $\lim_{x\to 2}(x^2+3) = 2^2+3 = 7$. That is, the limit of $f(x) = x^2 + 3$ as x tends to 2 is equal to $f(x)$ evaluated at $x = 2$. [That is, $f(2)$]. However, this process of evaluation, as noted earlier, will not always work because $f(x)$ may not even be defined at x_0. Nevertheless, it is true that **if f is a polynomial, then it is always possible to calculate the limit by evaluation**.

Theorem 9.1

Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_n x^n$ be a polynomial, where $a_0, a_1, \cdots, a_n$ are real numbers and n is a fixed positive integer. Then

$$\lim_{x\to x_0} P(x) = a_0 + a_1x_0 + a_2x_0^2 + \cdots + a_nx_0^n = P(x_0).$$

Example 9.7

Calculate $\lim_{x\to 3}(x^3 - 2x + 6)$.

Solution

$P(x) = x^3 - 2x + 6$ is a polynomial.

Hence, $\lim_{x\to 3} P(x) = P(3) = 3^3 - 2\times 3 + 6 = 27$.

Example 9.8

Calculate $\lim_{x\to x_0}(5)$ for any real number x_0.

Solution

$f(x) = 5$ is a polynomial (of degree 0).

Hence $\lim_{x\to x_0}(5) = f(x_0) = 5$.

The limit of a constant function is that constant.

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Theorem 9.2

Let I be an open interval containing $x_0 \in \mathbb{R}$.

Let $f, g : I \to \mathbb{R}$.

Suppose that c is a constant and the limits $\lim\limits_{x \to x_0} f(x)$ and $\lim\limits_{x \to x_0} g(x)$ exist. Then $\lim\limits_{x \to x_0}(cf(x)), \lim\limits_{x \to x_0}[f(x)+g(x)]$, $\lim\limits_{x \to x_0}[f(x) - g(x)]$, $\lim\limits_{x \to x_0}[f(x)g(x)]$ and $\lim\limits_{x \to x_0}\dfrac{f(x)}{g(x)}, g(x) \neq 0$, all exist. Moreover,

(i) $\lim\limits_{x \to x_0} cf(x) = c \lim\limits_{x \to x_0} f(x)$,

(ii) $\lim\limits_{x \to x_0}[f(x) \pm g(x)] = \lim\limits_{x \to x_0} f(x) \pm \lim\limits_{x \to x_0} g(x)$,

(iii) $\lim\limits_{x \to x_0}[f(x).g(x)] = \lim\limits_{x \to x_0} f(x) . \lim\limits_{x \to x_0} g(x)$ and

(iv) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to x_0} f(x)}{\lim\limits_{x \to x_0} g(x)}$, provided $\lim\limits_{x \to x_0} g(x) \neq 0$.

These results can be extended to any finite number of functions.

Example 9.9

Compute (i) : $\lim\limits_{x \to 8}(5x)$ (ii) $\lim\limits_{x \to -2}\left(-\dfrac{3}{2}x\right)$.

Solution

(i) $\lim\limits_{x \to 8}(5x) = 5\lim\limits_{x \to 8}(x) = 5 \times 8 = 40$.

(ii) $\lim\limits_{x \to -2}\left(-\dfrac{3}{2}x\right) = -\dfrac{3}{2}\lim\limits_{x \to -2}(x) = \left(-\dfrac{3}{2}\right)(-2) = 3$.

Example 9.10

Compute $\lim\limits_{x \to 0}\left[\dfrac{x^2+x}{x} + 4x^3 + 3\right]$.

Solution

$$\begin{aligned}
\lim_{x \to 0}\left[\frac{x^2+x}{x} + 4x^3 + 3\right] &= \lim_{x \to 0}\left(\frac{x^2+x}{x}\right) + \lim_{x \to 0}(4x^3+3) \\
&= \lim_{x \to 0}(x+1) + \lim_{x \to 0}(4x^3+3) \\
&= (0+1) + (0+3) \\
&= 4.
\end{aligned}$$

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Example 9.11

Calculate $\lim\limits_{x\to -1}(x^2-3)^{10}$.

Solution

$$\lim_{x\to -1}(x^2-3) = 1-3 = -2.$$

Therefore, $\lim\limits_{x\to -1}(x^2-3)^{10} = \lim\limits_{x\to -1}(x^2-3)\,(x^2-3)...(x^2-3)$ (10 times)

$$= \lim_{x\to -1}(x^2-3)\lim_{x\to -1}(x^2-3)...\lim_{x\to -1}(x^2-3) \text{ (10 times)}$$

$$= \left[\lim_{x\to -1}(x^2-3)\right]^{10} = (-2)^{10} = 2^{10} = 1024.$$

Note that $\lim\limits_{x\to -1}(x^2-3)^{10} = \left[\lim\limits_{x\to -1}(x^2-3)\right]^{10}$.

Theorem 9.3

If $\lim\limits_{x\to x_0} f(x)$ exists then $\lim\limits_{x\to x_0}[f(x)]^n$ exists and $\lim\limits_{x\to x_0}[f(x)]^n = \left[\lim\limits_{x\to x_0} f(x)\right]^n$.

Example 9.12

Calculate $\lim\limits_{x\to -2}(x^3-3x+6)\,(-x^2+15)$.

Solution

$$\lim_{x\to -2}(x^3-3x+6) = (-2)^3 - 3(-2) + 6 = -8+6+6 = 4$$

$$\lim_{x\to -2}(-x^2+15) = -(-2)^2 + 15 = -4+15 = 11$$

$$\lim_{x\to -2}(x^3-3x+6)\,(-x^2+15) = \lim_{x\to -2}(x^3-3x+6)\lim_{x\to -2}(-x^2+15) = 4\times 11 = 44.$$

Example 9.13

Calculate $\lim\limits_{x\to 3}\dfrac{(x^2-6x+5)}{x^3-8x+7}$.

Solution

$$\lim_{x\to 3}(x^2-6x+5) = 3^2 - 6\times 3 + 5 = -4$$

$$\lim_{x\to 3}(x^3-8x+7) = 3^3 - 8\times 3 + 7 = 10 \neq 0.$$

Therefore, $\lim\limits_{x\to 3}\dfrac{(x^2-6x+5)}{x^3-8x+7} = \dfrac{\lim\limits_{x\to 3}(x^2-6x+5)}{\lim\limits_{x\to 3}(x^3-8x+7)} = \dfrac{-4}{10} = -\dfrac{2}{5}$.

Caution

Do not use the limit theorem for the quotient if $\lim\limits_{x\to x_0} g(x) = 0$.

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Example 9.14

Compute $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}$.

Solution

Here $\lim\limits_{x\to 1}(x-1)=0$. In such cases, rationalise the numerator.

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{(\sqrt{x}-1)}{(\sqrt{x}-1)(\sqrt{x}+1)}=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}=\frac{\lim\limits_{x\to 1}(1)}{\lim\limits_{x\to 1}\left(\sqrt{x}+1\right)}=\frac{1}{2}.$$

Example 9.15

Find $\lim\limits_{t\to 0}\dfrac{\sqrt{t^2+9}-3}{t^2}$.

Solution

We can't apply the quotient theorem immediately. Use the algebra technique of rationalising the numerator.

$$\frac{\sqrt{t^2+9}-3}{t^2}=\frac{\left(\sqrt{t^2+9}-3\right)\left(\sqrt{t^2+9}+3\right)}{t^2\left(\sqrt{t^2+9}+3\right)}=\frac{t^2+9-9}{t^2\left(\sqrt{t^2+9}+3\right)}$$

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}=\lim_{t\to 0}\frac{t^2}{t^2\sqrt{t^2+9}+3}=\lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}.$$

Theorem 9.4

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}.$$

Proof

We know that $x^n-a^n=(x-a)(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+xa^{n-2}+a^{n-1})$

$$\begin{aligned}\lim_{x\to a}\frac{x^n-a^n}{x-a} &= \lim_{x\to a}\frac{(x-a)(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+xa^{n-2}+a^{n-1})}{(x-a)}\\ &= \lim_{x\to a}(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+xa^{n-2}+a^{n-1})\\ &= a^{n-1}+a^{n-1}+\cdots+a^{n-1}\ (n \text{ times})\\ \lim_{x\to a}\frac{x^n-a^n}{x-a} &= na^{n-1}.\end{aligned}$$

It is also true for any rational number n. ■

Example 9.16

Compute $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}$.

Solution

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{x^3-1^3}{x-1}=3(1)^{3-1}=3.$$

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Example 9.17

Calculate $\lim\limits_{t\to 1}\dfrac{\sqrt{t}-1}{t-1}$.

Solution

$$\lim_{x\to 1}\frac{\sqrt{t}-1}{t-1} = \lim_{x\to 1}\frac{t^{\frac{1}{2}}-1^{\frac{1}{2}}}{t-1} = \frac{1}{2}(1)^{\frac{1}{2}-1} = \frac{1}{2}.$$

Example 9.18

Find $\lim\limits_{x\to 0}\dfrac{(2+x)^5-2^5}{x}$.

Solution

Put $2 + x = y$ so that as $y \to 2$ as $x \to 0$.

Therefore, $\lim\limits_{x\to 0}\dfrac{(2+x)^5-2^5}{x} = \lim\limits_{y\to 2}\dfrac{y^5-2^5}{y-2} = 5(2^4) = 80$.

Example 9.19

Find the positive integer n so that $\lim\limits_{x\to 3}\dfrac{x^n-3^n}{x-3} = 27$.

Solution

$\lim\limits_{x\to 3}\dfrac{x^n-3^n}{x-3} = n.3^{n-1} = 27$

That is $n.3^{n-1} = 3\times 3^2 = 3\times 3^{3-1} \Rightarrow n = 3$.

Example 9.20

Find the relation between a and b if $\lim\limits_{x\to 3} f(x)$ exists where $f(x) = \begin{cases} ax+b & \text{if } x>3 \\ 3ax-4b+1 & \text{if } x<3 \end{cases}$.

Solution

$\lim\limits_{x\to 3^-} f(x) = 9a - 4b + 1$

$\lim\limits_{x\to 3^+} f(x) = 3a + b$. Now the existence of limit forces us to have

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x)$.

$\Rightarrow 9a - 4b + 1 = 3a + b$

$\Rightarrow 6a - 5b + 1 = 0.$

EXERCISE 9.2

Evaluate the following limits :

(1) $\lim\limits_{x\to 2}\dfrac{x^4-16}{x-2}$

(2) $\lim\limits_{x\to 1}\dfrac{x^m-1}{x^n-1}$, m and n are integers.

(3) $\lim\limits_{\sqrt{x}\to 3}\dfrac{x^2-81}{\sqrt{x}-3}$

(4) $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}, x>0$

(5) $\lim\limits_{x\to 5}\dfrac{\sqrt{x+4}-3}{x-5}$

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(6) $\lim\limits_{x\to 2}\dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$ (7) $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-x^2}{1-\sqrt{x}}$ (8) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$

(9) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$ (10) $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1}$ (11) $\lim\limits_{x\to 2}\dfrac{2-\sqrt{x+2}}{\sqrt[3]{2}-\sqrt[3]{4-x}}$

(12) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x^2}-1}{x}$ (13) $\lim\limits_{x\to 0}\dfrac{\sqrt{1-x}-1}{x^2}$ (14) $\lim\limits_{x\to 5}\dfrac{\sqrt{x-1}-2}{x-5}$

(15) $\lim\limits_{x\to a}\dfrac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}(a>b)$

9.2.4 Infinite limits and limits at infinity

Infinite Limits

Let $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be defined by $f(x)=\dfrac{1}{x^2}$.

Let us consider the problem of calculating $\lim\limits_{x\to 0}\dfrac{1}{x^2}$.

The following table gives the values of $\dfrac{1}{x^2}$ near 0.

$x\to 0^+$			$x\to 0^-$		
x	x^2	$\frac{1}{x^2}$	x	x^2	$\frac{1}{x^2}$
1	1	1	-1	1	1
0.5	0.25	4	-0.5	0.25	4
0.1	0.01	100	-0.1	0.01	100
0.01	0.0001	10,000	$-.01$	0.0001	10,000
0.001	0.000001	10,00,000	-0.001	0.000001	10,00,000
0.0001	0.00000001	10,00,00,000	-0.0001	0.00000001	10,00,00,000

The table values tell us that as x gets closer and closer to 0, $f(x)=\dfrac{1}{x^2}$ gets larger and larger. In fact, $\dfrac{1}{x^2}$ grows without bound as x approaches 0 from either side. In this situation we say that $f(x)$ **tends to infinity as x approaches zero** and write $\dfrac{1}{x^2}\to\infty$ as $x\to 0^-$ and $\dfrac{1}{x^2}\to\infty$ as $x\to 0^+$

and hence $\dfrac{1}{x^2}\to\infty$ as $x\to 0$.

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Geometrically, $x = 0$ namely the y-axis is a vertical asymptote to the curve representing $f(x) = \frac{1}{x^2}$.

The graph of the function $f(x) = \frac{1}{x^2}$ is shown in Fig. 9.22.

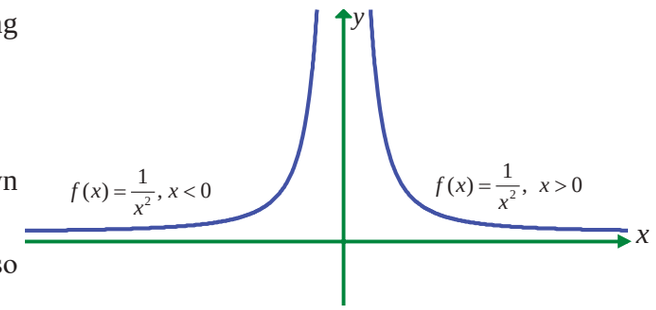

Fig. 9.22

Remember that the limit is infinite and so $\lim_{x \to 0} \frac{1}{x^2}$ does not exist. Students are cautioned that ∞ is a symbol for this behaviour of $f(x) = \frac{1}{x^2}$ and is not a new number.

Similarly, if we look at $f(x) = \frac{1}{x}$,

it is easy to see that, $\frac{1}{x} \to -\infty$ as $x \to 0^-$

and $\frac{1}{x} \to +\infty$ as $x \to 0^+$

which is geometrically clear from the graph of $f(x) = \frac{1}{x}$ (Fig. 9.23) .

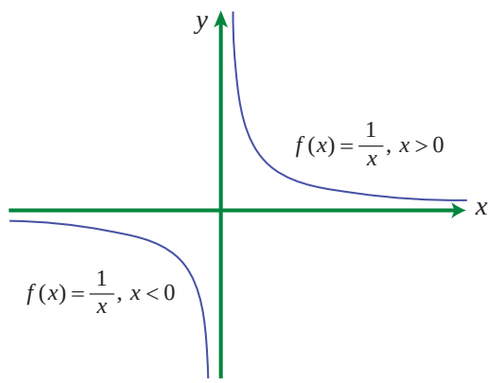

Fig. 9.23

In general we have the following intuitive definitions.

Definition 9.4

For a given $M > 0$, open intervals of the form (M, ∞) is called neighbourhood of ∞.

Similarly, for given $K < 0$, open intervals of the form $(-\infty, K)$ is called neighbourhood of $-\infty$.

Definition 9.5

We say, $f(x) \to \infty$ as x approaches x_0 if for given positive number M there is a neighbourhood of x_0, such that whenever x is in the neighbourhood of x_0, $f(x) > M$. i.e., $f(x) \in (M, \infty)$.

Similarly, $f(x) \to -\infty$ as x approaches x_0 if for a given $K < 0$ there is a neighbourhood of x_0 such that whenever x is in the neighbourhood of x_0, $f(x) < K$. i.e., $f(x) \in (-\infty, K)$.

To describe this situation symbolically, we write

$f(x) \to \infty$ as $x \to x_0$

$f(x) \to -\infty$ as $x \to x_0$

$f(x) \to \infty$ as $x \to x_0^-$

$f(x) \to -\infty$ as $x \to x_0^+$

and $f(x) \to \infty$ as $x \to x_0^+$

$f(x) \to -\infty$ as $x \to x_0^-$

are called **infinite limits**. If any one of the foregoing conditions hold, then the line $x = x_0$ is a **vertical asymptote** for the graph of $f(x)$.

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Example 9.21

Calculate $\lim\limits_{x\to 0}\dfrac{1}{(x^2+x^3)}$.

Solution

One can tabulate values of x near 0 (from either side) and conclude $f(x)=\dfrac{1}{x^2+x^3}$ grows without bound and hence $f(x)\to\infty$ as $x\to 0$.

To calculate this limit without making a table, we first divide the numerator and denominator by x^2. This division can be done, since in the calculation of the limit $x\neq 0$ and hence $x^2\neq 0$. We can have

$$\lim_{x\to 0}\frac{1}{x^2+x^3} = \lim_{x\to 0}\frac{\frac{1}{x^2}}{\frac{x^2+x^3}{x^2}} = \frac{\lim\limits_{x\to 0}\left(\frac{1}{x^2}\right)}{\lim\limits_{x\to 0}(1+x)}.$$

Now $\dfrac{1}{x^2}\to\infty$ as $x\to 0$ and $\lim\limits_{x\to 0}(1+x)=1$.

Thus the numerator grows without bound while the denominator approaches 1, implying that $\dfrac{1}{(x^2+x^3)}$ does tend to infinity.

This example illustrates how a difficult calculation can be greatly simplified by a few algebraic manipulations.

Example 9.22

Evaluate $\lim\limits_{x\to 2}\dfrac{1}{(x-2)^3}$.

Solution

From the graph of $f(x)=\dfrac{1}{(x-2)^3}$,

clearly, $\dfrac{1}{(x-2)^3}\to-\infty$ as $x\to 2^-$ and

$\dfrac{1}{(x-2)^3}\to\infty$ as $x\to 2^+$.

Hence the limit does not exist.

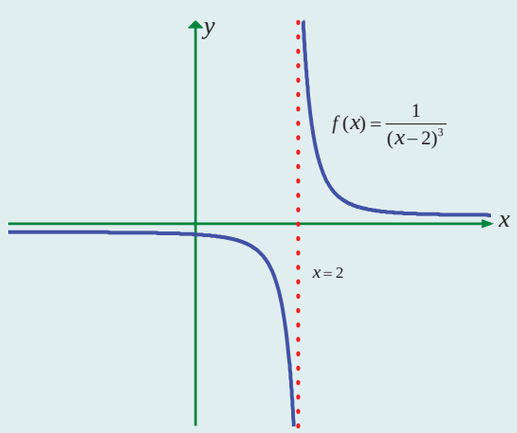

Fig. 9.24

In general

(i) If n is an even positive integer then

$\dfrac{1}{(x-a)^n}\to\infty$ as $x\to a$

$\dfrac{1}{(x-a)^n}\to\infty$ as $x\to a^-$

$\dfrac{1}{(x-a)^n}\to\infty$ as $x\to a^+$

(ii) If n is an odd positive integer, then

$\dfrac{1}{(x-a)^n}\to-\infty$ as $x\to a^-$

$\dfrac{1}{(x-a)^n}\to+\infty$ as $x\to a^+$.

The line $x=a$ becomes a vertical asymptote.

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9.2.5 Limits at infinity

In the previous section we investigated infinite limits and vertical asymptotes. There, we let x approach a number and the result was that the values of y became arbitrarily large (very large positive or very large negative). In this section, we let x become arbitrarily large (positive or negative) and see what happens to y.

Let's begin by investigating the behaviour of $f:\mathbb{R}\to\mathbb{R}$ defined by

$f(x) = \dfrac{x^2-1}{x^2+1}$ as x becomes large.

We tabulate the values of this function

x	$f(x)$
0	-1
±1	0
±2	0.6000000
±3	0.800000
±4	0.882353
±5	0.923077
±10	0.980198
±50	0.999200
±100	0.999800
±1000	0.999998

As x grows larger and larger (large positive or large negative) you can see that the values of $f(x)$ gets closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x\to\pm\infty}\frac{x^2-1}{x^2+1} = 1.$$

If we look at the graph :

Geometrically, (see Fig. 9.25) this situation also leads us to have

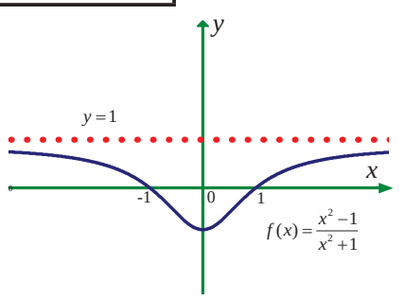

Fig. 9.25

Definition 9.6

The line $y = l$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to-\infty} f(x) = l \quad \text{or} \quad \lim_{x\to+\infty} f(x) = l.$$

Illustration 9.4

If $f:\mathbb{R}\to\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ is defined by $f(x) = \tan^{-1}x$,

find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

Solution

If we look at the graph of $y = \tan^{-1}x$,

$$\lim_{x\to-\infty}\tan^{-1}x = -\frac{\pi}{2},$$

$$\lim_{x\to+\infty}\tan^{-1}x = \frac{\pi}{2}.$$

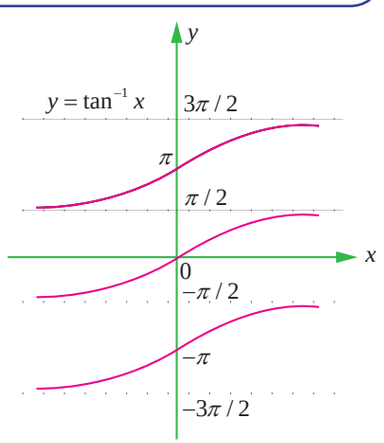

Fig. 9.26

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Illustration 9.5

Calculate $\lim\limits_{x\to\infty}\dfrac{2x^2-2x+3}{x^2+4x+4}$.

Solution

If we could try to use limit theorems to calculate this limit, we end up in the following situations.

$(2x^2-2x+3)\to\infty$ as $x\to\infty$

$(x^2+4x+3)\to\infty$ as $x\to\infty$

$\lim\limits_{x\to\infty}\left(\dfrac{2x^2-2x+3}{x^2+4x+3}\right)=\dfrac{\infty}{\infty}$ which is called an indeterminate form.

But actual calculation and tabulation gives the following :

x	$2x^2-2x+3$	x^2+4x+4	$\dfrac{2x^2-2x+3}{x^2+4x+4}$
1	3	9	0.3333
10	183	144	1.27083
100	19803	10404	1.90340
1000	1998003	1004004	1.99003
10000	199980003	100040004	1.99900

Table values show that as x becomes sufficiently large, $f(x)$ becomes closer and closer to 2. Then,

$\lim\limits_{x\to\infty}\left(\dfrac{2x^2-2x+3}{x^2+4x+4}\right)=2.$

Fortunately, we may simplify the problem by dividing the numerator and denominator by x^2. We have

$$\lim_{x\to\infty}\frac{2x^2-2x+3}{x^2+4x+4}=\lim_{x\to\infty}\left(\frac{2-\frac{2}{x}+\frac{3}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}}\right)$$

$$=\frac{2-0+0}{1+0+0}\quad\left(\text{since } \frac{1}{x}\to 0 \text{ as } x\to\infty,\ \frac{1}{x^2}\to 0 \text{ as } x\to\infty\right)$$

$$=2.$$

Note that the degree of both numerator and denominator expressions are the same.

In general, the limits as $x\to\pm\infty$ of rational expressions can be found by first dividing the numerator and denominator by the highest power of x that appears in the denominator, and then calculating the limit as $x\to\infty$ (or $x\to-\infty$) of both numerator and denominator.

Example 9.23

Calculate $\lim\limits_{x\to\infty}\dfrac{x^3+2x+3}{(5x^2+1)}$.

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Solution

Dividing by x^2

$$\lim_{x\to\infty}\frac{x^3+2x+3}{(5x^2+1)} = \lim_{x\to\infty}\frac{x+\frac{2}{x}+\frac{3}{x^2}}{5+\frac{1}{x^2}} \to \infty$$

That is, $\dfrac{x^3+2x+3}{(5x^2+1)} \to \infty$ as $x \to \infty$.

In other words, the limit does not exist.

Note that the degree of numerator is higher than that of the denominator.

Example 9.24

Calculate $\lim\limits_{x\to\infty}\dfrac{1-x^3}{3x+2}$.

Solution

Dividing by x, we get

$$\frac{1-x^3}{3x+2} = \frac{\frac{1}{x}-x^2}{3+\frac{2}{x}} \to -\infty \text{ as } x \to \infty .$$

Therefore the limit does not exist.

9.2.6 Limits of rational functions

If $R(x) = \dfrac{p(x)}{q(x)}$ and the degree of the polynomial $p(x)$ is greater than the degree of $q(x)$, then

$$\frac{p(x)}{q(x)} \to +\infty \text{ or } -\infty \text{ as } x \to \infty.$$

If the degree of $q(x)$ is greater than the degree of $p(x)$, then

$$\lim_{x\to\infty}\frac{p(x)}{q(x)} = 0.$$

Finally, if the degree of $p(x)$ is equal to the degree of $q(x)$, then

$$\lim_{x\to\infty}\frac{p(x)}{q(x)} = \frac{\text{coeffiecent of highest power of } x \text{ in } p(x)}{\text{coefficient of highest power of } x \text{ in } q(x)}.$$

Remark

We reemphasize that statements such as $f(x) \to \infty$ as $x \to a$, $f(x) \to -\infty$ as $x \to a$, and $f(x) \to \infty$ as $x \to \infty$, $f(x) \to -\infty$ as $x \to \infty$ mean that the limits do not exist. The symbol ∞ does not represent a number and should not be treated as a number.

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9.2.7 Applications of limits

Example 9.25

Alcohol is removed from the body by the lungs, the kidneys, and by chemical processes in liver. At moderate concentration levels, the majority work of removing the alcohol is done by the liver; less than 5% of the alcohol is eliminated by the lungs and kidneys. The rate r at which the liver processes alcohol from the bloodstream is related to the blood alcohol concentration x by a rational function of the form $r(x) = \dfrac{\alpha x}{x+\beta}$ for some positive constants α and β. Find the maximum possible rate of removal.

Solution

As the alcohol concentration x increases the rate of removal increases.

Therefore, the maximum possible rate of removal $= \lim\limits_{x\to\infty} r(x)$

$$= \lim_{x\to\infty} \frac{\alpha x}{x+\beta} = \lim_{x\to\infty} \frac{\alpha}{\left(1+\dfrac{\beta}{x}\right)} = \alpha.$$

9.2.8 Sandwich Theorem

Sandwich theorem is also known as squeeze theorem. As shown in the figure 9.27, if $f(x)$ is 'squeezed' or 'sandwiched' between $g(x)$ and $h(x)$ for all x close to x_0, and if we know that the functions g and h have a common limit l as $x \to x_0$, it stands to reason that f also approaches l as $x \to x_0$.

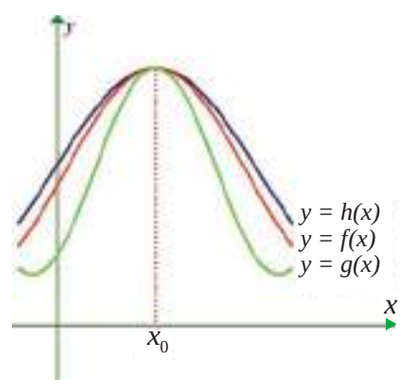

Fig. 9.27

Theorem 9.5 (Sandwich Theorem)

If $f, g, h : I \subseteq \mathbb{R} \to \mathbb{R}$ such that $g(x) \le f(x) \le h(x)$ for all x in a deleted neighbourhood of x_0 contained in I, and if

$$\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = l, \quad \text{then} \quad \lim_{x\to x_0} f(x) = l.$$

Example 9.26

According to Einstein's theory of relativity, the mass m of a body moving with velocity v is $m = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$, where m_0 is the initial mass and c is the speed of light. What happens to m as $v \to c^-$. Why is a left hand limit necessary?

Solution

$$\lim_{v\to c^-}(m) = \lim_{v\to c^-} \frac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}} = \frac{m_0}{\sqrt{\lim\limits_{v\to c^-}\left(1-\dfrac{v^2}{c^2}\right)}}$$

For $h > 0$, $c - h < v < c$. This implies, $(c-h)^2 < v^2 < c^2$.

That is, $\dfrac{(c-h)^2}{c^2} < \dfrac{v^2}{c^2} < 1$. That is, $\lim\limits_{h\to 0}\dfrac{(c-h)^2}{c^2} < \lim\limits_{h\to 0}\dfrac{v^2}{c^2} < \lim\limits_{h\to 0} 1$.

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That is, $1<\lim_{h\to 0}\frac{v^2}{c^2}<1$. That is, $1<\lim_{v\to c^-}\frac{v^2}{c^2}<1$. By Sandwich theorem, $\lim_{v\to c^-}=1$.

Therefore, $\lim_{v\to c^-}(m)\to\infty$.

That is, the mass becomes very very large (infinite) as $v\to c^-$.

The left hand limit is necessary. Otherwise as $v\to c^+$ makes $1-\frac{v^2}{c^2}<0$ and consequently we cannot find the mass.

Example 9.27

The velocity in ft/sec of a falling object is modeled by $r(t)=-\sqrt{\frac{32}{k}}\frac{1-e^{-2t\sqrt{32k}}}{1+e^{-2t\sqrt{32k}}}$, where k is a constant that depends upon the size and shape of the object and the density of the air. Find the limiting velocity of the object, that is, find $\lim_{t\to\infty}r(t)$.

Solution

$$\lim_{t\to\infty}r(t) = \lim_{t\to\infty}-\sqrt{\frac{32}{k}}\frac{1-e^{-2t\sqrt{32k}}}{1+e^{-2t\sqrt{32k}}}$$

$$=-\sqrt{\frac{32}{k}}\lim_{t\to\infty}\frac{1-e^{-2t\sqrt{32k}}}{1+e^{-2t\sqrt{32k}}}$$

$$=-\sqrt{\frac{32}{k}}\frac{(1-0)}{(1+0)}=-\sqrt{\frac{32}{k}}\text{ ft/sec.}$$

Example 9.28

Suppose that the diameter of an animal's pupils is given by $f(x)=\frac{160x^{-0.4}+90}{4x^{-0.4}+15}$, where x is the intensity of light and $f(x)$ is in *mm*. Find the diameter of the pupils with (a) minimum light (b) maximum light.

Solution

(a) For minimum light it is enough to find the limit of the function when $x\to 0^+$.

$$\lim_{x\to 0^+}f(x) = \lim_{x\to 0^+}\frac{160x^{-0.4}+90}{4x^{-0.4}+15}=\lim_{x\to 0^+}\frac{160+90x^{0.4}}{4+15x^{0.4}}$$

$$=\frac{160}{4}=40\text{mm}.$$

(b) For maximum light, it is enough to find the limit of the function when $x\to\infty$

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{160x^{-0.4}+90}{4x^{-0.4}+15}=\frac{90}{15}=6\text{mm}$$

That is, the pupils have a limiting size of 6mm, as the intensity of light is very large.

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EXERCISE 9.3

(1) (a) Find the left and right limits of $f(x)=\dfrac{x^2-4}{(x^2+4x+4)(x+3)}$ at $x=-2$.

(b) $f(x)=\tan x$ at $x=\dfrac{\pi}{2}$.

Evaluate the following limits

(2) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2(x^2-6x+9)}$ (3) $\lim\limits_{x\to\infty}\dfrac{3}{x-2}-\dfrac{2x+11}{x^2+x-6}$ (4) $\lim\limits_{x\to\infty}\dfrac{x^3+x}{x^4-3x^2+1}$

(5) $\lim\limits_{x\to\infty}\dfrac{x^4-5x}{x^2-3x+1}$ (6) $\lim\limits_{x\to\infty}\dfrac{1+x-3x^3}{1+x^2+3x^3}$ (7) $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{2x^2-1}-\dfrac{x^2}{2x+1}\right)$

(8) Show that

(i) $\lim\limits_{n\to\infty}\dfrac{1+2+3+\ldots+n}{3n^2+7n+2}=\dfrac{1}{6}$ (ii) $\lim\limits_{n\to\infty}\dfrac{1^2+2^2+\ldots+(3n)^2}{(1+2+\ldots+5n)(2n+3)}=\dfrac{9}{25}$

(iii) $\lim\limits_{n\to\infty}\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\ldots+\dfrac{1}{n(n+1)}=1$

(9) An important problem in fishery science is to estimate the number of fish presently spawning in streams and use this information to predict the number of mature fish or "recruits" that will return to the rivers during the reproductive period. If S is the number of spawners and R the number of recruits, "Beverton-Holt spawner recruit function" is $R(S)=\dfrac{S}{(\alpha S+\beta)}$ where α and β are positive constants. Show that this function predicts approximately constant recruitment when the number of spawners is sufficiently large.

(10) A tank contains 5000 litres of pure water. Brine (very salty water) that contains 30 grams of salt per litre of water is pumped into the tank at a rate of 25 litres per minute. The concentration of salt water after t minutes (in grams per litre) is $C(t)=\dfrac{30t}{200+t}$.
What happens to the concentration as $t\to\infty$?

Example 9.29

Evaluate $\lim\limits_{x\to 0}x^2\sin\left(\dfrac{1}{x}\right)$.

Solution

We know that $-1\le\sin\dfrac{1}{x}\le 1\Rightarrow -x^2\le x^2\sin\dfrac{1}{x}\le x^2$

Take $g(x)=-x^2$, $f(x)=x^2\sin\dfrac{1}{x}$; $h(x)=x^2$

Then $\lim\limits_{x\to 0}g(x)=\lim\limits_{x\to 0}(-x^2)=0$ and

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$$\lim_{x\to 0} h(x) = \lim_{x\to 0}(x^2) = 0\,.$$

By Sandwich theorem,

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

We could have wrongly concluded had we resorted to applying the limit theorems, namely

$$\lim_{x\to 0} x^2\left(\sin\frac{1}{x}\right) = \lim_{x\to 0}(x^2)\lim_{x\to 0}\left(\sin\frac{1}{x}\right)$$

Now, $\lim\limits_{x\to 0}\sin\frac{1}{x}$ does not exist, $\lim\limits_{x\to 0} x^2 = 0$ and hence $\lim\limits_{x\to 0} x^2\sin\left(\frac{1}{x}\right)$ leading us into trouble.

Note that, if $a \le f(x) \le a$, then $\lim\limits_{x\to x_0} f(x) = a$.

Example 9.30

Prove that $\lim\limits_{x\to 0}\sin x = 0$.

Solution

Since $-x \le \sin x \le x$ for all $x \ge 0$

$$\lim_{x\to 0}(-x) = 0 \text{ and}$$

$$\lim_{x\to 0}(x) = 0.$$

By Sandwich theorem

$$\lim_{x\to 0}\sin x = 0.$$

Example 9.31

Show that $\lim\limits_{x\to 0^+} x\left[\left\lfloor\frac{1}{x}\right\rfloor + \left\lfloor\frac{2}{x}\right\rfloor + \cdots + \left\lfloor\frac{15}{x}\right\rfloor\right] = 120$.

Solution

$$\frac{1}{x} - 1 \le \left\lfloor\frac{1}{x}\right\rfloor \le \frac{1}{x} + 1$$

$$\frac{2}{x} - 1 \le \left\lfloor\frac{2}{x}\right\rfloor \le \frac{2}{x} + 1$$

$$\vdots$$

$$\frac{15}{x} - 1 \le \left\lfloor\frac{15}{x}\right\rfloor \le \frac{15}{x} + 1$$

Summing, we get,

$$\frac{120}{x} - 15 \le \left\lfloor\frac{1}{x}\right\rfloor + \left\lfloor\frac{2}{x}\right\rfloor + \cdots + \left\lfloor\frac{15}{x}\right\rfloor \le \frac{120}{x} + 15$$

$$120 - 15x \le x\left[\left\lfloor\frac{1}{x}\right\rfloor + \left\lfloor\frac{2}{x}\right\rfloor + \cdots + \left\lfloor\frac{15}{x}\right\rfloor\right] \le 120 + 15x$$

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$$\lim_{x\to 0^+}(120-15x) \le \lim_{x\to 0^+} x\left[\left\lfloor\frac{1}{x}\right\rfloor+\left\lfloor\frac{2}{x}\right\rfloor+\cdots+\left\lfloor\frac{15}{x}\right\rfloor\right] \le \lim_{x\to 0^+}(120+15x)$$

$$120 \le \lim_{x\to 0^+} x\left[\left\lfloor\frac{1}{x}\right\rfloor+\left\lfloor\frac{2}{x}\right\rfloor+\cdots+\left\lfloor\frac{15}{x}\right\rfloor\right] \le 120$$

$$\lim_{x\to 0^+} x\left[\left\lfloor\frac{1}{x}\right\rfloor+\left\lfloor\frac{2}{x}\right\rfloor+\cdots+\left\lfloor\frac{15}{x}\right\rfloor\right]=120.$$

9.2.9 Two special Trigonometrical limits

Result 9.1

(a) $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ (b) $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0$.

Proof

We use a circular sector to prove the result.

Consider the circle with centre (0, 0) and radius 1. Any point on this circle is $P(\cos\theta, \sin\theta)$.

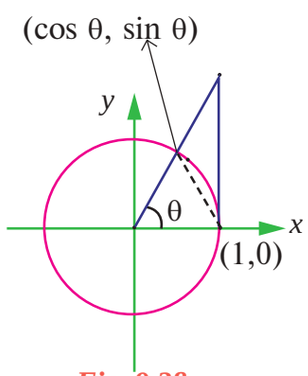
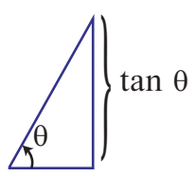
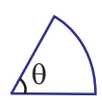


Fig. 9.28



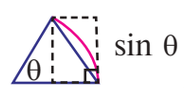
Area of triangle $\frac{\tan\theta}{2}$

Fig. 9.29



Area of sector $\frac{\theta}{2}$

Fig. 9.30



Area of triangle $\frac{\sin\theta}{2}$

Fig. 9.31

By area property $\frac{\tan\theta}{2}\ge\frac{\theta}{2}\ge\frac{\sin\theta}{2}$.

Multiplying each expression by $\frac{2}{\sin\theta}$ produces $\frac{1}{\cos\theta}\ge\frac{\theta}{\sin\theta}\ge 1$ and taking reciprocals

$\cos\theta\le\frac{\sin\theta}{\theta}\le 1$.

Because $\cos(-\theta)=\cos\theta$ and $\frac{\sin(-\theta)}{-\theta}=\frac{\sin\theta}{\theta}$ one can conclude that this inequality is valid for all non-zero θ in the open interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

We know that $\lim_{\theta\to 0}\cos\theta=1; \lim_{\theta\to 0}(1)=1$ and applying Sandwich theorem we get $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$. ■

(b) $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0$.

$$1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

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$$\frac{1-\cos\theta}{\theta} = \left(\sin\frac{\theta}{2}\right)\frac{\sin\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)}$$

$$\text{Therefore, } \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\left(\sin\frac{\theta}{2}\right).\lim_{\theta\to 0}\left(\frac{\frac{\sin\theta}{2}}{\frac{\theta}{2}}\right)$$

$$= 0 \times 1 = 0.$$

9.2.10 Some important other limits

Result 9.2

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1.$$

Proof

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots \qquad \text{(from sequences and series)}$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\left(\frac{\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots}{x}\right)$$

$$= \lim_{x\to 0}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots\right) = 1.$$

Result 9.3

$$\lim_{x\to 0}\frac{a^x-1}{x} = \log a, \ a>0$$

Proof

We know that $a^x, \log_a x$ are inverses of each other.

Since $\log f(x)$ is the inverse of $\exp(f(x))$, $\exp(\log f(x)) = f(x)$.

Therefore, $a^x = \exp(\log a^x)$

$= e^{x\log a}$

$$\text{Therefore, } \frac{a^x-1}{x} = \frac{e^{x\log a}-1}{x\log a}\times\log a$$

$$\text{Now as } x\to 0,\ y = x\log a \to 0$$

$$\text{Therefore, } \lim_{x\to 0}\frac{a^x-1}{x} = \lim_{y\to 0}\frac{e^y-1}{y}\times\log a = \log a\lim_{y\to 0}\left(\frac{e^y-1}{y}\right) = \log a$$

$$\left(\text{since } \lim_{x\to 0}\frac{e^x-1}{x} = 1\right).$$

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Result 9.4

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=1.$$

Proof

Take $\log(1+x) = y$

Then $y\to 0$ as $x\to 0$ and

$$1+x = e^y$$

$$x = e^y-1$$

Therefore, $$\lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{y\to 0}\frac{y}{e^y-1}$$

$$= \lim_{y\to 0}\frac{1}{\left(\frac{e^y-1}{y}\right)}=\frac{1}{1}=1.$$

Some important limits without proof

Results 9.5 to 9.9

(5) $\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$

(6) $\lim_{x\to 0}\frac{\tan^{-1}x}{x}=1$

(7) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ exists and this limit is e.

(8) $\lim_{x\to 0}(1+x)^{1/x}=e$

(9) $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x=e^k.$

This number e is also known as transcendental number in the sense that e never satisfies a polynomial (algebraic) equation of the form

$$a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n=0.$$

Example 9.32

Evaluate : $\lim_{x\to 0}(1+\sin x)^{2\operatorname{cosec} x}$

Solution

Let $\sin x=\frac{1}{y}$

As $x\to 0,\ y\to\infty$ and

$$\lim_{x\to 0}(1+\sin x)^{2\operatorname{cosec} x} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{2y}=\left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}\right]^2 = e^2.$$

Example 9.33

Evaluate : $\lim_{x\to\infty}\left(\frac{x+2}{x-2}\right)^x.$

Solution

$$\lim_{x\to\infty}\left(\frac{x+2}{x-2}\right)^x = \lim_{x\to\infty}\left(\frac{x-2+4}{x-2}\right)^{x-2+2}=\lim_{x\to\infty}\left(1+\frac{4}{x-2}\right)^{(x-2)+2}$$

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Let $y = x-2$, as $x \to \infty$, $y \to \infty$ and (Let $y = x-2$, Then as $x \to \infty$, $y \to \infty$)

$$\lim_{x\to\infty}\left(\frac{x+2}{x-2}\right)^x = \lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y+2} = \lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y}.\lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{2}$$

$$= e^4.1 = e^4 .$$

Example 9.34

Evaluate : $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{4\sqrt{2}-(\cos x+\sin x)^5}{1-\sin 2x}$.

Solution

$$\frac{4\sqrt{2}-(\cos x+\sin x)^5}{1-\sin 2x} = \frac{2^{\frac{5}{2}}-\left[(\cos x+\sin x)^2\right]^{\frac{5}{2}}}{1-\sin 2x}$$

$$= \frac{2^{\frac{5}{2}}-(1+\sin 2x)^{\frac{5}{2}}}{2-(1+\sin 2x)}$$

$$= \frac{2^{\frac{5}{2}}-(1+\sin 2x)^{\frac{5}{2}}}{2-(1+\sin 2x)}$$

Therefore, $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{2^{\frac{5}{2}}-[(\cos x+\sin x)^2]^{5/2}}{2-[1+\sin 2x]} = \lim\limits_{x\to\frac{\pi}{4}} \dfrac{2^{\frac{5}{2}}-\left[1+\sin 2x)\right]^{\frac{5}{2}}}{2-(1+\sin 2x)}$.

Take $y = 1 + \sin 2x$. As $x \to \dfrac{\pi}{4}$, $y \to 2$

$$= \lim_{y\to 2}\frac{2^{\frac{5}{2}}-y^{\frac{5}{2}}}{2-y}$$

$$= \frac{5}{2}.2^{\frac{5}{2}-1} = \frac{5}{2}\times 2^{\frac{3}{2}} = 5\sqrt{2} .$$

Example 9.35

Do the limits of following functions exist as $x \to 0$? State reasons for your answer.

(i) $\dfrac{\sin|x|}{x}$ (ii) $\dfrac{\sin x}{|x|}$ (iii) $\dfrac{x\lfloor x\rfloor}{\sin|x|}$ (iv) $\dfrac{\sin\left(x-\lfloor x\rfloor\right)}{x-\lfloor x\rfloor}$.

Solution

(i)
$$f(x) = \begin{cases} \dfrac{\sin(-x)}{x} & \text{if } -1<x<0 \\ \dfrac{\sin x}{x} & \text{if } 0<x<1 \end{cases}$$

Therefore, $\lim\limits_{x\to 0^-} f(x) = -1$ and

$\lim\limits_{x\to 0^+} f(x) = +1$.

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Hence the limit does not exist. Note that $f(0^-) \neq f(0^+)$.

(ii)
$$\frac{\sin x}{|x|} = \begin{cases} \dfrac{\sin(x)}{-x} & \text{if } -1 < x < 0 \\ \dfrac{\sin x}{x} & \text{if } 0 < x < 1 \end{cases}$$

Therefore,
$$\lim_{x \to 0^-} f(x) = -1$$
$$\lim_{x \to 0^+} f(x) = 1$$

Hence the limit does not exist.

(iii)
$$f(x) = \frac{x\lfloor x \rfloor}{\sin|x|} = \begin{cases} \dfrac{-x}{\sin(-x)} & \text{if } -1 < x < 0 \\ \dfrac{x.0}{\sin x} & \text{if } 0 < x < 1 \end{cases}$$
$$= \begin{cases} \dfrac{x}{\sin x} & \text{if } -1 < x < 0 \\ 0 & \text{if } 0 < x < 1 \end{cases}$$

Therefore,
$$\lim_{x \to 0^-} f(x) = +1$$
$$\lim_{x \to 0^+} f(x) = 0.$$

Hence the limit does not exist.

(iv)
$$\frac{\sin\left(x - \lfloor x \rfloor\right)}{x - \lfloor x \rfloor} = \begin{cases} \dfrac{\sin(x-(-1))}{x-(-1)} & \text{if } -1 < x < 0 \\ \dfrac{\sin(x-0)}{x-0} & \text{if } 0 < x < 1 \end{cases}$$
$$f(x) = \begin{cases} \dfrac{\sin(x+1)}{(x+1)} & \text{if } -1 < x < 0 \\ \dfrac{\sin x}{x} & \text{if } 0 < x < 1 \end{cases}$$
$$\lim_{x \to 0^-} f(x) = \frac{\sin 1}{1} = \sin 1$$
$$\lim_{x \to 0^+} f(x) = 1.$$

Hence the limit does not exist.

EXERCISE 9.4

Evaluate the following limits:

(1) $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^{7x}$

(2) $\lim\limits_{x \to 0} (1 + x)^{1/3x}$

(3) $\lim\limits_{x \to \infty} \left(1 + \dfrac{k}{x}\right)^{\frac{m}{x}}$

(4) $\lim\limits_{x \to \infty} \left(\dfrac{2x^2 + 3}{2x^2 + 5}\right)^{8x^2+3}$

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(5) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x+2}$

(6) $\lim_{x\to 0}\frac{\sin^3\left(\frac{x}{2}\right)}{x^3}$

(7) $\lim_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}$

(8) $\lim_{x\to 0}\frac{\tan 2x}{\sin 5x}$

(9) $\lim_{\alpha\to 0}\frac{\sin(\alpha^n)}{(\sin\alpha)^m}$

(10) $\lim_{x\to 0}\frac{\sin(a+x)-\sin(a-x)}{x}$

(11) $\lim_{x\to 0}\frac{\sqrt{x^2+a^2}-a}{\sqrt{x^2+b^2}-b}$

(12) $\lim_{x\to 0}\frac{2\arcsin x}{3x}$

(13) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

(14) $\lim_{x\to 0}\frac{\tan 2x}{x}$

(15) $\lim_{x\to 0}\frac{2^x-3^x}{x}$

(16) $\lim_{x\to 0}\frac{3^x-1}{\sqrt{x+1}-1}$

(17) $\lim_{x\to 0}\frac{1-\cos^2 x}{x\sin 2x}$

(18) $\lim_{x\to\infty} x\left[3^{\frac{1}{x}}+1-\cos\left(\frac{1}{x}\right)-e^{\frac{1}{x}}\right]$

(19) $\lim_{x\to\infty}\{x[\log(x+a)-\log(x)]\}$

(20) $\lim_{x\to\pi}\frac{\sin 3x}{\sin 2x}$

(21) $\lim_{x\to\pi}(1+\sin x)^{2\operatorname{cosec} x}$

(22) $\lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

(23) $\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}$

(24) $\lim_{x\to\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$

(25) $\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$

(26) $\lim_{x\to 0}\frac{e^{ax}-e^{bx}}{x}$

(27) $\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3}$

(28) $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$

9.3 Continuity

One of the chief features in the behaviour of functions is the property known as continuity. It reflects mathematically the general trait of many phenomena observed by us in nature. For instance, we speak of the continuous expansion of a rod on heating, of the continuous growth of an organism, of a continuous flow, or a continuous variation of atmospheric temperature etc.

The idea of continuity of a function stems from the geometric notion of "no breaks in a graph". In fact, the name itself derives from the Latin continuere, "to hang together". Nevertheless, to identify continuity with "no breaks in a graph" or "a hanging together" has serious drawbacks, at least from the point of view of applying the concept to the analysis of functions. Accordingly, a premature use of the graph to gain insight into the meaning of continuity is advised against as gravely misleading. However, we will realise later that for functions with interval domains, continuity means essentially that the graph may be traced without lifting the point of the pencil.

The proper and effective way of attitude which allows us to put the concept to work is to correlate continuity with limit. Loosely speaking, to possess the property of continuity will mean "to

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have a favourable limit". In order to formulate the concept of continuity in terms of limit we must focus our attention at a point. Both continuity and limit are primarily concepts defined at a point, but continuity acquires a global character in a pointwise way.

For motivation, consider the following physical situation. A thermometer T measures temperatures along a given hot wire L.

To each point x on the hot wire L is assigned a temperature readings $t(x)$ on thermometer, T. Suppose, to fix ideas, the temperature recordings are observed to be the same, 250°F, as one moves along the wire L until a point x_0 is reached on L.

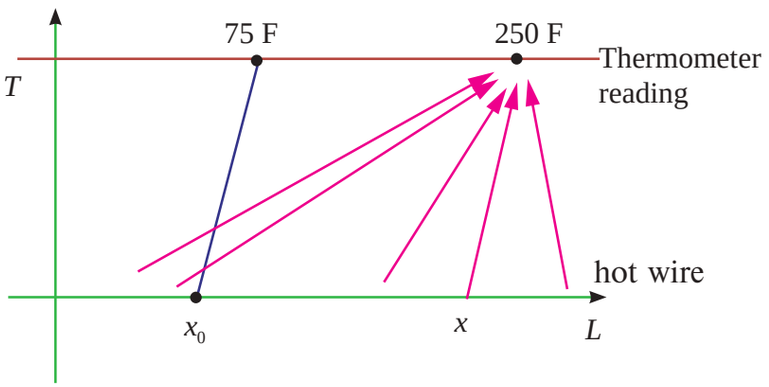

Fig. 9.32

Then suppose that at x_0 the temperature drops suddenly to near room temperature, say 75°F, as if an insulation were at x_0. But, beyond x_0 suppose readings of 250°F are again observed. In function notation we are assuming that

$$t(x) = \begin{cases} 250^\circ F & \text{if } x \neq x_0 \\ 75^\circ F & \text{if } x = x_0 \end{cases}$$

Thus the point x_0 stands out as singular point ("singular" means "special" or "unusual"). Analyzing the range of temperature readings, we should say that the approach of x to x_0 had no bearing on the approach of the corresponding $t(x)$ values to $t(x_0)$. Briefly, a jump occurs at x_0. Thus we would be led to say that the temperature function "lacks continuity at x_0". For, we would have expected $t(x_0)$ to be 250°F since the x values neighbouring x_0 showed $t(x)$ = 250°F. We now abstract the notion of continuity, and demand that images be "close" when pre-images are "close". In our example the points on the hot wire were the pre-images, while the temperature readings there were the corresponding images.

The students should reflect on the intuitive idea of continuity by considering instead the contrasting idea of lack of continuity, or more simply "discontinuity" as manifested in every day experiences of abrupt changes which could be headed "then suddenly!". A few that come readily to mind are listed below along with functions which correspond as mathematical models:

(1) Switching on a light : light intensity as a function of time.

(2) Collision of a vehicle : Velocity as a function of time.

(3) Switching off a radio : Sound intensity as a function of time.

(4) Busting of a balloon: Radius as a function of air input.

(5) Breaking of a string : Tension as a function of length.

(6) Cost of postage : Postage as a function of weight.

(7) Income tax : Tax-rate as a function of taxable income.

(8) Age count in years : Age in whole years as a function of time.

(9) Cost of insurance premium : Premium as a function of age.

Actually, examples (1) – (5) are not quite accurate. For example, light intensity has a transparent but continuous passage from zero luminosity to positive luminosity. Indeed, nature appears to abhor

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discontinuity. On the other hand, (6) – (9) are in fact discontinuous and really show jumps at certain points.

Students are advised to relate the mathematical definition of continuity corresponds closely with the meaning of the word continuity in everyday language. A continuous process is one that takes place gradually, without interruption or abrupt change. That is, there are no holes, jumps or gaps. Following figure identifies three values of x at which the graph of a function f is not continuous. At all other points in the interval (a, b), the graph of f is uninterrupted and continuous.

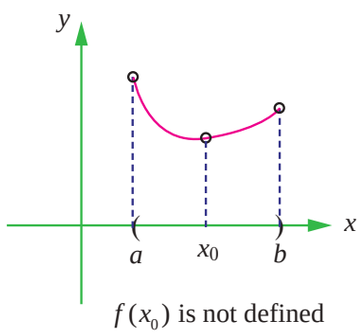

Fig. 9.33

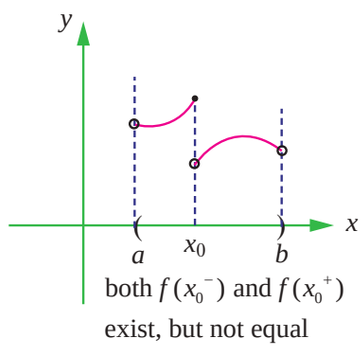

Fig. 9.34

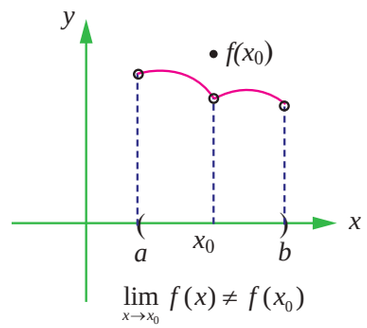

Fig. 9.35

Looking at the above graphs (9.33 to 9.35), three conditions exist for which the graph of f is not continuous at $x = x_0$.

It appears that continuity at $x = x_0$ can be destroyed by any one of the following three conditions:

(1) The function is not defined at $x = x_0$.

(2) The limit of $f(x)$ does not exist at $x = x_0$.

(3) The limit of $f(x)$ exists at $x = x_0$, but, it is not equal to $f(x_0)$.

Now let us look at the illustrative examples

Illustration 9.6

(i) $f(x) = x^2+3$ (ii) $f(x) = \dfrac{16-x^2}{4+x}$

(i) As $x \to 2$, the one sided limits are

$$\lim_{x\to 2^-} f(x) = 7$$

$$\lim_{x\to 2^+} f(x) = 7$$

and hence $\lim\limits_{x\to 2} f(x) = 7$ and moreover $f(2)$ is defined and $f(2) = 7 = \lim\limits_{x\to 2} f(x)$. In this case $f(x)$ is continuous at $x = 2$.

(ii) The one sided limits are : $\lim\limits_{x\to -4^-} f(x) = 8$

$$\lim_{x\to -4^+} f(x) = 8$$

and therefore $\lim\limits_{x\to -4} f(x) = 8$ but $f(-4)$ does not exist.

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Note that although $\lim_{x \to -4} f(x)$ exists, the function value at -4, namely $f(-4)$ is not defined. Thus the existence of $\lim_{x \to -4} f(x)$ has no bearing on the existence of $f(-4)$.

Now we formally define continuity as in

Definition 9.7

Let I be an open interval in $\mathbb{R}$ containing x_0. Let $f : I \to \mathbb{R}$. Then f is said to be continuous at x_0 if it is defined in a neighbourhood of this point and if the limit of this function, as the independent variable x tends to x_0, exists and is equal to the value of the function at $x = x_0$.

Thus three requirements have to be satisfied for the continuity of a function $y = f(x)$ at $x = x_0$:

(i) $f(x)$ must be defined in a neighbourhood of x_0 (i.e., $f(x_0)$ exists);

(ii) $\lim_{x \to x_0} f(x)$ exists ;

(iii) $f(x_0) = \lim_{x \to x_0} f(x)$.

The condition (iii) can be reformulated as $\lim_{\Delta x \to 0} \left[f(x_0 + \Delta x) - f(x_0) \right] = 0$ and the continuity of f at x_0 can be restated as follows :

Definition 9.8

A function $y = f(x)$ is said to be continuous at a point x_0 (or at $x = x_0$) if it is defined in some neighbourhood of x_0 and if $\lim_{\Delta x \to 0} \left[f(x_0 + \Delta x) - f(x_0) \right] = 0$.

The condition (iii) can also be put in the form $\lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0} x \right)$. Thus, if the symbol of the limit and the symbol of the function can be interchanged, the function is continuous at the limiting value of the argument.

9.3.1 Examples of functions Continuous at a point

(1) Constant function is continuous at each point of $\mathbb{R}$.

Let $f(x) = k, k \in \mathbb{R}$ is constant. If $x_0 \in \mathbb{R}$, then $f(x_0) = k$.

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (k) = k.$$

(2) Power functions with positive integer exponents are continuous at every point of $\mathbb{R}$

If $f(x) = x^n$, domain of f is $\mathbb{R} = (-\infty, \infty)$ and $\lim_{x \to x_0} x^n = x_o^n, x_0 \in \mathbb{R}$ by the limit theorem.

(3) Polynomial functions, $p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n, \quad a_0 \neq 0$

are continuous at every point of $\mathbb{R}$. By limit theorem,

$$\lim_{x \to x_0} p(x) = a_0 x_0^n + a_1 x_0^{n-1} + \cdots + a_{n-1} x_0 + a_n = p(x_0).$$

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(4) Quotients of polynomials namely rational functions of the form

$R(x) = \dfrac{p(x)}{q(x)}$, are continuous at every point where $q(x) \neq 0$, and

$$\lim_{x \to x_0} R(x) = \lim_{x \to x_0} \frac{p(x)}{q(x)}$$

$$= \frac{\lim\limits_{x \to x_0} p(x)}{\lim\limits_{x \to x_0} q(x)} = \frac{p(x_0)}{q(x_0)} = R(x_0).$$

(5) The circular functions $\sin x$ and $\cos x$ are continuous at every point of their domain $\mathbb{R} = (-\infty, \infty)$ since $\lim\limits_{x \to x_0} \sin x = \sin x_0$, $\lim\limits_{x \to x_0} \cos x = \cos x_0$.

As a consequence, $\tan x$, $\cot x$, $\operatorname{cosec} x$, $\sec x$ are continuous on their proper domains in view of the reciprocal and quotient rules in the algebra of limits.

(6) The nth root functions, $f(x) = x^{\frac{1}{n}}$ are continuous in their proper domain since $\lim\limits_{x \to x_0} \left(x^{\frac{1}{n}} \right) = x_0^{\frac{1}{n}}$.

(7) The reciprocal function $f(x) = \dfrac{1}{x}$ is not defined at 0 and hence it is not continuous at 0. It is continuous at each point of $\mathbb{R} - \{0\}$.

(8) $h(x) = \begin{cases} x+1, & x \leq 0 \\ x^2+1, & x > 0 \end{cases}$

The domain of h is all of real numbers and

$$\lim_{x \to 0^-} h(x) = \lim_{x \to 0^-} (x+1) = 1 = h(0)$$

$$\lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} (x^2+1) = 1 = h(0).$$

Thus $h(x)$ is continuous at $x = 0$.

Indeed, $h(x)$ is continuous at each point of $(-\infty, 0)$ and each point of $(0, \infty)$ and hence h is continuous in the whole of $(-\infty, \infty)$.

(9) The greatest integer function $f(x) = \lfloor x \rfloor$ is not continuous at $x = 0$.

For, $\lim\limits_{x \to 0^-} \lfloor x \rfloor = -1$ and

$\lim\limits_{x \to 0^+} \lfloor x \rfloor = 0$

It is discontinuous at each integer point. In fact,

$\lim\limits_{x \to n^-} \lfloor x \rfloor = n - 1$ and

$\lim\limits_{x \to n^+} \lfloor x \rfloor = n.$

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(10) The modulus function

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$

is continuous at all points of the real line $\mathbb{R}$.

In particular,

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0,$$

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} (x) = 0, \text{ and}$$

$$\lim_{x \to 0^-} f(x) = 0 = \lim_{x \to 0^+} f(x) = 0 = f(0).$$

(11) The exponential function $f(x) = e^x$ is continuous on $\mathbb{R}$.

(12) The logarithmic function $f(x) = \log x$ $(x > 0)$ in continuous in $(0, \infty)$

9.3.2 Algebra of continuous functions

If f and g are continuous at x_0 then

(1) $f + g$ is continuous at $x = x_0$,

(2) $f - g$ is continuous at $x = x_0$,

(3) $f \,.\, g$ is continuous at $x = x_0$, and

(4) $\dfrac{f}{g}$ is continuous at $x = x_0$ $(g(x) \neq 0)$.

(5) Composite function theorem on continuity.

If f is continuous at $g(x_0)$ and g is continuous at x_0 then fog is continuous at x_0.

Continuity in a closed interval

Definition 9.9

A function $f : [a, b] \to \mathbb{R}$ is said to be continuous on the closed interval $[a, b]$ if it is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \text{ and } \lim_{x \to b^-} f(x) = f(b).$$

That is, the function f is continuous from the right at a and continuous from the left at b, and is continuous at each point $x_0 \in (a, b)$.

Illustration 9.7

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.

The domain of definition of f is the closed interval $[-1, 1]$.

(f is defined if $1 - x^2 \geq 0$)

For any point $c \in (-1, 1)$

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$$\lim_{x\to c} f(x) = \lim_{x\to c}\sqrt{1-x^2} = \left[\lim_{x\to c}(1-x^2)\right]^{\frac{1}{2}}$$

$$= (1-c^2)^{\frac{1}{2}} = f(c).$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(1-x^2)^{\frac{1}{2}} = 0 = f(1).$$

$$\lim_{x\to -1^-} f(x) = \left[\lim_{x\to -1^-}(1-x^2)\right]^{\frac{1}{2}} = 0 = f(-1).$$

Thus f is continuous on $[-1, 1]$. One can also solve this problem using composite function theorem.

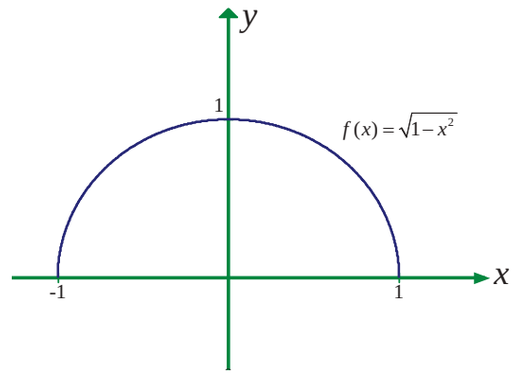

Fig. 9.36

Example 9.36

Describe the interval(s) on which each function is continuous.

(i) $f(x) = \tan x$

(ii) $g(x) = \begin{cases} \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(iii) $h(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution

(i) The tangent function $f(x) = \tan x$ is undefined at $x = (2n+1)\dfrac{\pi}{2}, n \in Z$.

At all other points it is continuous, so $f(x) = \tan x$ is continuous on each of the open intervals

$$\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

(ii) The function $y = \dfrac{1}{x}$ is continuous at all points of $\mathbb{R}$ except at $x = 0$ where it is undefined.

The function $g(x) = \sin\dfrac{1}{x}$ is continuous at all points except $x = 0$, where $\lim\limits_{x\to 0} g(x)$ does not exist. So, g is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$

(iii) The function $h(x)$ is defined at all points of the real line $\mathbb{R} = (-\infty, \infty)$; for any $x_0 \neq 0$,

$$\lim_{x\to x_0} h(x) = \left(\lim_{x\to x_0} x\sin\frac{1}{x}\right)$$

$$= x_0 \sin\frac{1}{x_0} = h(x_0)$$

For $x_0 = 0$

$$h(x) = x.\sin\frac{1}{x}$$

$$-x \leq x\sin\frac{1}{x} \leq x$$

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$$g(x) = -x,\ f(x) = x\sin\frac{1}{x}, h(x) = x$$

$$\lim_{x\to 0} g(x) = 0,\ \lim_{x\to 0} h(x) = 0$$

and have $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0.$

By Sandwich theorem

$$\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0 = h(0).$$

Therefore $h(x)$ is continuous in the entire real line.

Example 9.37

A tomato wholesaler finds that the price of a newly harvested tomatoes is ₹0.16 per kg if he purchases fewer than 100 kgs each day. However, if he purchases at least 100 kgs daily, the price drops to ₹0.14 per kg. Find the total cost function and discuss the cost when the purchase is 100 kgs.

Solution

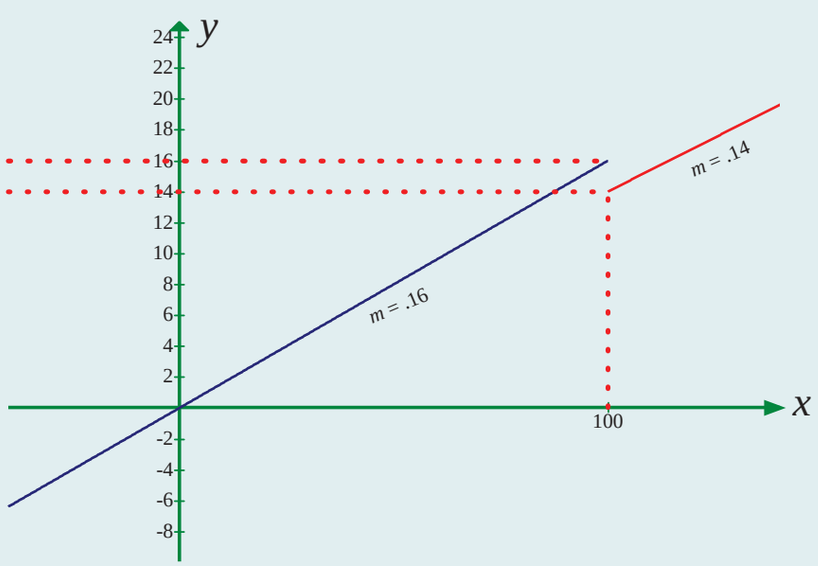

Fig.9.37

Let x denote the number of kilograms bought per day and C denote the cost. Then,

$$C(x) = \begin{cases} 0.16x, & \text{if } 0 \le x < 100 \\ 0.14x, & \text{if } x \ge 100 \end{cases}.$$

The sketch of this function is shown in Fig. 9.37.

It is discontinuous at $x = 100$ since $\lim\limits_{x\to 100^-} C(x) = 16$ and $\lim\limits_{x\to 100^+} C(x) = 14$.

Note that $C(100) = 14$. Thus, $\lim\limits_{x\to 100^-} C(x) = 16 \ne 14 = \lim\limits_{x\to 100^+} C(x) = C(100)$.

Note also that the function jumps from one finite value 14 to another finite value 16.

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9.3.3 Removable and Jump Discontinuities

Let us look at the following functions :

(i) $f(x) = \dfrac{\sin x}{x}$

(ii) $g(x) = C(x)$, where $C(x)$ is as defined in Example 9.38.

The function $f(x)$ is defined at all points of the real line except $x = 0$. That is, $f(0)$ is undefined, but $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ exists. If we redefine the function $f(x)$ as

$$h(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

h is defined at all points of the real line including $x = 0$. Moreover, h is continuous at $x = 0$ since

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1 = h(0).$$

Note that $h(x) = f(x)$ for all $x \neq 0$. Even though the original function $f(x)$ fails to be continuous at $x = 0$, the redefined function became continuous at 0. That is, we could remove the discontinuity by redefining the function. Such discontinuous points are called removable discontinuities. This example leads us to have the following.

Definition 9.10

A function f defined on an interval $I \subseteq \mathbb{R}$ is said to have removable discontinuity at $x_0 \in I$ if there is a function $h : I \to \mathbb{R}$ such that $h(x) = \begin{cases} f(x), & \text{if } x \neq x_0 \\ \lim\limits_{x\to x_0} f(x), & \text{if } x = x_0 \end{cases}$.

Note that for removable discontinuity, $\lim\limits_{x\to x_0} f(x)$ must exist.

Now if we examine the function $g(x) = C(x)$ (see Example 9.38), eventhough it is defined at all points of $[0, \infty)$, $\lim\limits_{x\to 100} g(x)$ does not exist and it has a jump of height $\lim\limits_{x\to 100^+} g(x) - \lim\limits_{x\to 100^-} g(x) = 16 - 14 = 2$, which is finite. Since $\lim\limits_{x\to 100} g(x)$ does not exist, it is not continuous at $x = 100$. Such discontinuities are called jump discontinuities. Thus we have the following :

Definition 9.11

Let f be a function defined on an interval $I \subseteq \mathbb{R}$. Then f is said to have jump discontinuity at a point $x_0 \in I$ if f is defined at x_0,

$$\lim_{x\to x_0^-} f(x) \text{ and } \lim_{x\to x_0^+} f(x) \text{ exist but}$$

$$\lim_{x\to x_0^-} f(x) \neq \lim_{x\to x_0^+} f(x).$$

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Example 9.38

Determine if f defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is continuous in $\mathbb{R}$.

Solution

By Sandwitch theorem $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$ and $f(0) = 0$ by the definition of $f(x)$. Hence it is continuous at $x = 0$. For other values it is clearly continuous and hence continuous in $\mathbb{R}$.

EXERCISE 9.5

(1) Prove that $f(x) = 2x^2 + 3x - 5$ is continuous at all points in $\mathbb{R}$.

(2) Examine the continuity of the following :

(i) $x + \sin x$ (ii) $x^2 \cos x$ (iii) $e^x \tan x$

(iv) $e^{2x} + x^2$ (v) $x.\ln x$ (vi) $\frac{\sin x}{x^2}$

(vii) $\frac{x^2 - 16}{x + 4}$ (viii) $|x + 2| + |x - 1|$ (ix) $\frac{|x-2|}{|x+1|}$

(x) $\cot x + \tan x$

(3) Find the points of discontinuity of the function f, where

(i) $f(x) = \begin{cases} 4x+5, & \text{if } x \le 3 \\ 4x-5, & \text{if } x > 3 \end{cases}$ (ii) $f(x) = \begin{cases} x+2, & \text{if } x \ge 2 \\ x^2, & \text{if } x < 2 \end{cases}$

(iii) $f(x) = \begin{cases} x^3-3, & \text{if } x \le 2 \\ x^2+1, & \text{if } x > 2 \end{cases}$ (iv) $f(x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{4} \\ \cos x, & \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$

(4) At the given point x_0 discover whether the given function is continuous or discontinuous citing the reasons for your answer :

(i) $x_0 = 1$, $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$ (ii) $x_0 = 3$, $f(x) = \begin{cases} \frac{x^2-9}{x-3}, & \text{if } x \neq 3 \\ 5, & \text{if } x = 3 \end{cases}$

(5) Show that the function $\begin{cases} \frac{x^3-1}{x-1}, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1 \end{cases}$ is continuous on $(-\infty, \infty)$

(6) For what value of α is this function $f(x) = \begin{cases} \frac{x^4-1}{x-1}, & \text{if } x \neq 1 \\ \alpha, & \text{if } x = 1 \end{cases}$ continuous at $x = 1$?

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(7) Let $f(x) = \begin{cases} 0, & \text{if } x < 0 \\ x^2, & \text{if } 0 \le x < 2 \\ 4, & \text{if } x \ge 2 \end{cases}$ .Graph the function. Show that $f(x)$ continuous on $(-\infty, \infty)$.

(8) If f and g are continuous functions with $f(3) = 5$ and $\lim_{x \to 3} \left[2f(x) - g(x)\right] = 4$, find $g(3)$.

(9) Find the points at which f is discontinuous. At which of these points f is continuous from the right, from the left, or neither? Sketch the graph of f.

(i) $f(x) = \begin{cases} 2x+1, & \text{if } x \le -1 \\ 3x & \text{if } -1 < x < 1 \\ 2x-1, & \text{if } x \ge 1 \end{cases}$ (ii) $f(x) = \begin{cases} (x-1)^3, & \text{if } x < 0 \\ (x+1)^3, & \text{if } x \ge 0 \end{cases}$

(10) A function f is defined as follows :

$$f(x) = \begin{cases} 0 & \text{for } x < 0; \\ x & \text{for } 0 \le x < 1; \\ -x^2 + 4x - 2 & \text{for } 1 \le x < 3; \\ 4 - x & \text{for } x \ge 3 \end{cases}$$

Is the function continuous?

(11) Which of the following functions f has a removable discontinuity at $x = x_0$? If the discontinuity is removable, find a function g that agrees with f for $x \ne x_0$ and is continuous on $\mathbb{R}$.

(i) $f(x) = \dfrac{x^2 - 2x - 8}{x + 2}$, $x_0 = -2$.

(ii) $f(x) = \dfrac{x^3 + 64}{x + 4}$, $x_0 = -4$.

(iii) $f(x) = \dfrac{3 - \sqrt{x}}{9 - x}$, $x_0 = 9$.

(12) Find the constant b that makes g continuous on $(-\infty, \infty)$.

$$g(x) = \begin{cases} x^2 - b^2 & \text{if } x < 4 \\ bx + 20 & \text{if } x \ge 4 \end{cases}$$

(13) Consider the function $f(x) = x \sin \dfrac{\pi}{x}$. What value must we give $f(0)$ in order to make the function continuous everywhere?

(14) The function $f(x) = \dfrac{x^2 - 1}{x^3 - 1}$ is not defined at $x = 1$. What value must we give $f(1)$ inorder to make $f(x)$ continuous at $x = 1$?

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(15) State how continuity is destroyed at $x = x_0$ for each of the following graphs.

(a)

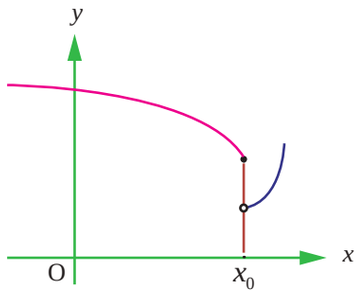

Fig. 9.38

(b)

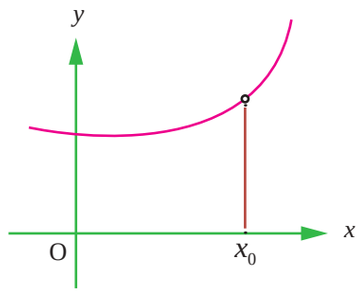

Fig. 9.39

(c)

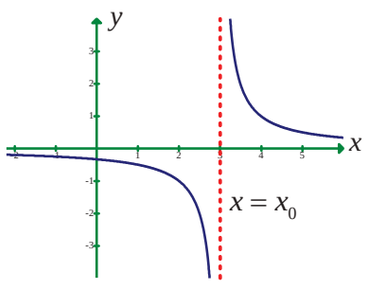

Fig. 9.40

(d)

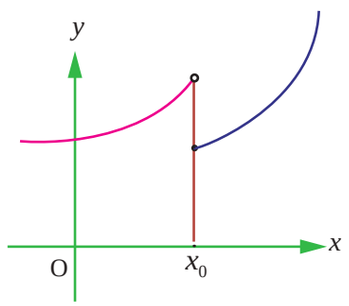

Fig. 9.41

EXERCISE 9.6

Choose the correct or the most suitable answer from the given four alternatives.

(1) $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$

(1) 1 (2) 0 (3) ∞ (4) $-\infty$

(2) $\lim\limits_{x\to\pi/2} \dfrac{2x-\pi}{\cos x}$

(1) 2 (2) 1 (3) -2 (4) 0

(3) $\lim\limits_{x\to 0} \dfrac{\sqrt{1-\cos 2x}}{x}$

(1) 0 (2) 1 (3) $\sqrt{2}$ (4) does not exist

(4) $\lim\limits_{\theta\to 0} \dfrac{\sin\sqrt{\theta}}{\sqrt{\sin\theta}}$

(1) 1 (2) -1 (3) 0 (4) 2

(5) $\lim\limits_{x\to\infty} \left(\dfrac{x^2+5x+3}{x^2+x+3}\right)^x$ is

(1) e^4 (2) e^2 (3) e^3 (4) 1

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(6) $\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{2x+1}=$

(1) 1 (2) 0 (3) – 1 (4) $\frac{1}{2}$

(7) $\lim_{x\to 0}\frac{a^x-b^x}{x}=$

(1) $\log ab$ (2) $\log\left(\frac{a}{b}\right)$ (3) $\log\left(\frac{b}{a}\right)$ (4) $\frac{a}{b}$

(8) $\lim_{x\to 0}\frac{8^x-4^x-2^x+1^x}{x^2}=$

(1) 2 log 2 (2) $2(\log 2)^2$ (3) log 2 (4) 3 log 2

(9) If $f(x)=x(-1)^{\left\lfloor\frac{1}{x}\right\rfloor}$, $x\le 0$, then the value of $\lim_{x\to 0} f(x)$ is equal to

(1) – 1 (2) 0 (3) 2 (4) 4

(10) $\lim_{x\to 3}\lfloor x\rfloor=$

(1) 2 (2) 3 (3) does not exist (4) 0

(11) Let the function f be defined by $f(x)=\begin{cases}3x & 0\le x\le 1\\ -3x+5 & 1<x\le 2\end{cases}$, then

(1) $\lim_{x\to 1} f(x)=1$ (2) $\lim_{x\to 1} f(x)=3$

(3) $\lim_{x\to 1} f(x)=2$ (4) $\lim_{x\to 1} f(x)$ does not exist

(12) If $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x)=\lfloor x-3\rfloor+|x-4|$ for $x\in\mathbb{R}$, then $\lim_{x\to 3^-} f(x)$ is equal to

(1) – 2 (2) – 1 (3) 0 (4) 1

(13) $\lim_{x\to 0}\frac{xe^x-\sin x}{x}$ is

(1) 1 (2) 2 (3) 3 (4) 0

(14) If $\lim_{x\to 0}\frac{\sin px}{\tan 3x}=4$, then the value of p is

(1) 6 (2) 9 (3) 12 (4) 4

(15) $\lim_{\alpha\to\pi/4}\frac{\sin\alpha-\cos\alpha}{\alpha-\frac{\pi}{4}}$ is

(1) $\sqrt{2}$ (2) $\frac{1}{\sqrt{2}}$ (3) 1 (4) 2

(16) $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\ldots+\frac{n}{n^2}\right)$ is

(1) $\frac{1}{2}$ (2) 0 (3) 1 (4) ∞

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(17) $\lim_{x\to 0}\frac{e^{\sin x}-1}{x}=$

(1) 1 (2) e (3) $\frac{1}{e}$ (4) 0

(18) $\lim_{x\to 0}\frac{e^{\tan x}-e^{x}}{\tan x-x}=$

(1) 1 (2) e (3) $\frac{1}{2}$ (4) 0

(19) The value of $\lim_{x\to 0}\frac{\sin x}{\sqrt{x^2}}$ is

(1) 1 (2) – 1 (3) 0 (4) limit does not exist

(20) The value of $\lim_{x\to k^-} x-\lfloor x\rfloor$, where k is an integer is

(1) – 1 (2) 1 (3) 0 (4) 2

(21) At $x=\frac{3}{2}$ the function $f(x)=\frac{|2x-3|}{2x-3}$ is

(1) continuous (2) discontinuous (3) differentiable (4) non-zero

(22) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=\begin{cases} x & x \text{ is irrational} \\ 1-x & x \text{ is rational}\end{cases}$ then f is

(1) discontinuous at $x=\frac{1}{2}$ (2) continuous at $x=\frac{1}{2}$

(3) continuous everywhere (4) discontinuous everywhere

(23) The function $f(x)=\begin{cases} \frac{x^2-1}{x^3+1} & x\neq -1 \\ P & x=-1\end{cases}$ is not defined for $x=-1$. The value of $f(-1)$ so that the function extended by this value is continuous is

(1) $\frac{2}{3}$ (2) $-\frac{2}{3}$ (3) 1 (4) 0

(24) Let f be a continuous function on [2, 5]. If f takes only rational values for all x and $f(3)=12$, then $f(4.5)$ is equal to

(1) $\frac{f(3)+f(4.5)}{7.5}$ (2) 12 (3) 17.5 (4) $\frac{f(4.5)-f(3)}{1.5}$

(25) Let a function f be defined by $f(x)=\frac{x-|x|}{x}$ for $x\neq 0$ and $f(0)=2$. Then f is

(1) continuous nowhere (2) continuous everywhere

(3) continuous for all x except x = 1 (4) continuous for all x except x = 0

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SUMMARY

In this chapter we have acquired the knowledge of

- Limit of a function $y = f(x)$ as x approaches x_0 from the lower values of x_0.
- Limit of $y = f(x)$ as x approaches x_0 from the higher values of x_0.
- Limit of a function as x approaches x_0, in the deleted neighbourhood of x_0 exists if and only if $\lim\limits_{x \to x_0^-} f(x) = L = \lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0} f(x) = L.$
- $\lim\limits_{x \to x_0} f(x) = L$ also means that $f(x)$ converges to L as x approaches x_0 from either side of x_0 except at $x = x_0$.
- If $\lim\limits_{x \to x_0} f(x)$ and $\lim\limits_{x \to x_0} g(x)$ exist then

  (i) $\lim\limits_{x \to x_0} \left[f(x) \pm g(x) \right] = \lim\limits_{x \to x_0} f(x) \pm \lim\limits_{x \to x_0} g(x)$

  (ii) $\lim\limits_{x \to x_0} \left[f(x).g(x) \right] = \lim\limits_{x \to x_0} f(x). \lim\limits_{x \to x_0} g(x)$

  (iii) $\lim\limits_{x \to x_0} \left[\dfrac{f(x)}{g(x)} \right] = \dfrac{\lim\limits_{x \to x_0} f(x)}{\lim\limits_{x \to x_0} g(x)}$ if $g(x) \neq 0$ and $\lim\limits_{x \to x_0} g(x) \neq 0$.
- Limit of $f(x)$ as x approaches x_0 does not exist if either $f(x) \to \pm\infty$ as $x \to x_0^-$ or $f(x) \to \pm\infty$ as $x \to x_0^+$ or $\lim\limits_{x \to x_0^-} f(x) = l_1 \neq l_2 = \lim\limits_{x \to x_0^+} f(x)$
- (M, ∞) is the neighborhood of $+\infty$, $M > 0$

  $(-\infty, K)$ is the neighborhood of $-\infty$, $K < 0$.
- If $f(x) \to \pm\infty$ as $x \to x_0$ then $x = x_0$ is a vertical asymptote.
- The line $y = l_1$ (or l_2) is a horizontal asymptote of the curve $y = f(x)$ if either $f(x) \to l_1$ as $x \to \infty$ or $f(x) \to l_2$ as $x \to -\infty$.
- $f(x)$ is continuous at x_0 if and only if

  (i) $\lim\limits_{x \to x_0} f(x) = f(x_0)$

  (ii) $\lim\limits_{\Delta x \to 0} [f(x_0 + \Delta x) - f(x_0)] = 0$

  (iii) $\lim\limits_{x \to x_0} f(x) = f\left(\lim\limits_{x \to x_0} x\right)$.
- Jump and removable discontinuities.

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ICT CORNER 9(a)

Limits and continuity

Expected Outcome

Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code.

GeoGebra Workbook called "XI Standard Limits" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Limits basic". A continuous function is given. You can select the limit at a desired point by moving the slider "a". Then move the lines $x=h$(nearest to a point) both left and right side to check $f(h)$ by moving the slider "h"

Compare this with the definition given in book.

Step1

Step2

Browse in the link:

Matrices and Determinants: https://ggbm.at/cpknpvvh

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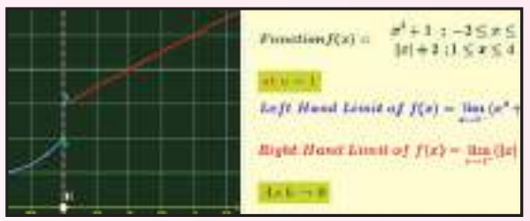
ICT CORNER 9(b)

Limits and continuity

Expected Outcome



Step 1

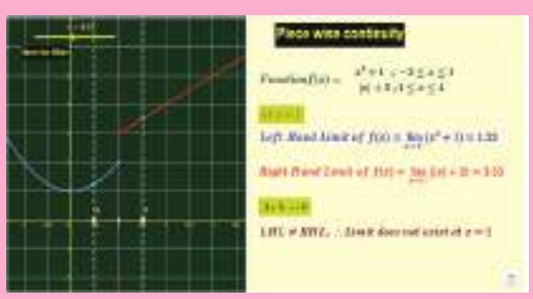
Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI Standard Limits" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Piece-wise limit". Piece-wise function is given. Move the lines $x=h$(nearest to $x = 1$) both left and right side to check $f(h)$ by moving the slider "h"

Compare this with the definition given in book.

Step1



Step2

Browse in the link:

Matrices and Determinants: https://ggbm.at/cpknpvvh

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Chapter 10 Differential Calculus- Differentiability and Methods of Differentiation

BCASNL

"Take what you need,
do what you should, you will get what you want"

- Leibnitz

10.1 Introduction

In this chapter we discuss the concept of derivative and related concepts and develop tools necessary for solving real life problems. In this connection, let us look at the following problem of finding average velocity.

Almost everyone has an intuitive notion of speed or velocity as a rate at which a distance is covered in a certain length of time. When, say, a bus travels 60 km in one hour, the average velocity of the bus must have been 60 km/h. Of course, it is difficult to maintain this rate of 60 km/h for the entire trip because the bus slows down for towns and speeds up when it passes cars. In other words, the velocity changes with time. If a bus company's schedule demands that the bus travel 60 km from one town to another in one hour, the driver knows instinctively that he must compensate for velocities or speeds greater than this at other points in the journey. Knowing that the average velocity is 60 km/h does not, however, answer the question: What is the velocity of the bus at a particular instant?

In general, this average velocity or average speed of a moving object is the time rate of change of position defined by

$$v_{ave} = \frac{\text{distance travelled}}{\text{time of travel}}$$

Consider a runner who finishes a 10 km race in an elapsed time of 1 h 15 min (1.25 h). The runner's average velocity or average speed for this race is

$$v_{ave} = \frac{10}{1.25} = 8.$$

Usain Bolt's average speed

$$\frac{\Delta y}{\Delta x} = \frac{100m}{9.58s} = 10.4$$

How fast is Usain Bolt right now? → Calculus

But suppose we now wish to determine the runner's exact velocity v at the instant the runner is one-half into the race. If the distance run in the time interval from 0 h to 0.5 h is measured to be 5 km, then $v_{ave} = \frac{5}{0.5} = 10$.

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Again this number is not a measure or necessarily such a good indicator, of the instantaneous rate v at which the runner is moving 0.5 h into the race. If we determine that rate at 0.6 h the runner is 5.7 km from the starting line, then the average velocity from 0 h to 0.6 h is

$$v_{ave} = \frac{5.7-5}{0.6-0.5} = 7km/h$$

Gottfried Wilhelm Leibnitz
(1646 - 1716)

The latter number is a more realistic measure of the rate v. By "shrinking" the time interval between 0.5 h and the time that corresponds to a measured position close to 5 km, we expect to obtain even better approximations to the runner's velocity at time 0.5 h.

This problem of finding velocities leads us to deal with the general problem of finding the derivative of a general mathematical model represented by the analytic equation, $y = f(x)$ Consequently, we will move towards in achieving the following objectives and subsequently deal with the analysis of derivatives.

Learning Objectives

On completion of this chapter, the students are expected to

- acquire the concept of a derivative as limit of quotients.
- visualise the concept of derivative geometrically.
- understand derivative as a process of measuring changes.
- realise derivative as a tool to measure slopes of tangents to curves / rates of changes.
- understand different methods of differentiation.
- apply calculus as a tool to solve everyday real life problems.

10.2 The concept of derivative

Calculus grew out of four major problems that mathematicians were working on during the seventeenth century.

(1) The tangent line problem

(2) The velocity and acceleration problem

(3) The minimum and maximum problem

(4) The area problem

We take up the above problems 1 and 2 for discussion in this chapter while the last two problems are dealt with in the later chapters.

10.2.1 The tangent line problem

What does it mean to say that a line is tangent to a curve at a point? For a circle, the tangent line at a point P is the line that is perpendicular to the radial line at a point P, as shown in fig. 10.1.

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For a general curve, however, the problem is more difficult, for example, how would you define the tangent lines shown in the following figures 10.2 to 10.4.

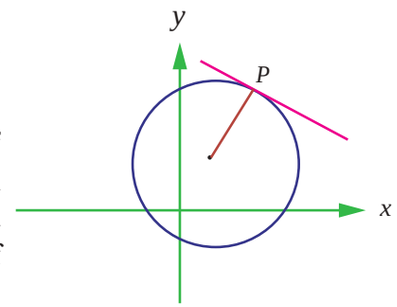

Fig. 10.1

You might say that a line is tangent to a curve at a point P if it touches, but does not cross, the curve at point P. This definition would work for the first curve (Fig. 10.2), but not for the second (Fig. 10.3). Or you might say that a line is tangent to a curve if the line touches or intersects the curve exactly at one point. This definition would work for a circle but not for more general curves, as the third curve shows (Fig. 10.4).

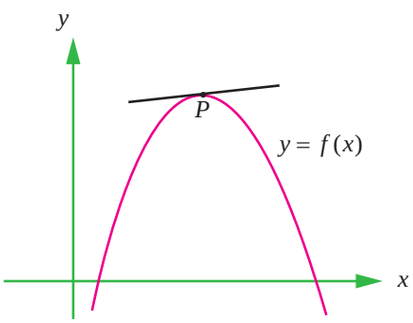

Fig. 10.2

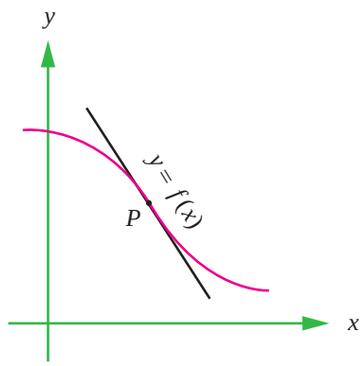

Fig. 10.3

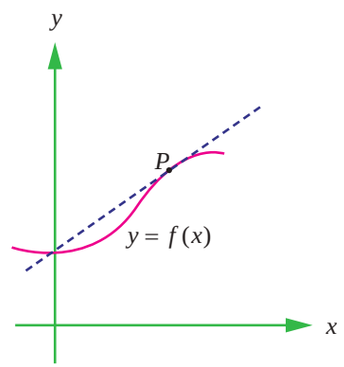

Fig. 10.4

Essentially, the problem of finding the tangent line at a point P boils down to the problem of finding the slope of the tangent line at point P. You can approximate this slope using a secant line through the point of tangency and a second point on the curve as in the following Fig. 10.5.

Let $P(x_0, f(x_0))$ be the point of tangency and $Q(x_0 + \Delta x, f(x_0 + \Delta x))$ be the second point.

The slope of the secant line through the two points is given by substitution into the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{sec} = \frac{f(x_0 + \Delta x) - f(x_0)}{(x_0 + \Delta x) - x_0} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}.$$

That is, $m_{\sec} = \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$, which is the slope of the secant line.

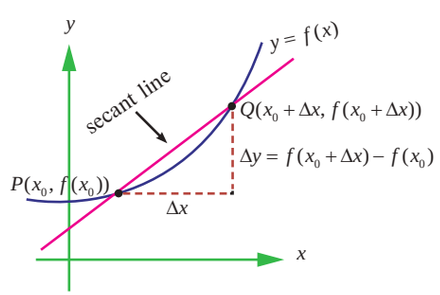

Fig. 10.5

The right hand side of this equation is a **difference quotient.** The denominator Δx is the **change in x** (increment in x), and the numerator $\Delta y = f(x_0 + \Delta x) - f(x_0)$ is the **change in y**.

The beauty of this procedure is that you can obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency.

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Tangent line approximation

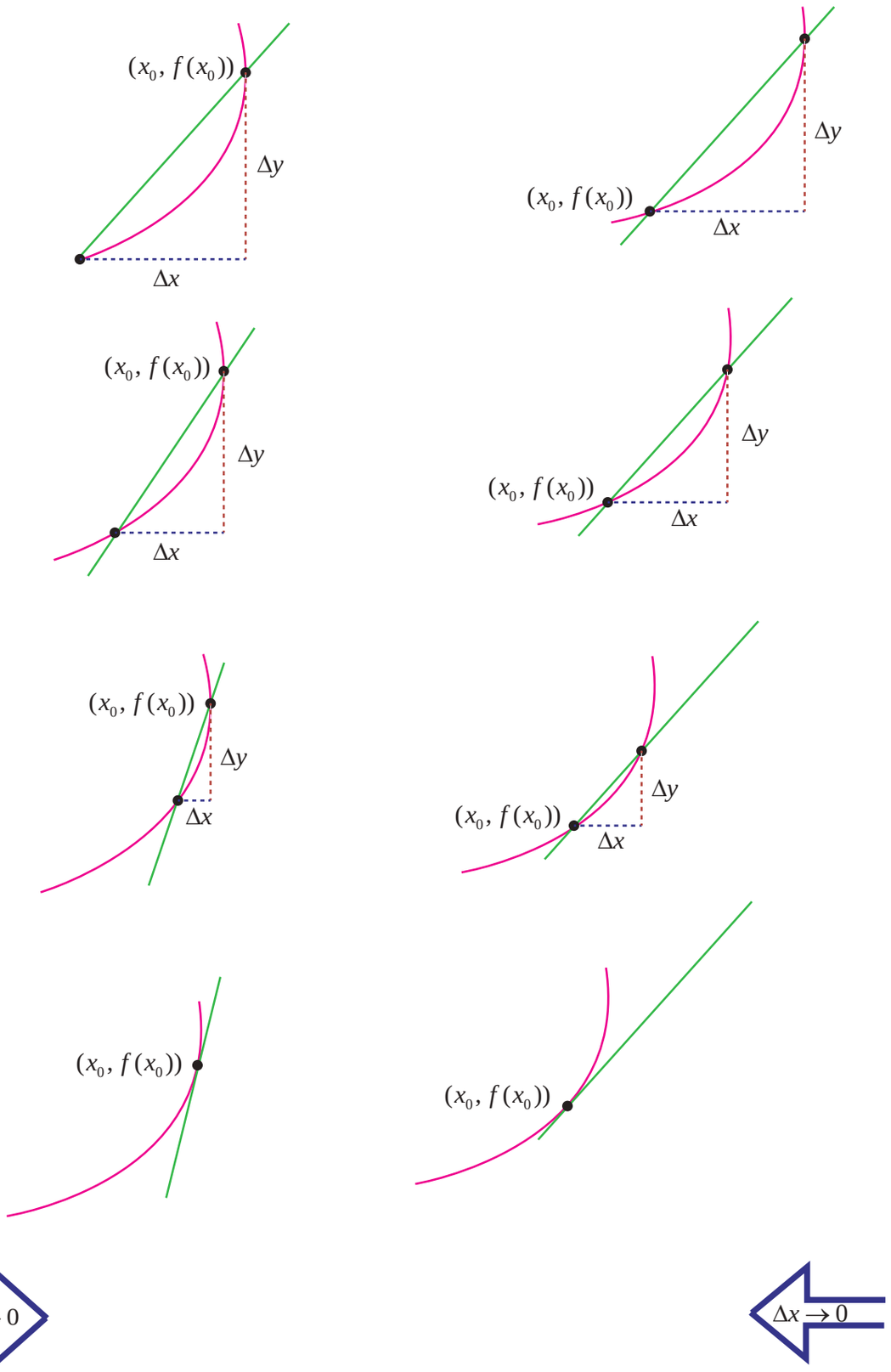

Fig. 10.6 to 10.13

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Illustration 10.1

Let us make an attempt to find the slope of the tangent line to the graph of $f(x) = x^2$ at (1, 1).

As a start, let us take $\Delta x = 0.1$ and find the slope of the secant line through (1,1) and $(1.1, (1.1)^2)$.

(i) $f(1.1) = (1.1)^2 = 1.21$

(ii) $\Delta y = f(1.1) - f(1)$

$= 1.21 - 1 = 0.21$

(iii) $\dfrac{\Delta y}{\Delta x} = \dfrac{0.21}{0.1} = 2.1$

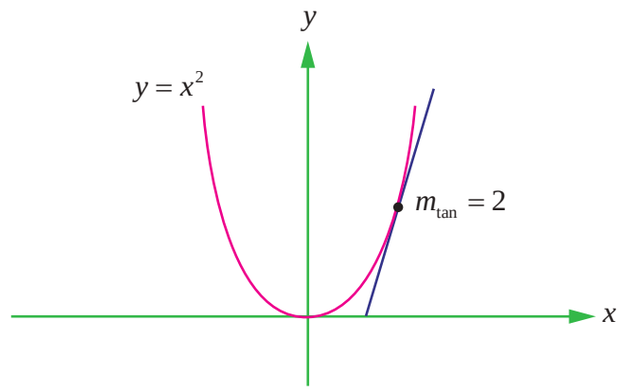

Fig. 10.14

Tabulate the successive values to the right and left of 1 as follows :

Δx	$1 + \Delta x$	$f(1)$	$f(1 + \Delta x)$	Δy	$\Delta y / \Delta x$
0.1	1.1	1	1.21	0.21	2.1
0.01	1.01	1	1.0201	1.0201	2.01
0.001	1.001	1	1.002001	0.002001	2.001
– 0.1	0.9	1	0.81	– 0.19	1.9
– 0.01	0.99	1	0.9801	– 0.0199	1.99
– 0.001	0.999	1	0.998001	– 0.001999	1.999

Clearly, $\lim\limits_{\Delta x \to 0^-} \dfrac{\Delta y}{\Delta x} = 2$; $\lim\limits_{\Delta x \to 0^+} \dfrac{\Delta y}{\Delta x} = 2$.

This shows that $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = 2$.

Thus the slope of the tangent line to the graph of $y = x^2$ at (1, 1) is $m_{tan} = 2$.

On the basis of the Fig.10.6 to 10.13, Illustration 10.1, and our intuition, we are prompted to say that if a graph of a function $y = f(x)$ has a tangent line L at a point P, then L must be the line that is the **limit** of the secants PQ through P and Q as $Q \to P$ ($\Delta x \to 0$). Moreover, the slope m_{tan} of L should be the **limiting** value of the values m_{sec} as $\Delta x \to 0$. This is summarised as follows:

Definition 10.1 (Tangent line with slope m)

Let f be defined on an open interval containing x_0, and if the limit

$\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = m_{tan}$ exists, then the line passing through $(x_0, f(x_0))$ with slope m is the **tangent line** to the graph of f at the point $(x_0, f(x_0))$.

The slope of the tangent line at $(x_0, f(x_0))$ is also called the **slope of the curve** at that point.

The definition implies that if a graph admits tangent line at a point $(x_0, f(x_0))$ then it is unique since a point and a slope determine a single line.

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The conditions of the definition can be formulated in 4 steps :

(i) Evaluate f at x_0 and $x_0 + \Delta x$: $f(x_0)$ and $f(x_0 + \Delta x)$

(ii) Compute Δy : $\Delta y = f(x_0 + \Delta x) - f(x_0)$

(iii) Divide Δy by $\Delta x\ (\neq 0)$: $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$

(iv) Compute the limit as $\Delta x \to 0 (\neq 0)$: $m_{\tan} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.

The computation of the slope of the graph in the Illustration 10.1 can be facilitated using the definitions.

(i) $f(1) = 1^2 = 1$. For any $\Delta x \neq 0$

$$f(1 + \Delta x) = (1 + \Delta x)^2 = 1 + 2\Delta x + (\Delta x)^2$$

(ii) $$\Delta y = f(1 + \Delta x) - f(1) = 2\Delta x + (\Delta x)^2 = \Delta x\,(2 + \Delta x)$$

(iii) $$\frac{\Delta y}{\Delta x} = \frac{\Delta x(2 + \Delta x)}{\Delta x} = 2 + \Delta x\,.$$

(iv) $$m_{\tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2 + \Delta x) = 2 + 0 = 2\,.$$

Example 10.1

Find the slope of the tangent line to the graph of $f(x) = 7x + 5$ at any point $(x_0, f(x_0))$.

Solution

Step (i) $f(x_0) = 7x_0 + 5$.

For any $\Delta x \neq 0$,

$$f(x_0 + \Delta x) = 7(x_0 + \Delta x) + 5 = 7x_0 + 7\Delta x + 5$$

Step (ii) $$\Delta y = f(x_0 + \Delta x) - f(x_0) = (7x_0 + 7\Delta x + 5) - (7x_0 + 5) = 7\Delta x$$

Step (iii) $$\frac{\Delta y}{\Delta x} = 7$$

Thus, at any point on the graph of $f(x) = 7x + 5$, we have

Step (iv) $$m_{\tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (7) = 7.$$

Note that for a linear graph, $\dfrac{\Delta y}{\Delta x}$ is a constant, depends neither on x_0 nor on the increment Δx.

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Example 10.2

Find the slope of tangent line to the graph of $f(x) = -5x^2 + 7x$ at $(5, f(5))$.

Solution

Step (i) $f(5) = -5(5)^2 + 7 \times 5 = -125 + 35 = -90.$

For any $\Delta x \neq 0$,

$$f(5 + \Delta x) = -5(5 + \Delta x)^2 + 7(5 + \Delta x) = -90 - 43\Delta x - 5(\Delta x)^2.$$

Step (ii)
$$\begin{aligned}\Delta y &= f(5 + \Delta x) - f(5)\\ &= -90 - 43\Delta x - 5(\Delta x)^2 + 90 = -43\Delta x - 5(\Delta x)^2\\ &= \Delta x\,[-43 - 5\Delta x].\end{aligned}$$

Step (iii)
$$\frac{\Delta y}{\Delta x} = -43 - 5\Delta x$$

Step (iv)
$$m_{\tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -43.$$

10.2.2 Velocity of Rectilinear motion

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the **position function** of the object. In the time interval from $t = t_0$ to $t = t_0 + \Delta t$, the change in position is $f(t_0 + \Delta t) - f(t_0)$. The average velocity over this time interval is

$$v_{avg} = \frac{\text{displacement}}{\text{time}} = \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t} = \frac{\text{Change in } s}{\text{Change in } t} = \frac{\Delta s}{\Delta t}$$ which is same as the slope of the secant line PQ in fig. 10.16.

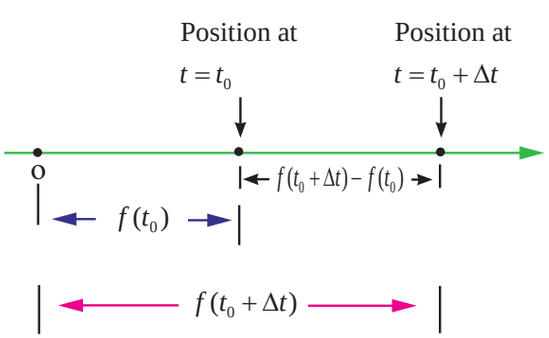

Fig. 10.15

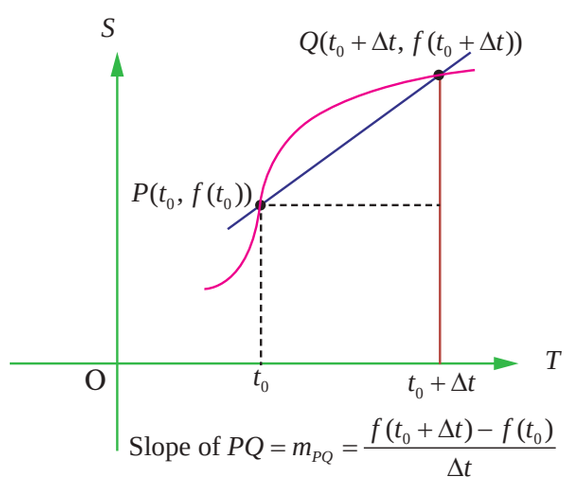

Fig. 10.16

In this time interval Δt (from t_0 to $t_0 + \Delta t$) the motion may be of entirely different types for the same distance covered (traversed). This is illustrated graphically by the fact that we can draw entirely different curves C_1, C_2, C_3 ... between the points P and Q in the plane. These curves are the graphs of quite different motions in the given time intervals, all the motions having the same average velocity $\dfrac{\Delta s}{\Delta t}$.

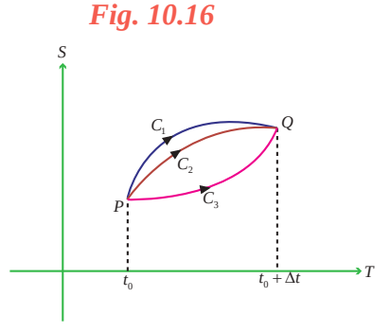

Fig. 10.17

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Now suppose we compute the average velocities over shorter and shorter time intervals $[t_0, t_0 + \Delta t]$. In other words, we let Δt approach 0. Then we define the velocity (or instantaneous velocity) $v(t_0)$ at time $t = t_0$ as the limit of these average velocities.

$$v(t_0) = \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

This means that the velocity at time $t = t_0$ is equal to the slope of the tangent line at P.

Illustration 10.2

The distance s travelled by a body falling freely in a vacuum and the time t of descent are variables. They depend on each other. This dependence is expressed by the law of the free fall :

$$s = \frac{1}{2}gt^2 \text{ (absence of initial velocity), } g \text{ is the gravitational constant.}$$

Step (i) $f(t_0 + \Delta t) = \frac{1}{2}g(t_0 + \Delta t)^2 = \frac{1}{2}g\left(t_0^2 + 2t_0\Delta t + (\Delta t)^2\right)$

Step (ii) $\Delta s = f(t_0 + \Delta t) - f(t_0)$

$$= \frac{1}{2}g\left[(t_0^2 + 2t_0\Delta t + (\Delta t)^2\right] - \frac{1}{2}gt_0^2$$

$$= g\,\Delta t\left[t_0 + \frac{1}{2}\Delta t\right]$$

BCJNQ9

Step (iii) $\frac{\Delta s}{\Delta t} = \frac{g\,\Delta t\left[t_0 + \frac{1}{2}\Delta t\right]}{\Delta t} = g\left[t_0 + \frac{1}{2}\Delta t\right]$

Step (iv) $v(t_0) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = gt_0$.

It is clear from this that the velocity is completely defined by the instant t_0. It is proportional to the time of motion (of the fall).

10.2.3 The derivative of a Function

We have now arrived at a crucial point in the study of calculus. The limit used to define the slope of a tangent line or the instantaneous velocity of a freely falling body is also used to define one of the two fundamental operations of calculus – **differentiation**.

Definition 10.2

Let f be defined on an open interval $I \subseteq \mathbb{R}$ containing the point x_0, and suppose that $\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists. Then f is said to be **differentiable** at x_0 and the **derivative** of f at x_0, denoted by $f'(x_0)$, is given by

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

For all x for which this limit exists,

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ is a function of } x.$$

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Be sure you see that the derivative of a function of x is also a function of x. This "new" function gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided the graph has a tangent line at this point.

The process of finding the derivative of a function is called **differentiation**. A function is **differentiable** at x if its derivative exists at x and is **differentiable on an open interval** (a, b) if it is differentiable at every point in (a, b).

In addition to $f'(x)$, which is read as 'f prime of x' or 'f dash of x', other notations are used to denote the derivative of $y = f(x)$. The most common notations are

$$f'(x), \frac{dy}{dx}, y', \frac{d}{dx}[f(x)], D_x[y] \text{ or } Dy \text{ or } y_1. \text{ Here } \frac{d}{dx} \text{ or } D \text{ is the differential operator.}$$

The notation $\frac{dy}{dx}$ is read as "derivative of y with respect to x" or simply "$dy - dx$", or we should rather read it as "*Dee y Dee x*" or "*Dee Dee x* of *y*". But it is cautioned that we should not regard $\frac{dy}{dx}$ as the quotient $dy \div dx$ and should not refer it as "*dy by dx*". The symbol $\frac{dy}{dx}$ is known as **Leibnitz symbol.**

10.2.4 One sided derivatives (left hand and right hand derivatives)

For a function $y = f(x)$ defined in an open interval (a, b) containing the point x_0, the **left hand and right hand derivatives** of f at $x = x_0$ are respectively denoted by $f'(x_0^-)$ and $f'(x_0^+)$, are defined as

$$f'(x_0^-) = \lim_{\Delta x \to 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

and $$f'(x_0^+) = \lim_{\Delta x \to 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}, \text{ provided the limits exist.}$$

That is, the function is differentiable from the left and right. As in the case of the existence of limits of a function at x_0, it follows that $f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists if and only if both $f'(x_0^-) = \lim_{\Delta x \to 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ and $f'(x_0^+) = \lim_{\Delta x \to 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exist and $f'(x_0^-) = f'(x_0^+)$.

Therefore $f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ if and only if $f'(x_0^-) = f'(x_0^+)$.

If any one of the condition fails then f is not differentiable at x_0.

In terms of $h = \Delta x > 0$,

$$f'(x_0^+) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ and}$$

$$f'(x_0^-) = \lim_{h \to 0} \frac{f(x_0 - h) - f(x_0)}{h}.$$

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Definition 10.3

A function f is said to be **differentiable in the closed interval** $[a,b]$ if it is differentiable on the open interval (a,b) and at the end points a and b,

$$f'(a) = \lim_{\Delta x \to 0^+} \frac{f(a+\Delta x)-f(a)}{\Delta x} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}, \quad h>0$$

$$f'(b) = \lim_{\Delta x \to 0^-} \frac{f(b+\Delta x)-f(b)}{\Delta x} = \lim_{h\to 0}\frac{f(b-h)-f(b)}{-h}, \quad h>0.$$

If f is differentiable at $x = x_0$, then $f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$, where $x = x_0 + \Delta x$ and $\Delta x \to 0$ is equivalent to $x \to x_0$. This alternative form is some times more convenient to be used in computations.

As a matter of convenience, if we let $h = \Delta x$, then $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, provided the limit exists.

10.3 Differentiability and Continuity

Illustration 10.3

Test the differentiability of the function $f(x) = |x-2|$ at $x = 2$.

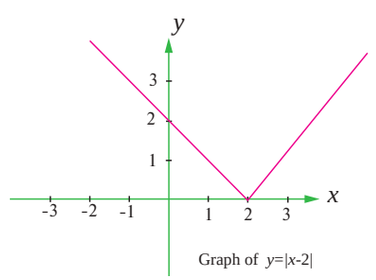

Fig. 10.18

Solution

We know that this function is continuous at $x = 2$.

But
$$f'(2^-) = \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$$
$$= \lim_{x\to 2^-}\frac{|x-2|-0}{x-2}$$
$$= \lim_{x\to 2^-}\frac{|x-2|}{x-2} = \lim_{x\to 2^-}\frac{-(x-2)}{(x-2)} = -1 \text{ and}$$
$$f'(2^+) = \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$$
$$= \lim_{x\to 2^+}\frac{|x-2|-0}{x-2} = \lim_{x\to 2^+}\frac{(x-2)}{(x-2)} = 1$$

Since the one sided derivatives $f'(2^-)$ and $f'(2^+)$ are not equal, $f'(2)$ does not exist. That is, f is not differentiable at $x = 2$. At all other points, the function is differentiable.

If $x_0 \neq 2$ is any other point then

$$f'(x_0) = \lim_{x\to x_0}\frac{|x-x_0|}{x-x_0} = \begin{cases} 1 & \text{if } x > x_0 \\ -1 & \text{if } x < x_0 \end{cases}$$

Thus $$f'(2) = \begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \end{cases}$$

The fact that $f'(2)$ does not exist is reflected geometrically in the fact that the curve $y = |x-2|$ does not have a tangent line at (2, 0). Note that the curve has a sharp edge at (2, 0).

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Illustration 10.4

Examine the differentiability of $f(x) = x^{\frac{1}{3}}$ at $x = 0$.

Solution

Let $f(x) = x^{\frac{1}{3}}$. Clearly, there is no hole (or break) in the graph of this function and hence it is continuous at all points of its domain.

Let us check whether $f'(0)$ exists.

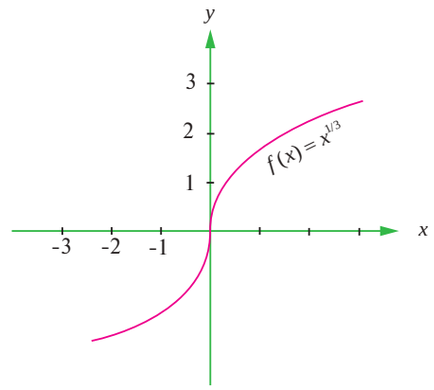

Fig. 10.19

Now $$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x^{\frac{1}{3}}-0}{x}$$

$$= \lim_{x\to 0} x^{\frac{-2}{3}} = \lim_{x\to 0}\frac{1}{x^{\frac{2}{3}}} \to \infty$$

Therefore, the function is not differentiable at $x = 0$. From the Fig. 10.19, further we conclude that the tangent line is vertical at $x = 0$. So f is not differentiable at $x = 0$.

If a function is continuous at a point, then it is not necessary that the function is differentiable at that point.

Example 10.3

Show that the greatest integer function $f(x) = \lfloor x \rfloor$ is not differentiable at any integer?

Solution

The greatest integer function $f(x) = \lfloor x \rfloor$ is not continuous at every integer point n, since $\lim_{x\to n^-}\lfloor x \rfloor = n-1$ and $\lim_{x\to n^+}\lfloor x \rfloor = n$. Thus $f'(n)$ does not exist.

What can you say about the differentiability of this function at other points?

Illustration 10.5

Let $f(x) = \begin{cases} x & x \le 0 \\ 1+x & x > 0 \end{cases}$

Compute $f'(0)$ if it exists.

Solution

Look at the graph drawn.

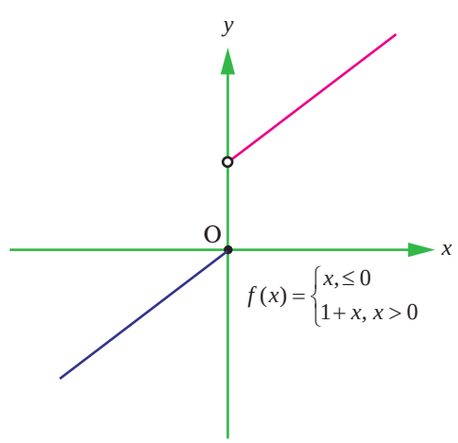

Fig. 10.20

$$f'(0^-) = \lim_{\Delta x\to 0^-}\frac{f(0+\Delta x)-f(0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0^-}\frac{f(\Delta x)}{\Delta x}$$

$$f'(0^-) = \lim_{\Delta x\to 0^-}\frac{\Delta x}{\Delta x} = 1$$

$$f'(0^+) = \lim_{\Delta x\to 0^+}\frac{1+\Delta x}{\Delta x}$$

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$$= \lim_{\Delta x \to 0^+} \left(1 + \frac{1}{\Delta x}\right) \to \infty$$

Therefore $f'(0)$ does not exist.

Here we observe that the graph of f has a jump at $x = 0$. That is $x = 0$ is a jump discontinuity.

The above illustrations and examples can be summarised to have the following conclusions.

A function f is not differentiable at a point x_0 belonging to the domain of f if one of the following situations holds:

(i) f has a vertical tangent at x_0.

(ii) The graph of f comes to a point at x_0 (either a sharp edge $\vee$ or a sharp peak $\wedge$)

(iii) f is discontinuous at x_0.

A function fails to be differentiable under the following situations :

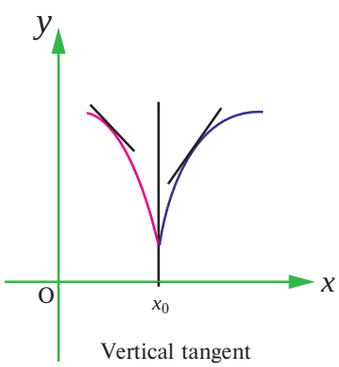

Fig. 10.21

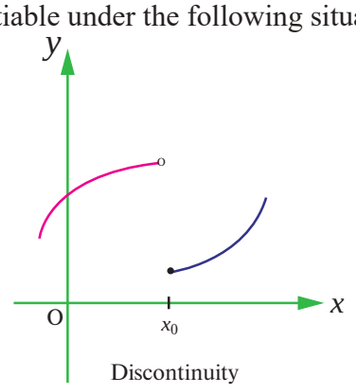

Fig. 10.22

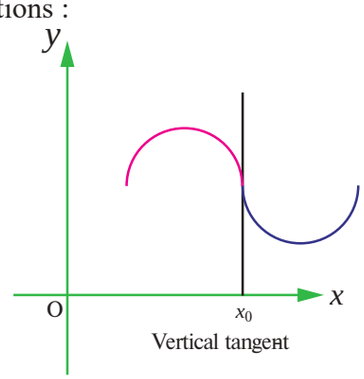

Fig. 10.23

We have seen in illustration 10.3 and 10.4, the function $f(x) = |x-2|$ and $f(x) = x^{\frac{1}{3}}$ are respectively continuous at $x = 2$ and $x = 0$ but not differentiable there, whereas in Example 10.3 and Illustration 10.5, the functions $f(x) = \lfloor x \rfloor$ and $f(x) = \begin{cases} x & x \le 0 \\ 1+x & x > 0 \end{cases}$ are respectively not continuous at any integer $x = n$ and $x = 0$ respectively and not differentiable too. The above argument can be condensed and encapsuled to state: **Discontinuity implies non-differentiability.**

Theorem 10.1 (Differentiability implies continuity)

If f is differentiable at a point $x = x_0$, then f is continuous at x_0.

Proof

Let $f(x)$ be a differentiable function on an interval (a, b) containing the point x_0. Then $f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists, in the sense that $f'(x_0)$ is a unique real number.

Now $\lim_{\Delta x \to 0} \left[f(x_0 + \Delta x) - f(x_0) \right] = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \times \Delta x$

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$$= \lim_{\Delta x \to 0}\left[\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\right]\times \lim_{\Delta x\to 0}(\Delta x)$$

$$= f'(x_0)\times 0 = 0\,.$$

This implies, f is continuous at $x = x_0$.

Derivatives from first principle

The process of finding the derivative of a function using the conditions stated in the definition of derivatives is known as derivatives from first principle.

EXERCISE 10.1

(1) Find the derivatives of the following functions using first principle.

(i) $f(x) = 6$ (ii) $f(x) = -4x + 7$ (iii) $f(x) = -x^2 + 2$

(2) Find the derivatives from the left and from the right at $x = 1$ (if they exist) of the following functions. Are the functions differentiable at $x = 1$?

(i) $f(x) = |x - 1|$ (ii) $f(x) = \sqrt{1-x^2}$ (iii) $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

(3) Determine whether the following function is differentiable at the indicated values.

(i) $f(x) = x\,|\,x\,|$ at $x = 0$ (ii) $f(x) = |\,x^2 - 1|$ at $x = 1$

(iii) $f(x) = |x| + |x - 1|$ at $x = 0, 1$ (iv) $f(x) = \sin|\,x\,|$ at $x = 0$

(4) Show that the following functions are not differentiable at the indicated value of x.

(i) $f(x) = \begin{cases} -x+2, & x \le 2 \\ 2x-4, & x > 2 \end{cases}$; $x = 2$ (ii) $f(x) = \begin{cases} 3x, & x < 0 \\ -4x, & x \ge 0 \end{cases}$, $x = 0$

(5) The graph of f is shown below. State with reasons that x values (the numbers), at which f is not differentiable.

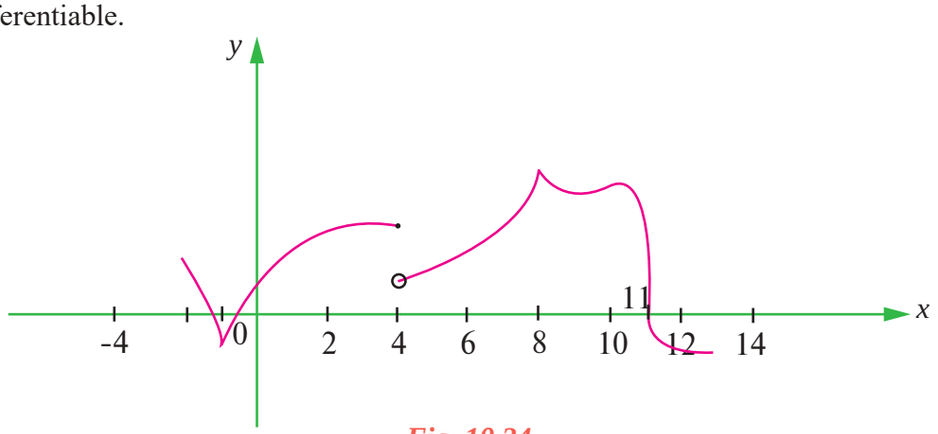

Fig. 10.24

(6) If $f(x) = |x + 100| + x^2$, test whether $f'(-100)$ exists.

(7) Examine the differentiability of functions in $\mathbb{R}$ by drawing the diagrams.

(i) $|\sin x|$ (ii) $|\cos x|$.

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10.4 Differentiation Rules

If f is a real valued function of a real variable defined on an open interval I and if $y = f(x)$ is a differentiable function of x, then $\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. In general finding such direct derivatives using first principle is extremely laborious and difficult operation in the majority of cases. But if we know, once and for all, the derivatives of all the basic elementary functions, together with rules of differentiating the algebra of functions and functions of a function, we can find the derivative of any element – any function without carrying out limit process each time. Hence the operation of differentiation can be made automatic for the class of functions that concern us.

Now we divert our attention to the rules for differentiation of a sum, product and quotient.

Theorem 10.2

The derivative of the sum of two (or more) differentiable functions is equal to the sum of their derivatives. That is, if u and v are two differentiable functions then $\boxed{\frac{d}{dx}(u+v) = \frac{d}{dx}u + \frac{d}{dx}v}$

Proof

Let u and v be two real valued functions defined and differentiable on an open interval $I \subseteq \mathbb{R}$. Let $y = u + v$, then $y = f(x)$ is a function defined on I, and by hypothesis

$$u'(x) = \frac{du}{dx} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

$$v'(x) = \frac{dv}{dx} = \lim_{\Delta x \to 0} \frac{v(x+\Delta x) - v(x)}{\Delta x} \text{ exist.}$$

Now,
$$f(x+\Delta x) = u(x+\Delta x) + v(x+\Delta x)$$

$$f(x+\Delta x) - f(x) = u(x+\Delta x) - u(x) + v(x+\Delta x) - v(x).$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{v(x+\Delta x) - v(x)}{\Delta x}.$$

This implies, $$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{(x+\Delta x) - v(x)}{\Delta x}.$$

That is, $$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = u'(x) + v'(x).$$

That is, $$f'(x) = u'(x) + v'(x).$$

or $$(u+v)'(x) = u'(x) + v'(x).$$

That is, $$\boxed{\frac{d}{dx}(u+v) = \frac{d}{dx}u + \frac{d}{dx}v}.$$

This can be extended to finite number of differentiable functions $u_1, u_2, \ldots, u_n$ and $(u_1 + u_2 + \ldots + u_n)' = u_1' + u_2' + \ldots + u_n'$.

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Theorem 10.3

Let u and v be two differentiable functions. Then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

Proof

Let u and v be the given two differentiable functions so that

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x)-u(x)}{\Delta x} = \frac{du}{dx} \text{ and } \lim_{\Delta x \to 0} \frac{v(x+\Delta x)-v(x)}{\Delta x} = \frac{dv}{dx}.$$

Let $y = f(x) = u.v$

Then $f(x + \Delta x) = u(x + \Delta x)\, v(x + \Delta x)$, and

$f(x + \Delta x) - f(x) = u(x + \Delta x)\, v(x + \Delta x) - u(x).v(x)$

$$= v(x + \Delta x)\,\left[u(x+\Delta x)-u(x)\right] + u(x)\left[v(x+\Delta x)-v(x)\right].$$

This implies, $\frac{f(x+\Delta x)-f(x)}{\Delta x} = u(x)\frac{\left[v(x+\Delta x)-v(x)\right]}{\Delta x} + v(x+\Delta x)\frac{\left[u(x+\Delta x)-u(x)\right]}{\Delta x}$.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = u(x)\lim_{\Delta x \to 0}\frac{v(x+\Delta x)-v(x)}{\Delta x} + \lim_{\Delta x \to 0} v(x+\Delta x)\lim_{\Delta x \to 0}\frac{u(x+\Delta x)-u(x)}{\Delta x}$$

$= u(x)v'(x) + v(x)u'(x)$ (since v is continuous, $\lim_{\Delta x \to 0} v(x+\Delta x) = v(x)$.

That is, $f'(x) = \boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$.

or $\boxed{(uv)' = uv' + vu'}$

Similarly $(uvw)' = uvw' + uv'w + u'vw$.

This can be extended to a finite number of differentiable functions $u_1, u_2, \ldots\ u_n$, using induction :

$$(u_1.u_2\ldots,u_n)' = u_1u_2\ldots u_{n-1}u_n' + u_1u_2\ldots u_{n-1}'u_n + \cdots + u_1'u_2\cdots u_n.$$

Theorem 10.4 (Quotient Rule)

Let u and v be two differentiable functions with $v(x) \neq 0$. Then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

Proof

Let $y = f(x) = \frac{u}{v}$, u and v are differentiable functions of x and where $v(x) \neq 0$.

Now $f(x+\Delta x) = \frac{u(x+\Delta x)}{v(x+\Delta x)}$

This implies, $f(x+\Delta x) - f(x) = \frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)}$

$$= \frac{v(x)u(x+\Delta x) - u(x)v(x+\Delta x)}{v(x+\Delta x)v(x)}.$$

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This implies, $$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\dfrac{v(x)\left[u(x+\Delta x)-u(x)\right]}{\Delta x} - \dfrac{u(x)\left[v(x+\Delta x-v(x)\right]}{\Delta x}}{v(x+\Delta x)v(x)}$$

This implies, $$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{v(x)\lim\limits_{\Delta x\to 0}\dfrac{u(x+\Delta x)-u(x)}{\Delta x} - u(x)\lim\limits_{\Delta x\to 0}\dfrac{v(x+\Delta x)-v(x)}{\Delta x}}{v(x)\lim\limits_{\Delta x\to 0} v(x+\Delta x)}$$

$$= \frac{v(x)u'(x)-u(x)v'(x)}{v(x)v(x)} \qquad \left(\because \lim_{\Delta x\to 0} v(x+\Delta x) = v(x)\right)$$

This implies, $$f'(x) = \frac{v(x)u'(x)-u(x)v'(x)}{\left[v(x)\right]^2}.$$

That is, $$\frac{d}{dx}(f) = \frac{vu'-uv'}{v^2}$$

or $$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}}.$$

Theorem 10.5 (Chain Rule / Composite Function Rule or Function of a Function Rule)

Let $y=f(u)$ be a function of u and in turn let $u=g(x)$ be a function of x so that $y = f\left(g(x)\right) = (fog)(x)$. Then $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$.

Proof

In the above function $u = g(x)$ is known as the inner function and f is known as the outer function. Note that, ultimately, y is a function of x.

Now $\Delta u = g(x+\Delta x) - g(x)$

Therefore, $$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\times\frac{\Delta u}{\Delta x} = \frac{f(u+\Delta u)-f(u)}{\Delta u}\times\frac{g(x+\Delta x)-g(x)}{\Delta x}.$$

Note that $\Delta u \to 0$ as $\Delta x \to 0$

Therefore, $$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta u}\times\frac{\Delta y}{\Delta x}\right)$$

$$= \lim_{\Delta u\to 0}\left(\frac{\Delta y}{\Delta u}\right).\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)$$

$$= \lim_{\Delta u\to 0}\frac{f(u+\Delta u)-f(u)}{\Delta u}\times\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$$= f'(u)\times u'(x)$$

$$= f'\left(g(x)\right)g'(x) \quad \text{or} \quad \boxed{\frac{d}{dx}\left(f(g(x)\right) = f'\left(g(x)\right)g'(x)}.$$

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Thus, to differentiate a function of a function $y = f\left(g(x)\right)$, we have to take the derivative of the outer function f regarding the argument $g(x) = u$, and multiply the derivative of the inner function $g(x)$ with respect to the independent variable x. The variable u is known as **intermediate argument**.

Theorem 10.6

Let $f(x)$ be a differentiable function and let $y = k\,f(x), k \neq 0$. Then $\frac{d}{dx}(k\,f(x)) = k\frac{d}{dx}f(x)$.

Proof

Since f is differentiable, $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$.

$$\text{Let } y = h(x) = k\,f(x)$$

$$h(x + \Delta x) = k\,f(x + \Delta x)$$

$$h(x + \Delta x) - h(x) = kf(x + \Delta x) - k\,f(x)$$

$$= k\left[f(x+\Delta x) - f(x)\right].$$

$$\frac{h(x+\Delta x) - h(x)}{\Delta x} = k\frac{\left[f(x+\Delta x) - f(x)\right]}{\Delta x}.$$

$$\text{This implies, } \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} = \lim_{\Delta x \to 0} k\,\frac{\left[f(x+\Delta x) - f(x)\right]}{\Delta x}$$

$$= k \lim_{\Delta x \to 0} \frac{\left[f(x+\Delta x) - f(x)\right]}{\Delta x}$$

$$= kf'(x) = k\frac{d}{dx}f(x)$$

$$\frac{d}{dx}\left(h(x)\right) = k\frac{d}{dx}f(x).$$

$$\text{That is, } \boxed{\frac{d}{dx}\left(kf(x)\right) = k\frac{d}{dx}f(x)}.$$

10.4.1 Derivatives of basic elementary functions

We shall now find the derivatives of all the basic elementary functions; we start with the constant function.

(1) The derivative of a constant function is zero.

$$\text{Let } y = f(x) = k, \text{ } k \text{ is a constant.}$$

$$\text{Then } f(x + \Delta x) = k \text{ and}$$

$$f(x + \Delta x) - f(x) = k - k = 0.$$

$$\text{This implies, } \frac{f(x+\Delta x) - f(x)}{\Delta x} = 0$$

$$\text{This implies, } \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = 0.$$

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That is, $f'(x) = 0$.

or $\boxed{\frac{d}{dx}(k)=0}$.

(2) The power function $y = x^n$, $n > 0$ is an integer.

Let $f(x) = x^n$.

Then, $f(x+\Delta x) = (x+\Delta x)^n$ and

$f(x+\Delta x) - f(x) = (x+\Delta x)^n - x^n$.

This implies, $$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{(x+\Delta x)-x}$$

This implies, $$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{(x+\Delta x)^n - x^n}{(x+\Delta x)-x}$$

$$= \lim_{y\to x}\frac{y^n - x^n}{y-x} = nx^{n-1}, \text{ where } y = x+\Delta x \text{ and } y\to x \text{ as } \Delta x\to 0.$$

That is, $\frac{d}{dx}f(x) = nx^{n-1}$ or $\boxed{\frac{d}{dx}(x^n)=nx^{n-1}}$.

Corollary 10.1

When $n=\frac{p}{q},(p,q)=1$, $\frac{d}{dx}\left(x^{\frac{p}{q}}\right)=\frac{p}{q}x^{\frac{p}{q}-1}$.

Corollary 10.2

For any real number α, $\frac{d}{dx}(x^\alpha)=\alpha x^{\alpha-1}$.

For instance,

(1) $\frac{d}{dx}(5) = 0$ since 5 is a constant.

(2) $\frac{d}{dx}(x^3) = 3x^2$, by power function rule.

(3) $\frac{d}{dx}\left(x^{\frac{3}{2}}\right) = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{\frac{1}{2}}$, by power function rule.

(4) $\frac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$, by power function rule.

(5) $\frac{d}{dx}\left(x^{\frac{2}{3}}\right) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{\frac{-1}{3}}, (x\neq 0)$, by power function rule.

(6) $\frac{d}{dx}(100x^9) = 100\frac{d}{dx}(x^9) = 100\times 9x^{9-1} = 900x^8$ by theorem 10.6.

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(3) Derivative of the logarithmic function

The natural logarithm of x is denoted by $\log_e x$ or $\log x$ or $\ln x$

$$\text{Let } y = f(x) = \log x$$

$$\text{Now } f(x+\Delta x) = \log(x+\Delta x) \text{ and}$$

$$f(x+\Delta x) - f(x) = \log(x+\Delta x) - \log x$$

$$= \log\left(\frac{x+\Delta x}{x}\right).$$

$$= \log\left(1+\frac{\Delta x}{x}\right).$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\log\left(1+\frac{\Delta x}{x}\right)}{\Delta x}.$$

$$\text{We know that } \lim_{\alpha\to 0}\frac{\log(1+\alpha k)}{\alpha} = k\lim_{\alpha\to 0}\frac{\log(1+k\alpha)}{k\alpha} = k$$

$$\text{Therefore, } \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\log\left(1+\frac{\Delta x}{x}\right)}{\Delta x} = \frac{1}{x}.$$

$$\text{That is, } \boxed{\frac{d}{dx}(\log x) = \frac{1}{x}}.$$

Corollary 10.3

$$\text{If } y = f(x) = \log_a x \text{ then } f'(x) = \frac{1}{(\log a)x}.$$

$$\text{We have } f(x) = \log_a x = \log_a e \times \log_e x = (\log_a e)\log x$$

$$\text{Therefore, } \frac{d}{dx}(f(x)) = \frac{d}{dx}(\log_a e \times \log x)$$

$$= (\log_a e)\frac{d}{dx}(\log x) \qquad \text{(by constant multiple rule)}$$

$$= \log_a e.\frac{1}{x} \text{ (or) } \frac{1}{(\log a)x}.$$

(4) Derivative of the exponential function

$$\text{Let } y = a^x.$$

$$\text{Then } f(x+\Delta x) - f(x) = a^{x+\Delta x} - a^x$$

$$= a^x(a^{\Delta x} - 1) \text{ and}$$

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$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = a^x\left(\frac{a^{\Delta x}-1}{\Delta x}\right).$$

We know that $\lim\limits_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x} = \log a$.

Therefore, $\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = a^x \lim\limits_{\Delta x\to 0}\left(\frac{a^{\Delta x}-1}{\Delta x}\right)$

$$= a^x \times \log a$$

or $\boxed{\frac{d}{dx}(a^x) = a^x \log a}$.

In particular, $\frac{d}{dx}(e^x) = e^x \log e$

$$\boxed{\frac{d}{dx}(e^x) = e^x}.$$

(5) The derivatives of the Trigonometric functions

(i) The sine function, sin*x*.

Let $y = f(x) = \sin x$.

Then $f(x+\Delta x) = \sin(x+\Delta x)$ and

$$f(x+\Delta x)-f(x) = \sin(x+\Delta x)-\sin x = 2\sin\frac{\Delta x}{2}\cos\left(x+\frac{x}{2}\right).$$

Now $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\cos\left(x+\frac{\Delta x}{2}\right)$.

This implies, $\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}.\lim\limits_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right)$

$= 1 \times \cos x$ (Since $\cos x$ is continuous, $\lim\limits_{\Delta x\to 0}\cos(x+\Delta x) = \cos x$) .

$= \cos x$

That is $\boxed{\frac{d}{dx}(\sin x) = \cos x}$.

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(ii) The cosine function, cos x

$$\text{Let } y = \cos x = \sin\left(x+\frac{\pi}{2}\right).$$

$$\text{Then, } \frac{dy}{dx} = \frac{d}{dx}\sin\left(x+\frac{\pi}{2}\right)$$

$$\text{Let } u = x+\frac{\pi}{2}$$

$$\frac{du}{dx} = 1+0 = 1$$

$$\text{Therefore, } \frac{dy}{dx} = \frac{d}{dx}(\sin u) = \frac{d}{dx}(\sin u)\frac{du}{dx} \text{ by Chain rule}$$

$$= \cos u \times 1 = \cos u = \cos\left(x+\frac{\pi}{2}\right) = -\sin x .$$

That is, $\boxed{\frac{d}{dx}(\cos x) = -\sin x}$.

(iii) The tangent function, tan x

$$\text{Let } y = f(x) = \tan x .$$

$$= \frac{\sin x}{\cos x}$$

$$\text{Therefore, } \frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \quad \text{(by quotient rule)}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} .$$

That is, $\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$.

(iv) The secant function, sec x

$$\text{Let } y = \sec x = \frac{1}{\cos x} = (\cos x)^{-1} .$$

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$$\text{Then } \frac{dy}{dx} = (-1)(\cos x)^{-2}(-\sin x) \quad \text{(by chain rule)}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}.\frac{\sin x}{\cos x} = \sec x.\tan x \ .$$

That is, $\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x}$.

(v) The cosecant function, cosec x

$$\text{Let } y = \operatorname{cosec} x = \frac{1}{\sin x} = (\sin x)^{-1} \ .$$

$$\frac{dy}{dx} = (-1)(\sin x)^{-2}(\cos x) \quad \text{(by chain rule)}$$

$$= -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x}.\frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x \ .$$

That is, $\boxed{\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x}$.

(vi) The cotangent function, cot x

$$\text{Let } y = \cot x = \frac{\cos x}{\sin x}.$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

That is, $\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$.

(6) The derivatives of the inverse trigonometric functions

(i) The derivative of arcsin x or $\sin^{-1} x$

$$\text{Let } y = f(x) = \sin^{-1} x.$$

$$\text{Then } y + \Delta y = f(x + \Delta x) = \sin^{-1}(x + \Delta x)$$

This implies, $x = \sin y$ and

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$$x + \Delta x = \sin(y + \Delta y).$$

$$\text{Therefore, } \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\sin(y+\Delta y) - \sin y} = \frac{1}{\dfrac{\sin(y+\Delta y)-\sin y}{\Delta y}}$$

As $\Delta x \to 0$, $\Delta y \to 0$ also, so that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\dfrac{\sin(y+\Delta y)-\sin y}{\Delta y}}$$

$$= \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

That is, $\boxed{\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}}$.

(ii) The derivative of arccos x or $\cos^{-1} x$

We know the identity :

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

$$\text{This implies, } \frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2}\right) = 0$$

$$\text{This implies, } \frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\cos^{-1} x) = 0$$

$$\text{Therefore, } \frac{1}{\sqrt{1-x^2}} + \frac{d}{dx}(\cos^{-1} x) = 0$$

or $\boxed{\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}}$.

(iii) The derivative of arctan x or $\tan^{-1} x$

$$\text{Let } y = f(x) = \tan^{-1} x. \qquad \ldots (1)$$

$$\text{This implies, } y + \Delta y = f(x + \Delta x) = \tan^{-1}(x + \Delta x) \qquad \ldots (2)$$

$$x = \tan y \text{ and}$$

$$x + \Delta x = \tan(y + \Delta y)$$

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$$\text{This implies,} \quad \Delta x = \tan(y + \Delta y) - \tan y$$

$$\text{Therefore,} \quad \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\tan(y+\Delta y) - \tan y}$$

$$= \frac{1}{\dfrac{\tan(y+\Delta y) - \tan y}{\Delta y}}.$$

As $\Delta x \to 0$, $\Delta y \to 0$ also, so that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{\dfrac{\lim\limits_{\Delta y \to 0} \tan(y+\Delta y) - \tan y}{\Delta y}} = \frac{1}{\dfrac{d}{dy}(\tan y)}$$

$$= \frac{1}{\sec^2 y}$$

$$= \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

That is, $\boxed{\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}}$.

(iv) The derivative of arccot x or $\cot^{-1} x$

We know the identity

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}.$$

$$\text{This implies,} \quad \frac{d}{dx}(\tan^{-1} x + \cot^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2}\right) = 0$$

$$\text{This implies,} \quad \frac{d}{dx}(\tan^{-1} x) + \frac{d}{dx}(\cot^{-1} x) = 0$$

$$\text{That is,} \quad \frac{d}{dx}(\cot^{-1} x) = -\frac{d}{dx}(\tan^{-1} x) = -\frac{1}{1+x^2}$$

That is, $\boxed{\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}}$.

(v) The derivative of arcsec x or $\boxed{\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}}$.

(vi) The derivative of arccosec x or $\boxed{\dfrac{d}{dx}(\operatorname{cosec}^{-1} x) = \dfrac{-1}{x\sqrt{x^2-1}}}$.

The proofs of (v) and (vi) are left as exercises.

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Example 10.7

Differentiate the following with respect to x :

(i) $y = x^3 + 5x^2 + 3x + 7$ (ii) $y = e^x + \sin x + 2$

(iii) $y = 4\text{cosec}\, x - \log x - 2e^x$ (iv) $y = \left(x - \frac{1}{x}\right)^2$ (v) $y = xe^x \log x$

(vi) $y = \frac{\cos x}{x^3}$ (vii) $y = \frac{\log x}{e^x}$

(viii) Find $f'(3)$ and $f'(5)$ if $f(x) = |x-4|$.

Solution

(i) $\frac{dy}{dx} = 3x^2 + 10x + 3.$ (ii) $\frac{dy}{dx} = e^x + \cos x.$

(iii) $\frac{dy}{dx} = -4\ \text{cosec}\ x.\cot x - \frac{1}{x} - 2e^x.$

(iv) $y = x^2 + \frac{1}{x^2} - 2 = x^2 + x^{-2} - 2$

$$\frac{dy}{dx} = 2x - 2x^{-2-1} = 2x - \frac{2}{x^3}.$$

(v) $\frac{dy}{dx} = xe^x\left(\frac{1}{x}\right) + e^x.\log x(1) + x\log x(e^x)$

$$= e^x + e^x \log x + xe^x \log x = e^x(1 + \log x + x\log x).$$

(vi) $y = \frac{\cos x}{x^3}$

$$\frac{dy}{dx} = \frac{x^3(-\sin x) - \cos x(3x^2)}{x^6} = \frac{-x^2(x\sin x + 3\cos x)}{x^6} = -\frac{(x\sin x + 3\cos x)}{x^4}.$$

(vii) $y = \frac{\log x}{e^x} = e^{-x}.\log x$

$$\frac{dy}{dx} = e^{-x}\left(\frac{1}{x}\right) + \log x(e^{-x})(-1)$$

$$= e^{-x}\left[\frac{1}{x} - \log x\right].$$

(viii) $f(x) = |x-4| = \begin{cases} -(x-4) & ; \ x < 4 \\ (x-4) & ; \ x \geq 4 \end{cases}$

$$f'(x) = \begin{cases} -1 \text{ if } x < 4 \\ +1 \text{ if } x \geq 4 \end{cases}$$

Therefore, $f'(3) = -1$

$f'(5) = 1$.

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EXERCISE 10.2

Find the derivatives of the following functions with respect to corresponding independent variables:

(1) $f(x) = x - 3\sin x$

(2) $y = \sin x + \cos x$

(3) $f(x) = x \sin x$

(4) $y = \cos x - 2\tan x$

(5) $g(t) = t^3 \cos t$

(6) $g(t) = 4\sec t + \tan t$

(7) $y = e^x \sin x$

(8) $y = \dfrac{\tan x}{x}$

(9) $y = \dfrac{\sin x}{1+\cos x}$

(10) $y = \dfrac{x}{\sin x + \cos x}$

(11) $y = \dfrac{\tan x - 1}{\sec x}$

(12) $y = \dfrac{\sin x}{x^2}$

(13) $y = \tan\theta(\sin\theta + \cos\theta)$

(14) $y = \operatorname{cosec} x \,.\, \cot x$

(15) $y = x \sin x \cos x$

(16) $y = e^{-x}.\log x$

(17) $y = (x^2+5)\log(1+x)e^{-3x}$

(18) $y = \sin x^\circ$

(19) $y = \log_{10} x$

(20) Draw the function $f'(x)$ if $f(x) = 2x^2 - 5x + 3$

10.4.2 Examples on Chain Rule

Example 10.8

Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

Solution

$$\text{Take } u = g(x) = x^2+1 \text{ and } f(u) = \sqrt{u}$$

$$\therefore F(x) = (fog)(x) = f(g(x))$$

$$\text{Since } f'(u) = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} \text{ and}$$

$$g'(x) = 2x, \text{ we get}$$

$$F'(x) = f'(g(x))g'(x)$$

$$= \frac{1}{2\sqrt{x^2+1}}.2x = \frac{x}{\sqrt{x^2+1}}.$$

Example 10.9

Differentiate : (i) $y = \sin(x^2)$ (ii) $y = \sin^2 x$

Solution

(i) The outer function is the sine function and the inner function is the squaring function.

$$\text{Let } u = x^2$$

$$\text{That is, } y = \sin u.$$

$$\text{Therefore, } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

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$$= \cos u \times (2x)$$
$$= \cos(x^2).2x$$
$$= 2x\cos(x^2).$$

(ii) $u = \sin x$

Then, $y = u^2$

and $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2u \times \cos x$$
$$= 2\sin x \,.\, \cos x$$
$$= \sin 2x.$$

Example 10.10

Differentiate : $y = (x^3 - 1)^{100}$.

Solution

Take $u = x^3 - 1$ so that

$y = u^{100}$

and $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 100u^{100-1} \times (3x^2 - 0)$$
$$= 100(x^3 - 1)^{99} \times 3x^2$$
$$= 300x^2(x^3 - 1)^{99}.$$

Example 10.11

Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

Solution

First we write : $f(x) = (x^2 + x + 1)^{\frac{-1}{3}}$

Then, $$f'(x) = -\frac{1}{3}(x^2 + x + 1)^{\frac{-1}{3}-1} \frac{d}{dx}(x^2 + x + 1)$$
$$= -\frac{1}{3}(x^2 + x + 1)^{\frac{-4}{3}} \times (2x + 1)$$
$$= -\frac{1}{3}(2x + 1)(x^2 + x + 1)^{\frac{-4}{3}}.$$

Example 10.12

Find the derivative of the function $g(t) = \left(\dfrac{t-2}{2t+1}\right)^9$.

Solution

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^{9-1} \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

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$$= 9\left(\frac{t-2}{2t+1}\right)^8\left[\frac{(2t+1)\frac{d}{dt}(t-2)-(t-2)\frac{d}{dt}(2t+1)}{(2t+1)^2}\right]$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8\left[\frac{(2t+1)\times 1-(t-2)\times 2}{(2t+1)^2}\right]$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8\left[\frac{2t+1-2t+4}{(2t+1)^2}\right]$$

$$= \frac{45(t-2)^8}{(2t+1)^{10}}\ .$$

Example 10.13

Differentiate $(2x + 1)^5\ (x^3 - x + 1)^4$.

Solution

Let $y = (2x + 1)^5\ (x^3 - x + 1)^4$

Take $u = 2x + 1$; $v = x^3 - x + 1$ so that

$y = u^5 \,.\, v^4$

$$\frac{dy}{dx} = u^5.\frac{d}{dx}(v^4)+v^4\frac{d}{dx}(u^5) \qquad \text{by Product Rule}$$

$$= u^5.4v^{4-1}\frac{dv}{dx}+v^4.5u^{5-1}\frac{du}{dx} \qquad \text{by Chain Rule}$$

$$= 4u^5.v^3\times(3x^2-1)+5v^4u^4\times 2$$

$$= 4(2x+1)^5(x^3-x+1)^3(3x^2-1)+10(x^3-x+1)^4(2x+1)^4$$

$$= (2x+1)^4(x^3-x+1)^3\left[4(2x+1)(3x^2-1)+10(x^3-x+1)\right]$$

$$= 2(2x+1)^4(x^3-x+1)^3(17x^3+6x^2-9x+3)\,.$$

Example 10.14

Differentiate : $y = e^{\sin x}$.

Solution

Take $u = \sin x$ so that

$y = e^u$

$$\frac{dy}{dx} = \frac{d(e^u)}{du}\times\frac{du}{dx} = e^u\times\cos x = \cos x\, e^{\sin x}.$$

Example 10.15

Differentiate 2^x.

Solution

Let $y = 2^x = e^{x\log 2}$.

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Take $u = (\log 2)x$ so that

$$y = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times \log 2 = \log 2\, e^{x \log 2}$$

$$= (\log 2) 2^x.$$

By using the differentiation formula for a^x, one can find the derivative directly.

Example 10.16

If $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, find y'.

Solution

$$y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$$

Let $\frac{1+x}{1-x} = t$

Then, $y = \tan^{-1} t$

$$\frac{dy}{dx} = \frac{d}{dt}(\tan^{-1} t).\frac{dt}{dx}$$

$$= \frac{1}{1+t^2}.\frac{(1-x).1-(1+x)(-1)}{(1-x)^2}$$

$$= \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}.\frac{(1-x)+(1+x)}{(1-x)^2} = \frac{1}{1+x^2}.$$

EXERCISE 10.3

Differentiate the following :

(1) $y = (x^2+4x+6)^5$ (2) $y = \tan 3x$ (3) $y = \cos(\tan x)$

(4) $y = \sqrt[3]{1+x^3}$ (5) $y = e^{\sqrt{x}}$ (6) $y = \sin(e^x)$

(7) $F(x) = (x^3+4x)^7$ (8) $h(t) = \left(t - \frac{1}{t}\right)^{\frac{3}{2}}$ (9) $f(t) = \sqrt[3]{1+\tan t}$

(10) $y = \cos(a^3+x^3)$ (11) $y = e^{-mx}$ (12) $y = 4 \sec 5x$

(13) $y = (2x-5)^4(8x^2-5)^{-3}$ (14) $y = (x^2+1)\sqrt[3]{x^2+2}$ (15) $y = xe^{-x^2}$

(16) $s(t) = \sqrt[4]{\frac{t^3+1}{t^3-1}}$ (17) $f(x) = \frac{x}{\sqrt{7-3x}}$ (18) $y = \tan(\cos x)$

(19) $y = \frac{\sin^2 x}{\cos x}$ (20) $y = 5^{\frac{-1}{x}}$ (21) $y = \sqrt{1+2\tan x}$

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(22) $y=\sin^3 x+\cos^3 x$ (23) $y=\sin^2(\cos kx)$ (24) $y=(1+\cos^2 x)^6$

(25) $y=\dfrac{e^{3x}}{1+e^x}$ (26) $y=\sqrt{x+\sqrt{x}}$ (27) $y=e^{x\cos x}$

(28) $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$ (29) $y=\sin\left(\tan\left(\sqrt{\sin x}\right)\right)$ (30) $y=\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

10.4.3 Implicit Differentiation

A function in which the dependent variable is expressed solely in terms of the independent variable x, namely, $y = f(x)$, is said to be an **explicit** function. For instance, $y=\frac{1}{2}x^3-1$ is an explicit function, whereas an equivalent equation $2y-x^3+2=0$ is said to define the function **implicitly** or y is an **implicit function** of x.

Now, as we know, the equation

$$x^2+y^2=4 \tag{1}$$

describes *a* a circle of radius 2 centered at the origin. Equation (1) is not a function since for any choice of x in the interval $-2<x<2$ there correspond two values of y, namely

$$f(x)=\sqrt{4-x^2}, -2\le x\le 2 \tag{2}$$

$$g(x)=-\sqrt{4-x^2}, -2\le x\le 2 \tag{3}$$

(2) represents the top half of the circle (1) and (3) represents the bottom half of the circle (1). By considering either the top half or bottom half, of the circle, we obtain a function. We say that (1) defines at least two implicit functions of x on the interval $-2\le x\le 2$.

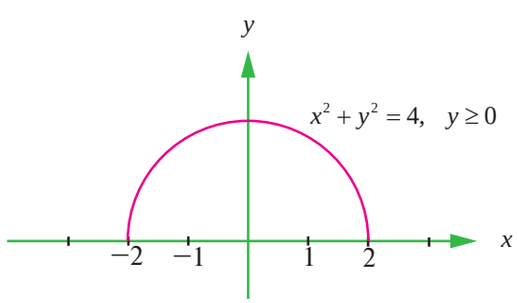

Fig. 10.25

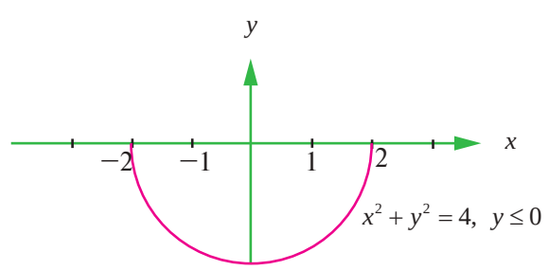

Fig. 10.26

Note that both equations

$x^2+[f(x)]^2=4$ and $x^2+[g(x)]^2=4$ are identities on the interval $-2\le x\le 2$.

In general, if an equation $F(x,y)=0$ defines a function f implicitly on some interval, then $F(x, f(x))=0$ is an identity on the interval. The graph of f is a portion (or all) of the graph of the equation $F(x,y)=0$.

A more complicated equation such as $x^4+x^2y^3-y^5=2x+1$ may determine several implicit functions on a suitably restricted interval of the x-axis and yet it may not be possible to solve for y in terms of x. However, in some cases we can determine the derivative $\dfrac{dy}{dx}$ by a process known

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as **implicit differentiation**. This process consists of differentiating both sides of an equation with respect to x, using the rules of differentiation and then solving for $\frac{dy}{dx}$. Since we think of y as being determined by the given equation as a differentiable function, the chain rule, in the form of the power rule for functions, gives the result.

$$\frac{d}{dx}(y^n) = n\,y^{n-1}\frac{dy}{dx}, \text{ where } n \text{ is an integer.}$$

Example 10.17

Find $\frac{dy}{dx}$ if $x^2 + y^2 = 1$.

Solution

We differentiate both sides of the equation,

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for the derivative yields

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 10.18

Find the slopes of the tangent lines to the graph of $x^2 + y^2 = 4$ at the points corresponding to $x = 1$.

Solution

Substituting $x = 1$ into the given equation yields $y^2 = 3$ or $y = \pm\sqrt{3}$.

Hence, there are tangent lines at $\left(1, \sqrt{3}\right)$ and $\left(1, -\sqrt{3}\right)$. Although $\left(1, \sqrt{3}\right)$ and $\left(1, -\sqrt{3}\right)$ are points on the graphs of two different implicit functions, we got the correct slope of each point. We have

$$\frac{dy}{dx}\, at \left(1, \sqrt{3}\right) = -\frac{1}{\sqrt{3}} \text{ and } \frac{dy}{dx} \text{ at } \left(1, -\sqrt{3}\right) = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Example 10.19

Find $\frac{dy}{dx}$ if $x^4 + x^2y^3 - y^5 = 2x + 1$.

Solution

Differentiating implicitly, we have

$$\frac{d}{dx}(x^4) + \frac{d}{dx}(x^2y^3) - \frac{d}{dx}(y^5) = \frac{d}{dx}(2x+1)$$

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This implies, $4x^3 + x^2\left(3y^2\dfrac{dy}{dx}\right) + (2x)y^3 - 5y^4\dfrac{dy}{dx} = 2 + 0$

This implies, $4x^3 + 3x^2y^2\dfrac{dy}{dx} + 2xy^3 - 5y^4\dfrac{dy}{dx} = 2$

This implies, $(3x^2y^2 - 5y^4)\dfrac{dy}{dx} = 2 - 4x^3 - 2xy^3$

or $\dfrac{dy}{dx} = \dfrac{2 - 4x^3 - 2xy^3}{3x^2y^2 - 5y^4}$.

Example 10.20

Find $\dfrac{dy}{dx}$ if $\sin y = y\cos 2x$.

Solution

We have $\sin y = y\cos 2x$.

Differentiating, $\dfrac{d}{dx}\sin y = \dfrac{d}{dx}(y\cos 2x)$

That is, $\cos y\dfrac{dy}{dx} = y(-2\sin 2x) + \cos 2x\dfrac{dy}{dx}$

This implies, $(\cos y - \cos 2x)\dfrac{dy}{dx} = -2y\sin 2x$

or $\dfrac{dy}{dx} = \dfrac{-2y\sin 2x}{\cos y - \cos 2x}$.

10.4.4 Logarithmic Differentiation

By using the rules for differentiation and the table of derivatives of the basic elementary functions, we can now find automatically the derivatives of any elementary function, except for one type, the simplest representative of which is the function $y = x^x$. Such functions are described as power-exponential and include, in general, any function written as a power whose base and index both depend on the independent variable.

In order to find by the general rules the derivative of the power-exponential function $y = x^x$, we take logarithms on both sides to get

$\log y = x \log x,\ x > 0$

Since this is an identity, the derivative of the left-hand side must be equal to the derivative of the right, we obtain by differentiating with respect to x (keeping in mind the fact that the left hand side is a function of function) :

$$\frac{1}{y}\frac{dy}{dx} = \log x + 1$$

Hence $\dfrac{dy}{dx} = y(\log x + 1) = x^x(\log x + 1)$

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The operation consists of first taking the logarithm of the function $f(x)$ (to base e) then differentiating is called **logarithmic differentiation** and its result

$$\boxed{\frac{d}{dx}(\log f(x)) = \frac{f'(x)}{f(x)}}.$$

is called the logarithmic derivative of $f(x)$.

The advantage in this method is that the calculation of derivatives of complicated functions involving products, quotients or powers can often be simplified by taking logarithms.

Example 10.21

Find the derivative of $y = \sqrt{x^2+4}\,.\sin^2 x\,.2^x$

Solution

Taking logarithm on both sides and using the law of logarithm,

we have, $\log y = \frac{1}{2}\log(x^2+4) + 2\log(\sin x) + x\log(2).$

$$\text{This implies,}\quad \frac{y'}{y} = \frac{1}{2}\frac{2x}{x^2+4} + 2.\frac{\cos x}{\sin x} + \log 2$$

$$= \frac{x}{x^2+4} + 2\cot x + \log 2$$

$$\text{Therefore,}\quad y' = \frac{dy}{dx} = y\left(\frac{x}{x^2+4} + 2\cot x + \log 2\right).$$

Example 10.22

Differentiate : $y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$.

Solution

Taking logarithm on both sides of the equation and using the rules of logarithm we have,

$\log y = \frac{3}{4}\log x + \frac{1}{2}\log(x^2+1) - 5\log(3x+2).$

Differentiating implicitly

$$\frac{y'}{y} = \frac{3}{4x} + \frac{1}{2}\frac{2x}{(x^2+1)} - \frac{5\times 3}{3x+2}$$

$$= \frac{3}{4x} + \frac{x}{(x^2+1)} - \frac{15}{3x+2}$$

$$\text{Therefore,}\quad \frac{dy}{dx} = y' = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right].$$

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Steps in Logarithmic Differentiation

(1) Take natural logarithm on both sides of an equation $y = f(x)$ and use the law of logarithms to simplify.

(2) Differentiate implicitly with respect to x.

(3) Solve the resulting equation for y'.

In general there are four cases for exponents and bases.

BCTJRH

(1) $\frac{d}{dx}(a^b) = 0$ (a and b are constants).

(2) $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1} f'(x)$

(3) $\frac{d}{dx}[a^{g(x)}] = a^{g(x)}(\log a)g'(x)$

(4) $\frac{d}{dx}[f(x)]^{g(x)} = [f(x)]^{g(x)}\left[\frac{g(x)f'(x)}{f(x)} + \log f(x).g'(x)\right]$

Example 10.23

Differentiate $y = x^{\sqrt{x}}$.

Solution

Take logarithm :

$$\log y = \sqrt{x}\log x$$

Differentiating implicitly,

$$\frac{y'}{y} = \sqrt{x}.\frac{1}{x} + \frac{1}{2\sqrt{x}}.\log x$$

$$= \frac{\log x + 2}{2\sqrt{x}}$$

Therefore, $\frac{d}{dx}(x^{\sqrt{x}}) = y' = x^{\sqrt{x}}\left(\frac{\log x + 2}{2\sqrt{x}}\right)$.

10.4.5 Substitution method

It is very much useful in some processes of differentiation, in particular the differentiation involving inverse trigonometrical functions.

Consider $f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

For this function $f'(x)$ can be found out by using function of a function rule. But it is laborious. Instead we can use the substitution method.

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$$\text{Take} \quad x = \tan\theta.$$

$$\text{Then} \quad \frac{2x}{1-x^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta \text{ and}$$

$$f(x) = \tan^{-1}(\tan 2\theta) = 2\theta$$

$$= 2\tan^{-1}x$$

$$f'(x) = \frac{2}{1+x^2}.$$

Example 10.24

If $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, find y'.

Solution

Let $x = \tan\theta$

$$\text{Then} \quad \frac{1+x}{1-x} = \frac{1+\tan\theta}{1-\tan\theta} = \tan\left(\frac{\pi}{4}+\theta\right).$$

$$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right] = \frac{\pi}{4}+\theta = \frac{\pi}{4}+\tan^{-1}x$$

$$y = \frac{\pi}{4}+\tan^{-1}x$$

$$y' = \frac{1}{1+x^2}.$$

Example 10.25

Find $f'(x)$ if $f(x) = \cos^{-1}(4x^3-3x)$.

Solution

$$\text{Let} \quad x = \cos\theta.$$

$$\text{Then} \quad 4x^3-3x = 4\cos^3\theta - 3\cos\theta = \cos 3\theta \text{ and}$$

$$f(x) = \cos^{-1}(\cos 3\theta) = 3\theta = 3\cos^{-1}x$$

$$\text{Therefore,} \quad f'(x) = 3\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{-3}{\sqrt{1-x^2}}.$$

10.4.6 Derivatives of variables defined by parametric equations

Consider the equations $x = f(t)$, $y = g(t)$.

These equations give a functional relationship between the variables x and y. Given the value of t in some domain $[a, b]$, we can find x and y.

If two variables x and y are defined separately as a function of an intermediating (auxiliary) variable t, then the specification of a functional relationship between x and y is described as **parametric** and the auxiliary variable is known as **parameter**.

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The operation of finding the direct connection between x and y without the presence of the auxiliary variable t is called **elimination of the parameter**. The question as to why should we deal with parametric equations is that two or more variables are reduced to a single variable, t.

For example, the equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$ This equation describes the relationship between x and y and the equations

$x = r\cos t$; $y = r\sin t$ are parametric equations.

Conversely, if we eliminate t, we get $x^2 + y^2 = r^2$

If y is regarded as a function of x then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{g'(t)}{f'(t)}$.

If we regard x as a function of y, then the derivative of x with respect to y is,

$$\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{f'(t)}{g'(t)}.$$

In the case of the circle, then $\dfrac{dy}{dx}$ is the slope of the tangent to the circle namely

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{r\cos t}{-r\sin t} = -\cot t \ .$$

Example 10.26

Find $\dfrac{dy}{dx}$ if $x = at^2$; $y = 2at$, $t \neq 0$.

Solution

We have $x = at^2$; $y = 2at$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2a}{2at} = \frac{1}{t}.$$

Example 10.27

Find $\dfrac{dy}{dx}$ if $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

Solution

We have $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

Now $\dfrac{dx}{dt} = a(1 - \cos t)$; $\dfrac{dy}{dt} = a\sin t$

Therefore, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\sin t}{a(1-\cos t)} = \frac{\sin t}{(1-\cos t)} \ .$$

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10.4.7 Differentiation of one function with respect to another function :

If $y = f(x)$ is differentiable, then the derivative of y with respect to x is

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} .$$

If f and g are differentiable functions of x and if $\frac{dg}{dx} = g'(x) \neq 0$, then

$$\frac{df}{dg} = \frac{\frac{df}{dx}}{\frac{dg}{dx}} = \frac{f'(x)}{g'(x)}.$$

Example 10.28

Find the derivative of x^x with respect to $x \log x$.

Solution

Take $\quad u = x^x, v = x \log x$

$$\log u = x \log x$$

$$\frac{1}{u}\frac{du}{dx} = x.\frac{1}{x} + 1.\log x = 1 + \log x$$

$$\frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x)$$

$$\frac{dv}{dx} = 1 + \log x$$

$$\frac{d(x^x)}{d(x \log x)} = \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = x^x .$$

Note that when g is the identity function $g(x) = x$ then $\frac{df}{dg}$ reduces to $\frac{df}{dx} = f'(x)$.

Example 10.29

Find the derivative of $\tan^{-1}(1 + x^2)$ with respect to $x^2 + x + 1$.

Solution :

Let

$$f(x) = \tan^{-1}(1 + x^2)$$

$$g(x) = x^2 + x + 1$$

$$\frac{df}{dg} = \frac{f'(x)}{g'(x)}$$

$$f'(x) = \frac{2x}{(x^4 + 2x^2 + 2)}$$

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$$g'(x) = 2x+1$$

$$\frac{df}{dg} = \frac{2x}{(2x+1)(x^4+2x^2+2)}.$$

Example 10.30

Differentiate $\sin(ax^2+bx+c)$ with respect to $\cos(lx^2+mx+n)$

Solution

Let $u = \sin(ax^2+bx+c)$

$$v = \cos(lx^2+mx+n)$$

$$\frac{du}{dv} = \frac{u'(x)}{v'(x)}$$

$$u'(x) = \cos(ax^2+bx+c)(2ax+b)$$

$$v'(x) = -\sin(lx^2+mx+n)(2lx+m)$$

$$\frac{du}{dv} = \frac{u'(x)}{v'(x)} = \frac{(2ax+b)\cos(ax^2+bx+c)}{-(2lx+m)\sin(lx^2+mx+n)}.$$

10.4.8 Higher order Derivatives

If $s = s(t)$ is the position function (displacement) of an object that moves in a straight line, we know that its first derivative has the simple physical interpretation as the velocity $v(t)$ of the object as a function of time :

$$v(t) = s'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{ds}{dt}.$$

The instantaneous rate of change of velocity with respect to time is called the acceleration $a(t)$ of the object. Then, the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

$$= \frac{d}{dt}(v(t))$$

$$= \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} = s''(t)$$

Thus, if f is a differentiable function of x, then its first derivative $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ has a very simple geometrical interpretation as the slope of a tangent to the graph of $y = f(x)$. Since f' is also a function of x, f' may have a derivative of its own, and if it exists, denoted by $(f')' = f''$ is,

$$f''(x) = \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}.$$

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$$= \frac{d}{dx}\left(f'(x)\right) = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right)$$

$$= \frac{d^2 f}{dx^2} = \frac{d^2 y}{dx^2}$$

Other notations are $D^2 f(x) = D^2 y = y_2 = y''$

We can interpret a second derivative as a rate of change of a rate of change. But its geometrical interpretation is not so simple. However, there is a close connection exists between the second derivative $f''(x)$ and the radius of curvature of the graph of $y = f(x)$ which you will learn in higher classes.

Similarly, if f'' exists, it might or might not be differentiable. If it is, then the derivative of f'' is called **third derivative** of f and is denoted by

$$f'''(x) = \frac{d^3 y}{dx^3} = y''' = y_3.$$

We can interpret the third derivative physically in case when the function is the position function $f(t)$ of an object that moves along a straight line. Because $s''' = (s'')' = a'(t)$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3 s}{dt^3}.$$

Thus, jerk is the rate of change of acceleration.

It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

Example 10.31

Find y', y'' and y''' if $y = x^3 - 6x^2 - 5x + 3$.

Solution

We have, $y = x^3 - 6x^2 - 5x + 3$ and

$$y' = 3x^2 - 12x - 5$$

$$y'' = 6x - 12$$

$$y''' = 6.$$

Example 10.32

Find y''' if $y = \frac{1}{x}$.

Solution

We have, $y = \frac{1}{x} = x^{-1}$

$$y' = -1x^{-2} = -\frac{1}{x^2}$$

$$y'' = (-1)(-2)x^{-3} = \frac{(-1)^2 2!}{x^3}.$$

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$$\text{and} \quad y''' = (-1)(-2)(-3)x^{-4} = \frac{(-1)^3 3!}{x^4} .$$

Example 10.33

Find f'' if $f(x) = x \cos x$.

Solution

$$\text{We have,} \quad f(x) = x \cos x.$$

$$\text{Now} \quad f'(x) = -x \sin x + \cos x, \text{ and}$$

$$f''(x) = -(x \cos x + \sin x) - \sin x$$

$$= -x \cos x - 2 \sin x.$$

Example 10.34

Find y'' if $x^4 + y^4 = 16$.

Solution

We have $x^4 + y^4 = 16$.

Differentiating implicitly, $4x^3 + 4y^3 y' = 0$

Solving for y' gives

$$y' = -\frac{x^3}{y^3} .$$

To find y'' we differentiate this expression for y' using the quotient rule and remembering that y is a function of x.

$$y'' = \frac{d}{dx}\left(\frac{-x^3}{y^3}\right) = \frac{-\left[y^3 \frac{d}{dx}(x^3) - x^3 \frac{d}{dx}(y^3)\right]}{(y^3)^2}$$

$$= -\frac{\left[y^3.3x^2 - x^3(3y^2 y')\right]}{y^6}$$

$$= -\frac{3x^2 y^3 - 3x^3 y^2\left(-\frac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3(x^2 y^4 + x^6)}{y^7} = \frac{-3x^2[x^4 + y^4]}{y^7}$$

$$= \frac{-3x^2(16)}{y^7} = \frac{-48x^2}{y^7} .$$

Example 10.35

Find the second order derivative if x and y are given by

$$x = a \cos t$$

$$y = a \sin t.$$

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Solution

Differentiating the function implicitly with respect to x, we get

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{-a\sin t} = -\frac{\cos t}{\sin t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{-\cos t}{\sin t}\right)$$

$$= \frac{d}{dt}\left(\frac{-\cos t}{\sin t}\right)\frac{dt}{dx} = -[-\text{cosec}^2 t]\times\frac{1}{x'(t)}$$

$$= \text{cosec}^2 t \times \frac{1}{-a\sin t}$$

$$= -\frac{\cos ec^3 t}{a}.$$

Example 10.36

Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 4$.

Solution

We have $x^2 + y^2 = 4$

As before, $\frac{dy}{dx} = -\frac{x}{y}$

Hence, by the quotient rule

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= -\frac{y.1 - x.\frac{dy}{dx}}{y^2}$$

$$= -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$

$$= -\frac{x^2 + y^2}{y^3} = -\frac{4}{y^3}.$$

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EXERCISE 10.4

Find the derivatives of the following (1 – 18) :

(1) $y = x^{\cos x}$

(2) $y = x^{\log x} + (\log x)^x$

(3) $\sqrt{xy} = e^{(x-y)}$

(4) $x^y = y^x$

(5) $(\cos x)^{\log x}$

(6) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(7) $\sqrt{x^2 + y^2} = \tan^{-1}\left(\frac{y}{x}\right)$

(8) $\tan(x + y) + \tan(x - y) = x$

(9) If $\cos(xy) = x$, show that $\frac{dy}{dx} = \frac{-(1 + y\sin(xy))}{x\sin xy}$

(10) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

(11) $\tan^{-1}\left(\frac{6x}{1-9x^2}\right)$

(12) $\cos\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$

(13) $x = a\cos^3 t$; $y = a\sin^3 t$

(14) $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

(15) $x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}$

(16) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(17) $\sin^{-1}(3x - 4x^3)$

(18) $\tan^{-1}\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)$.

(19) Find the derivative of $\sin x^2$ with respect to x^2.

(20) Find the derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\tan^{-1} x$.

(21) If $u = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ and $v = \tan^{-1} x$, find $\frac{du}{dv}$.

(22) Find the derivative with $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ with respect to $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$.

(23) If $y = \sin^{-1}x$ then find y''.

(24) If $y = e^{\tan^{-1} x}$, show that $(1+x^2)y'' + (2x-1)y' = 0$.

(25) If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, show that $(1-x^2)y_2 - 3xy_1 - y = 0$.

(26) If $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ then prove that at $\theta = \frac{\pi}{2}$, $y'' = \frac{1}{a}$.

(27) If $\sin y = x\sin(a+y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$, $a \neq n\pi$.

(28) If $y = (\cos^{-1} x)^2$, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$. Hence find y_2 when $x = 0$

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EXERCISE 10.5

BD3FTJ

Choose the correct or the most suitable answer from the given four alternatives.

(1) $\frac{d}{dx}\left(\frac{2}{\pi}\sin x^\circ\right)$ is

(1) $\frac{\pi}{180}\cos x^\circ$ (2) $\frac{1}{90}\cos x^\circ$ (3) $\frac{\pi}{90}\cos x^\circ$ (4) $\frac{2}{\pi}\cos x^\circ$

(2) If $y = f(x^2+2)$ and $f'(3)=5$, then $\frac{dy}{dx}$ at $x=1$ is

(1) 5 (2) 25 (3) 15 (4) 10

(3) If $y=\frac{1}{4}u^4$, $u=\frac{2}{3}x^3+5$, then $\frac{dy}{dx}$ is

(1) $\frac{1}{27}x^2(2x^3+15)^3$ (2) $\frac{2}{27}x(2x^3+5)^3$

(3) $\frac{2}{27}x^2(2x^3+15)^3$ (4) $-\frac{2}{27}x(2x^3+5)^3$

(4) If $f(x)=x^2-3x$, then the points at which $f(x)=f'(x)$ are

(1) both positive integers (2) both negative integers

(3) both irrational (4) one rational and another irrational

(5) If $y=\frac{1}{a-z}$, then $\frac{dz}{dy}$ is

(1) $(a-z)^2$ (2) $-(z-a)^2$ (3) $(z+a)^2$ ` (4) $-(z+a)^2$

(6) If $y=\cos(\sin x^2)$, then $\frac{dy}{dx}$ at $x=\sqrt{\frac{\pi}{2}}$ is

(1) – 2 (2) 2 (3) $-2\sqrt{\frac{\pi}{2}}$ (4) 0

(7) If $y=mx+c$ and $f(0)=f'(0)=1$, then $f(2)$ is

(1) 1 (2) 2 (3) 3 (4) – 3

(8) If $f(x)=x\tan^{-1}x$, then $f'(1)$ is

(1) $1+\frac{\pi}{4}$ (2) $\frac{1}{2}+\frac{\pi}{4}$ (3) $\frac{1}{2}-\frac{\pi}{4}$ (4) 2

(9) $\frac{d}{dx}(e^{x+5\log x})$ is

(1) $e^x.x^4(x+5)$ (2) $e^x.x(x+5)$ (3) $e^x+\frac{5}{x}$ (4) $e^x-\frac{5}{x}$

(10) If the derivative of $(ax-5)e^{3x}$ at $x=0$ is – 13, then the value of a is

(1) 8 (2) – 2 (3) 5 (4) 2

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(11) $x=\dfrac{1-t^2}{1+t^2}, y=\dfrac{2t}{1+t^2}$ then $\dfrac{dy}{dx}$ is

(1) $-\dfrac{y}{x}$ (2) $\dfrac{y}{x}$ (3) $-\dfrac{x}{y}$ (4) $\dfrac{x}{y}$

(12) If $x=a\sin\theta$ and $y=b\cos\theta$, then $\dfrac{d^2y}{dx^2}$ is

(1) $\dfrac{a}{b^2}\sec^2\theta$ (2) $-\dfrac{b}{a}\sec^2\theta$ (3) $-\dfrac{b}{a^2}\sec^3\theta$ (4) $-\dfrac{b^2}{a^2}\sec^3\theta$

(13) The differential coefficient of $\log_{10} x$ with respect to $\log_x 10$ is

(1) 1 (2) $-(\log_{10} x)^2$ (3) $(\log_x 10)^2$ ` (4) $\dfrac{x^2}{100}$

(14) If $f(x)=x+2$, then $f'(f(x))$ at $x=4$ is

(1) 8 (2) 1 (3) 4 (4) 5

(15) If $y=\dfrac{(1-x)^2}{x^2}$, then $\dfrac{dy}{dx}$ is

(1) $\dfrac{2}{x^2}+\dfrac{2}{x^3}$ (2) $-\dfrac{2}{x^2}+\dfrac{2}{x^3}$ (3) $-\dfrac{2}{x^2}-\dfrac{2}{x^3}$ (4) $-\dfrac{2}{x^3}+\dfrac{2}{x^2}$

(16) If $pv=81$, then $\dfrac{dp}{dv}$ at $v=9$ is

(1) 1 (2) – 1 (3) 2 (4) –2

(17) If $f(x)=\begin{cases} x-5 & \text{if } x\le 1 \\ 4x^2-9 & \text{if } 1<x<2 \\ 3x+4 & \text{if } x\ge 2 \end{cases}$, then the right hand derivative of $f(x)$ at $x = 2$ is

(1) 0 (2) 2 (3) 3 (4) 4

(18) It is given that $f'(a)$ exists, then $\lim\limits_{x\to a}\dfrac{xf(a)-af(x)}{x-a}$ is

(1) $f(a)-af'(a)$ (2) $f'(a)$ (3) $-f'(a)$ (4) $f(a)+af'(a)$

(19) If $f(x)=\begin{cases} x+1, & \text{when } x<2 \\ 2x-1 & \text{when } x\ge 2 \end{cases}$, then $f'(2)$ is

(1) 0 (2) 1 (3) 2 (4) does not exist

(20) If $g(x)=(x^2+2x+1)f(x)$ and $f(0)=5$ and $\lim\limits_{x\to 0}\dfrac{f(x)-5}{x}$=4, then $g'(0)$ is

(1) 20 (2) 14 (3) 18 (4) 12

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(21) If $f(x)=\begin{cases} x+2, & -1<x<3 \\ 5 & x=3 \\ 8-x & x>3 \end{cases}$, then at $x=3$, $f'(x)$ is

(1) 1 (2) – 1 (3) 0 (4) does not exist

(22) The derivative of $f(x)=x|x|$ at $x=-3$ is

(1) 6 (2) – 6 (3) does not exist (4) 0

(23) If $f(x)=\begin{cases} 2a-x, & \text{for } -a<x<a \\ 3x-2a & \text{for } x\geq a \end{cases}$, then which one of the following is true?

(1) $f(x)$ is not differentiable at $x = a$ (2) $f(x)$ is discontinuous at $x = a$

(3) $f(x)$ is continuous for all x in $\mathbb{R}$ (4) $f(x)$ is differentiable for all $x \geq a$

(24) If $f(x)=\begin{cases} ax^2-b, & -1<x<1 \\ \dfrac{1}{|x|}, & \text{elsewhere} \end{cases}$ is differentiable at $x = 1$, then

(1) $a=\frac{1}{2},\ b=\frac{-3}{2}$ (2) $a=\frac{-1}{2},\ b=\frac{3}{2}$ (3) $a=-\frac{1}{2},\ b=-\frac{3}{2}$ (4) $a=\frac{1}{2},\ b=\frac{3}{2}$

(25) The number of points in $\mathbb{R}$ in which the function $f(x)=|x-1|+|x-3|+\sin x$ is not differentiable, is

(1) 3 (2) 2 (3) 1 (4) 4

SUMMARY

In this chapter we have acquired the knowledge of

- Derivative as a rate of change. If $y=f(x)$ then the derivative of y with respect to x at x_0 is $\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$, provided the limit exists, where existence means
$$f'(x_0^+)=\lim_{\Delta x\to 0^+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\lim_{\Delta x\to 0^-}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=f'(x_0^-)$$
is a unique real number.
- Derivative of $y=f(x)$ at $x=x_0$ is $\left(\dfrac{dy}{dx}\right)_{x=x_0}=f'(x_0)=\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}$.
- Geometrical meaning of the derivative of $y=f(x)$ is the slope of the tangent to the curve $y=f(x)$ at $(x, f(x))$.
- Physical meaning of the derivative of $s=f(t)$ with respect to t is the rate of change of displacement, that is velocity. The second derivative is acceleration and the third derivative is jerk.
- Discontinuity of $y=f(x)$ at $x=x_0$ implies non-differentiability of $f(x)$ at $x=x_0$.

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- Non-existence of the derivative of $y = f(x)$ at $x = x_0$ implies that the graph of $y = f(x)$ fails to admit tangent at $(x_0, f(x_0))$.
- Geometrically, if the graph of $y = f(x)$ admits cups $(\vee)$ or caps $(\wedge)$ at $x = x_0$ then derivative at $x = x_0$ does not exist.
- Derivative should be understood as a process not as a set of rules.
- Differentiability implies continuity, but the converse is not true.
- The sum, difference, product and composite of differentiable function is differentiable, and the quotient of two differentiable function is differentiable wherever it is defined.

(i) $$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

(ii) $$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}$$

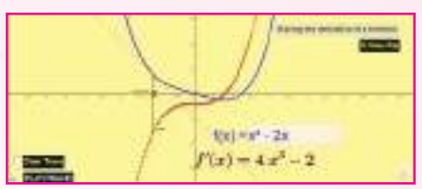
(iii) $$\frac{d}{dx}((fog)(x)) = f'(g(x))g'(x)$$

(iv) $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}, \text{ where } g(x) \neq 0.$$

Differential Calculus

ICT CORNER 10(a)

Expected Outcome

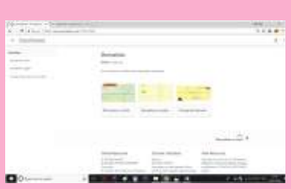


Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code.
GeoGebra Workbook called "Derivatives" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Tracing the derivative of a function". You can enter any function in *f*(*x*) box. You can see the function in blue colour and derivative in orange colour. Click play trace button to get animation of the locus of derivative(*x*, slope at *x*)
Observe the trace and find that derivative is the path of slope at each point on *f*(*x*).



Step1

Step2

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Browse in the link:

Derivatives: https://ggbm.at/fk3w5g8y

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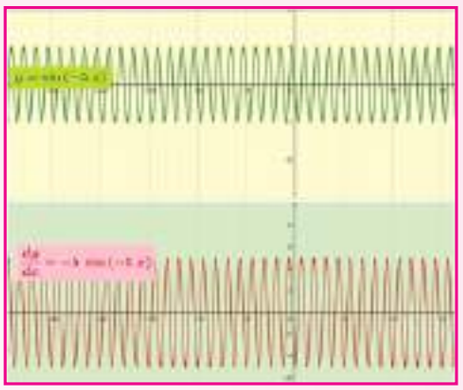
ICT CORNER 10(b)

Differential Calculus

Expected Outcome



Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code.

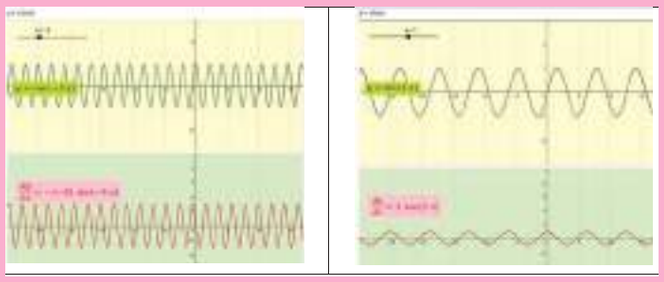
GeoGebra Workbook called "Derivatives" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Derivatives in graph". Some basic functions and their derivatives are given one under another

You can change "a" value by moving the slider and observe the changes in each function and its derivatives.

Step1



Step2

Browse in the link:

Derivatives: https://ggbm.at/fk3w5g8y

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Chapter 11 Integral Calculus

BDCBV7

It is worth noting that the notation facilitates discovery.
This, in most wonderful way, reduces the minds labour.
- Gottfried Wilhelm Leibnitz

11.1 Introduction

Newton

Leibnitz

Gottfried Wilhelm Leibnitz (1646-1716) and Sir Isaac Newton (1643-1727) independently discovered calculus in the mid-17th century. **Leibnitz,** a German philosopher, mathematician, and political adviser, importantly both as a metaphysician and as a logician, was a distinguished independent inventor of the Differential and Integral Calculus. Sir Isaac Newton had created an expression for the area under a curve by considering a momentary increase at a point. In effect, the fundamental theorem of calculus was built into his calculations. His work and discoveries were not limited to mathematics; he also developed theories in optics and gravitation.

One cannot imagine a world without differentiation and integration. In this century, we witnessed remarkable scientific advancement owing to the ingenious application of these two basic components of Mathematics. Calculus serve as unavoidable tool for finding solutions to the variety of problems that arise in physics, astronomy, engineering, chemistry, geology, biology, and social sciences.

Calculus deals principally with two geometric problems.

(i) The problem of finding **SLOPE** of the tangent line to the curve, is studied by the limiting process known as **differentiation** and

(ii) Problem of finding the **AREA** of a region under a curve is studied by another limiting process called **Integration**.

In chapters 9 and 10, we have studied the differential calculus. In this chapter let us study some fundamentals of integration.

Consider some simple situations illustrated below.

Situation 1

The shortest distance between two points A and B in a plane is the line segment joining the straight line A and B. Suppose it is required to find the line connecting two points A and B that do not lie on a vertical line such that a moving particle slides down on this line from A to B in the shortest time (minimum time). Most of us believe that the shortest distance path in Fig. 11.1(a) will take the shortest time.

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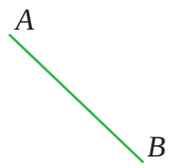
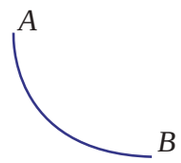


The shortest distance path between *A* and *B*
Fig. 11.1 (a)



The shortest time route between *A* and *B*
Fig. 11.1 (b)

Certainly this route is not the shortest time route joining the points *A* and *B*, because the velocity of the motion in the straight line (Fig. 11.1(a)) will be comparatively slow; whereas it take a curve that is steeper near *A* (Fig. 11.1(b)), even though the path becomes longer, a considerable portion of the distance will be covered at a greater speed. The solution to this problem is solved by **Integral calculus.** This is called Brachistochrone problem which initiates the study of calculus of variation using integral tool.

Situation 2

In elementary geometry we have learnt to evaluate the measure of the following regular shape of the figures given below by using known formulae.

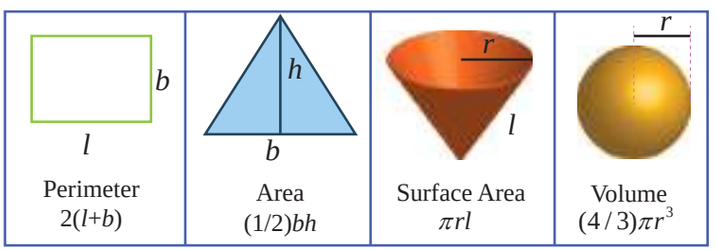


Fig. 11.2 (a)

How can the measure of the following figures given by functions be calculated?

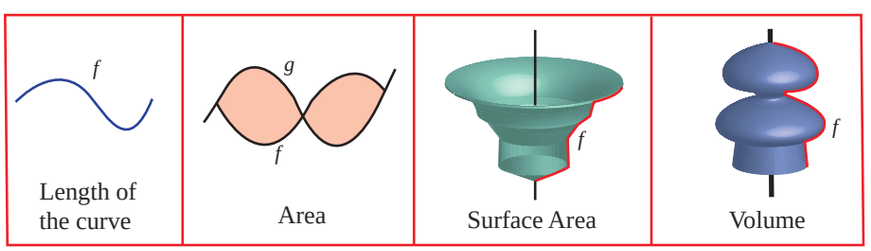


Fig. 11.2 (b)

Though the problems look so difficult, integral calculus solves it without any difficulties.

Situation 3

At a particular moment, a student needs to stop his speedy bike to avoid a collision with the barrier ahead at a distance 40 metres away from him. Immediately he slows (acceleration) down the bike applying brake at a rate of 8 meter/second^2. If the bike is moving at a speed of 24m/s, when the brakes are applied, will it stop before collision?

Fig. 11.3

Also look at the following problems that occur naturally in our life.

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- What speed has to be applied to fire a satellite upward so that it never returns to the earth?
- What is the radius of the smallest circular disk that can cover every isosceles triangle of given perimeter P?
- What volume of material is removed from a solid sphere of radius $2r$ if a hole of radius r is drilled through the centre?
- If a strain of bacteria grows at a rate proportional to the amount present and if the population doubles in one hour, how much will it increase at the end of two hours?

Integration will answer for all the above problems.

Learning Objectives

On completion of this chapter, the students are expected to

- understand the definition of an indefinite integral as a result of reversing the process of differentiation
- find the indefinite integrals of sums, differences and constant multiples of certain elementary functions.
- use the appropriate techniques to find the indefinite integrals of composite functions.
- apply integration to find the function, when the rate of change of function is given.

11.2 Newton-Leibnitz Integral

Integral calculus is mainly divided into indefinite integrals and definite integrals. In this chapter, we study indefinite integration, the process of obtaining a function from its derivative.

We are already familiar with inverse operations. $(+,-)$, $(\times,\div)$, $\left((\ )^n, \sqrt[n]{\ }\right)$ are some pairs of inverse operations. Similarly differentiation and integrations $(d, \int)$ are also inverse operations. In this section we develop the inverse operation of differentiation called 'antidifferentiation'.

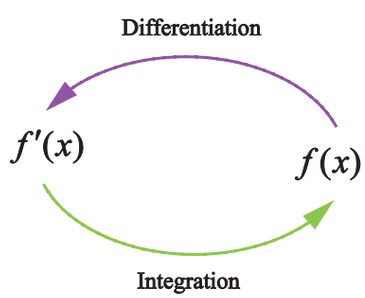

Fig. 11.4

Definition 11.1

A function $F(x)$ is called an **antiderivative** (Newton-Leibnitz integral or primitive) of a function $f(x)$ on an interval I if

$$F'(x) = f(x), \text{ for every value of } x \text{ in } I$$

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Illustration 11.1

If $F(x) = x^2 + 5$ then

$F'(x) = 2x$.

Thus if $f(x)$ is defined by

$f(x) = 2x$, then

we say that $f(x)$ is the derivative of $F(x)$ and that $F(x)$ is an antiderivative of $f(x)$

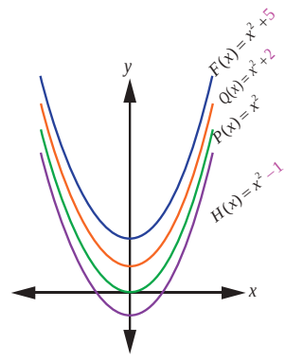

Fig. 11.5

Consider the following table

$F(x)$	$F'(x) = f(x)$	Antiderivative of $f(x) = 2x$
$P(x) = x^2 + 0$ $Q(x) = x^2 + 2$ $H(x) = x^2 - 1$	$P'(x) = 2x$ $Q'(x) = 2x$ $H'(x) = 2x$ $\Big\}$ $f(x) = 2x$	$F(x) = x^2 + ?$

We can see that the derivative of $F(x), P(x), Q(x)$ and $H(x)$ is $f(x)$, but in reverse the antiderivatives of $f(x) = 2x$ is not unique. That is the antiderivatives of $f(x)$ is a family of infinitely many functions.

Theorem 11.1

If $F(x)$ is a particular antiderivative of a function $f(x)$ on an interval *I*, then every antiderivative of $f(x)$ on *I* is given by

$$\int f(x)dx = F(x) + c$$

where c is called an arbitrary constant, and all antiderivatives of $f(x)$ on *I* can be obtained by assigning particular value to c.

The function ***f(x)*** is called **Integrand**.

The variable ***x*** in **dx** is called **variable of integration** or **integrator**.

The process of finding the integral is called **integration** or **antidifferentiation** (Newton-Leibnitz integral).

The peculiar integral sign $\int$ originates in an elongated S (like Σ) which stands for sum.

Often in applications involving differentiation it is desired to find a particular integral antiderivative that satisfies certain conditions called initial condition or boundary conditions.

For instance, if an equation involving $\frac{dy}{dx}$ is given as well as the initial condition that $y = y_1$ when $x = x_1$, then after the set of all antiderivatives is found, if x and y are replaced by x_1 and y_1, a particular value of the arbitrary constant is determined. With this value of c a particular antiderivative is obtained.

Illustration 11.2

Suppose we wish to find the particular antiderivative satisfying the equation

$$\frac{dy}{dx} = 2x$$

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and the initial condition that $y = 10$ when $x = 2$.

From the given equation

$$\frac{dy}{dx} = 2x$$

$$y = \int 2x\,dx$$

$$y = x^2 + c$$

We substitute $y = 10$ when $x = 2$, in the above equation

$$10 = 2^2 + c \;\Rightarrow\; c = 6$$

When this value c =6 is substituted we obtain

$$y = x^2 + 6$$

which gives the particular antiderivative desired.

11.3 Basic Rules of Integration

Standard results:

Since integration is the reverse process of differentiation, the basic integration formulae given below can be derived directly from their corresponding derivative formulae from earlier chapter.

Derivatives	Antiderivatives
$\frac{d}{dx}(c) = 0,$ where c is a constant	$\int 0\,dx = c$, where c is a constant
$\frac{d}{dx}(kx) = k,$ where k is a constant	$\int k\,dx = kx + c$ where c is a constant
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$ (Power rule)
$\frac{d}{dx}(\log x) = \left(\frac{1}{x}\right)$	$\int \frac{1}{x}\,dx = \log\lvert x\rvert + c$
$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$

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$\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x \, dx = -\cot x + c$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + c$
$\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$	$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$	$\int a^x dx = \frac{a^x}{\log a} + c$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

Example 11.1

Integrate the following with respect to x.

(i) x^{10} (ii) $\frac{1}{x^{10}}$ (iii) $\sqrt{x}$ (iv) $\frac{1}{\sqrt{x}}$

Solution

(i) We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1.$

Putting $n = 10$, we get

$$\int x^{10} dx = \frac{x^{10+1}}{10+1} + c = \frac{x^{11}}{11} + c$$

(ii) $\int \frac{1}{x^{10}} dx = \int x^{-10} dx = \frac{x^{-10+1}}{-10+1} + c = -\frac{1}{9x^9} + c$

(iii) $\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3} x^{\frac{3}{2}} + c$

(iv) $\int \frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2\sqrt{x} + c$

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Example 11.2

Integrate the following with respect to x.

(i) $\dfrac{1}{\cos^2 x}$ (ii) $\dfrac{\cot x}{\sin x}$ (iii) $\dfrac{\sin x}{\cos^2 x}$ (iv) $\dfrac{1}{\sqrt{1-x^2}}$

Solution

(i) $\displaystyle\int \frac{1}{\cos^2 x}dx = \int \sec^2 x\,dx = \tan x + c$

(ii) $\displaystyle\int \frac{\cot x}{\sin x}dx = \int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

(iii) $\displaystyle\int \frac{\sin x}{\cos^2 x}dx = \int \frac{\sin x}{\cos x}.\frac{1}{\cos x}dx = \int \tan x \sec x\,dx = \sec x + c$

(iv) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + c$

Example 11.3

Integrate the following with respect to x:

(i) $\dfrac{1}{e^{-x}}$ (ii) $\dfrac{x^2}{x^3}$ (iii) $\dfrac{1}{x^3}$ (iv) $\dfrac{1}{1+x^2}$

Solution

(i) $\displaystyle\int \frac{1}{e^{-x}}\,dx = \int e^x dx = e^x + c$

(ii) $\displaystyle\int \frac{x^2}{x^3}dx = \int \frac{1}{x}dx = \log|x| + c$

(iii) $\displaystyle\int \frac{1}{x^3}dx = \int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + c = -\frac{1}{2x^2} + c$

(iv) $\displaystyle\int \frac{1}{1+x^2}dx = \tan^{-1} x + c$

EXERCISE 11.1

Integrate the following with respect to x:

(1) (i) x^{11} (ii) $\dfrac{1}{x^7}$ (iii) $\sqrt[3]{x^4}$ (iv) $(x^5)^{\frac{1}{8}}$

(2) (i) $\dfrac{1}{\sin^2 x}$ (ii) $\dfrac{\tan x}{\cos x}$ (iii) $\dfrac{\cos x}{\sin^2 x}$ (iv) $\dfrac{1}{\cos^2 x}$

(3) (i) 12^3 (ii) $\dfrac{x^{24}}{x^{25}}$ (iii) e^x

(4) (i) $(1+x^2)^{-1}$ (ii) $\left(1-x^2\right)^{-\frac{1}{2}}$

11.4 Integrals of the Form $\int f(ax+b)dx$

We know that

$$\frac{d}{dx}\left[\frac{(x-a)^{10}}{10}\right] = (x-a)^9 \Rightarrow \int (x-a)^9 dx = \frac{(x-a)^{10}}{10} + c$$

$$\frac{d}{dx}[\sin(x+k)] = \cos(x+k) \Rightarrow \int \cos(x+k)dx = \sin(x+k) + c$$

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It is clear that whenever a constant is added or subtracted with the independent variable x, the fundamental formulae remain the same.

But

$$\frac{d}{dx}\left[\frac{1}{l}(e^{lx+m})\right] = e^{lx+m} \Rightarrow \int e^{lx+m}dx = \frac{1}{l}e^{(lx+m)} + c$$

$$\frac{d}{dx}\left[\frac{1}{a}\sin(ax+b)]\right] = \cos(ax+b) \Rightarrow \int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$$

Here, if any constant is multiplied with the independent variable x, then the same fundamental formula can be used after dividing it by the coefficient of x

That is, $\boxed{\text{if } \int f(x)dx = g(x)+c, \text{ then } \int f(ax+b)dx = \frac{1}{a}g(ax+b)+c}$

The above formula can also be derived by using substitution method, which will be studied later.

Example 11.4

Evaluate the following with respect to x:

(i) $\int (4x+5)^6\, dx$ (ii) $\int \sqrt{(15-2x)}\, dx$ (iii) $\int \frac{1}{(3x+7)^4} dx$

Solution

(i) $\int (4x+5)^6\, dx = \frac{1}{4}\frac{(4x+5)^{6+1}}{6+1} = \frac{(4x+5)^7}{28} + c$

(ii) $\int \sqrt{(15-2x)}\, dx = \int (15-2x)^{\frac{1}{2}}\, dx = \left(\frac{1}{-2}\right)\frac{(15-2x)^{\frac{1}{2}+1}}{(1/2)+1} = -\frac{(15-2x)^{\frac{3}{2}}}{3} + c$

(iii) $\int \frac{1}{(3x+7)^4} dx = \int (3x+7)^{-4}\, dx = \frac{1}{3}\frac{(3x+7)^{-4+1}}{-4+1} = -\frac{1}{9(3x+7)^3} + c$

Example 11.5

Integrate the following with respect to x:

(i) $\sin(2x + 4)$ (ii) $\sec^2(3+4x)$ (iii) $\operatorname{cosec}(ax+b)\cot(ax+b)$

Solution

(i) $\int \sin(2x + 4)\, dx = \left(\frac{1}{2}\right)(-\cos(2x+4)) + c = -\frac{1}{2}\cos(2x+4) + c$

(ii) $\int \sec^2(3+4x)\, dx = \frac{1}{4}\tan(3+4x) + c$

(iii) $\int \operatorname{cosec}(ax+b)\cot(ax+b)\, dx = \left(\frac{1}{a}\right)(-\operatorname{cosec}(ax+b)) + c = -\frac{1}{a}\operatorname{cosec}(ax+b) + c$

Example 11.6

Integrate the following with respect to x:

(i) e^{3x} (ii) e^{5-4x} (iii) $\frac{1}{(3x-2)}$ (iv) $\frac{1}{(5-4x)}$

Solution

(i) $\int e^{3x}\, dx = \frac{1}{3}e^{3x} + c$ (ii) $\int e^{5-4x}\, dx = -\frac{e^{5-4x}}{4} + c$

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(iii) $\int \frac{1}{(3x-2)}dx = \frac{1}{3}\log|(3x-2)|+c$ (iv) $\int \frac{1}{(5-4x)}dx = -\frac{1}{4}\log|(5-4x)|+c$

Example 11.7

Integrate the following with respect to x:

(i) $\frac{1}{1+(2x)^2}$ (ii) $\frac{1}{\sqrt{1-(9x)^2}}$ (iii) $\frac{1}{\sqrt{1-25x^2}}$

Solution

(i) $\int \frac{1}{1+(2x)^2}dx = \frac{1}{2}\tan^{-1}(2x)+c$ (ii) $\int \frac{1}{\sqrt{1-(9x)^2}}dx = \frac{1}{9}\sin^{-1}(9x)+c$

(iii) $\int \frac{1}{\sqrt{1-25x^2}}dx = \int \frac{1}{\sqrt{1-(5x)^2}}dx = \frac{1}{5}\sin^{-1}(5x)+c$

EXERCISE 11.2

Integrate the following functions with respect to x:

(1) (i) $(x+5)^6$ (ii) $\frac{1}{(2-3x)^4}$ (iii) $\sqrt{3x+2}$

(2) (i) $\sin 3x$ (ii) $\cos(5-11x)$ (iii) $\operatorname{cosec}^2(5x-7)$

(3) (i) e^{3x-6} (ii) e^{8-7x} (iii) $\frac{1}{6-4x}$

(4) (i) $\sec^2\frac{x}{5}$ (ii) $\operatorname{cosec}(5x+3)\cot(5x+3)$ (iii) $\sec(2-15x)\tan(2-15x)$

(5) (i) $\frac{1}{\sqrt{1-(4x)^2}}$ (ii) $\frac{1}{\sqrt{1-81x^2}}$ (iii) $\frac{1}{1+36x^2}$

11.5 Properties of Integrals

(1) If k is any constant, then $\int kf(x)dx = k\int f(x)dx$

(2) $\int (f_1(x) \pm f_2(x))dx = \int f_1(x)dx \pm \int f_2(x)dx$

BDL7WU

Note 11.1

The above two properties can be combined and extended as

$$\int (k_1 f_1(x) \pm k_2 f_2(x) \pm k_3 f_3(x) \pm \cdots \pm k_n f_n(x))\, dx$$

$$= k_1\int f_1(x)dx \pm k_2\int f_2(x)dx \pm k_3\int f_3(x)dx \pm \cdots \pm k_n\int f_n(x)dx.$$

That is, the integration of the linear combination of a finite number of functions is equal to the linear combination of their integrals

Example 11.8

Integrate the following with respect to x:

(i) $5x^4$ (ii) $5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}$ (iii) $2\cos x - 4\sin x + 5\sec^2 x + \operatorname{cosec}^2 x$

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Solution

(i) $\int 5x^4 dx = 5\int x^4 dx = 5\frac{x^{4+1}}{4+1} = 5\frac{x^5}{5} = x^5 + c.$

(ii) $\int\left(5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right)dx = 5\int x^2 dx - 4\int dx + 7\int \frac{1}{x}dx + 2\int \frac{1}{\sqrt{x}}dx$

$$= 5\frac{x^{2+1}}{2+1} - 4x + 7\log|x| + 2\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{5}{3}x^3 - 4x + 7\log|x| + 4\sqrt{x} + c$$

(iii) $\int(2\cos x - 4\sin x + 5\sec^2 x + \operatorname{cosec}^2 x)dx$

$$= 2\int \cos x\, dx - 4\int \sin x\, dx + 5\int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx$$

$$= 2\sin x + 4\cos x + 5\tan x - \cot x + c$$

Example 11.9

Evaluate the following integrals:

(i) $\frac{12}{(4x-5)^3} + \frac{6}{3x+2} + 16e^{4x+3}$ (ii) $\frac{15}{\sqrt{5x-4}} - 8\cot(4x+2)\operatorname{cosec}(4x+2)$

Solution

(i) $\int\left(\frac{12}{(4x-5)^3} + \frac{6}{3x+2} + 16e^{4x+3}\right)dx$

$$= 12\int \frac{1}{(4x-5)^3}dx + 6\int \frac{1}{3x+2}dx + 16\int e^{4x+3}dx$$

$$= 12\left(\frac{1}{4}\right)\left(-\frac{1}{2(4x-5)^2}\right) + 6\left(\frac{1}{3}\right)\log|3x+2| + 16\left(\frac{1}{4}\right)e^{4x+3} + c$$

$$= -\frac{3}{2(4x-5)^2} + 2\log|3x+2| + 4e^{4x+3} + c.$$

(ii) $\int\left(\frac{15}{\sqrt{5x-4}} - 8\cot(4x+2)\operatorname{cosec}(4x+2)\right)dx$

$$= 15\int \frac{1}{\sqrt{5x-4}}dx - 8\int \cot(4x+2)\operatorname{cosec}(4x+2)\,dx$$

$$= 15\left(\frac{1}{5}\right)(2\sqrt{5x-4}) - 8\left(\frac{1}{4}\right)(-\operatorname{cosec}(4x+2)) + c$$

$$= 6\sqrt{5x-4} + 2\operatorname{cosec}(4x+2) + c.$$

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EXERCISE 11.3

Integrate the following with respect to x:

(1) $(x+4)^5+\dfrac{5}{(2-5x)^4}-\operatorname{cosec}^2(3x-1)$

(2) $4\cos(5-2x)+9e^{3x-6}+\dfrac{24}{6-4x}$

(3) $\sec^2\dfrac{x}{5}+18\cos 2x+10\sec(5x+3)\tan(5x+3)$

(4) $\dfrac{8}{\sqrt{1-(4x)^2}}+\dfrac{27}{\sqrt{1-9x^2}}-\dfrac{15}{1+25x^2}$

(5) $\dfrac{6}{1+(3x+2)^2}-\dfrac{12}{\sqrt{1-(3-4x)^2}}$

(6) $\dfrac{1}{3}\cos\left(\dfrac{x}{3}-4\right)+\dfrac{7}{7x+9}+e^{\frac{x}{5}+3}$

11.6 Simple applications

So far in this section we have been using x as the variable of integration. In the case of applications, it is often convenient to use a different variable. For instance in the equation of motion the independent variable is time and the variable of integration is t.

In this section we discuss how integration is used to find the position and velocity of an object, given its acceleration and similar types of problems. Mathematically, this means that, starting with the derivative of a function, we must find the original function. Many common word which indicate derivative such as rate, growth, decay, marginal, change, varies, increase, decrease etc.

Example 11.10

If $f'(x)=3x^2-4x+5$ and $f(1)=3$, then find $f(x)$.

Solution

Given that $\quad f'(x)=\dfrac{d}{dx}\big(f(x)\big)=3x^2-4x+5$

Integrating on both sides with respect to x, we get

$$\int f'(x)\,dx=\int(3x^2-4x+5)\,dx$$
$$f(x)=x^3-2x^2+5x+c$$

To determine the constant of integration c, we have to apply the given information $f(1)=3$

$$f(1)=3\Rightarrow\ 3=(1)^3-2(1)^2+5(1)+c\Rightarrow c=-1$$

Thus $\quad f(x)=x^3-2x^2+5x-1.$

Example 11.11

A train started from Madurai Junction towards Coimbatore at 3pm (time $t=0$) with velocity $v(t)=20t+50$ kilometre per hour, where t is measured in hours. Find the distance covered by the train at 5pm.

Solution

In calculus terminology, velocity $v=\dfrac{ds}{dt}$ is rate of change of position with time, where s is the distance.The velocity of the train is given by

$$v(t)=20t+50$$

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Therefore, $\frac{ds}{dt} = 20t + 50$

To find the distance function s one has to integrate the derivative function.

That is, $s = \int (20t + 50)\, dt$

$$s = 10t^2 + 50t + c$$

The distance covered by the train is zero when time is zero. Let us use this initial condition $s = 0$ at $t = 0$ to determine the value c of the constant of integration.

$$\Rightarrow s = 10t^2 + 50t + c \quad \Rightarrow c = 0$$

Therefore, $s = 10t^2 + 50t$

The distance covered by the train in 2 hours (5pm-3pm) is given by substituting $t = 2$ in the above equation, we get

$$s = 10(2)^2 + 50(2) = 140 \text{ km.}$$

Example 11.12

The rate of change of weight of person w in kg with respect to their height h in centimetres is given approximately by $\frac{dw}{dh} = 4.364 \times 10^{-5} h^2$. Find weight as a function of height. Also find the weight of a person whose height is 150 cm.

Solution

The rate of change of weight with respect to height is

$$\frac{dw}{dh} = 4.364 \times 10^{-5} h^2$$

$$w = \int 4.364 \times 10^{-5} h^2 \; dh$$

$$w = 4.364 \times 10^{-5} \left(\frac{h^3}{3} \right) + c$$

One can obviously understand that the weight of a person is zero when height is zero.

Let us find the value c of the constant of integration by substituting the initial condition $w = 0$, at $h = 0$, in the above equation

$$w = 4.364 \times 10^{-5} \left(\frac{h^3}{3} \right) + c \Rightarrow c = 0$$

The required relation between weight and height of a person is

$$w = 4.364 \times 10^{-5} \left(\frac{h^3}{3} \right)$$

When the height h =150cm,

$$w = 4.364 \times 10^{-5} \left(\frac{150^3}{3} \right)$$

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When the height h = 150cm, the weight is $w = 49kg$ (approximately)
Therefore, the weight of the person whose height 150cm is 49 kg.

Example 11.13

A tree is growing so that, after t - years its height is increasing at a rate of $\frac{18}{\sqrt{t}}$ cm per year. Assume that when t = 0, the height is 5 cm.

(i) Find the height of the tree after 4 years.

(ii) After how many years will the height be 149 cm?

Solution

The rate of change of height h with respect to time t is the derivative of h with respect to t.

$$\text{Therefore, } \frac{dh}{dt} = \frac{18}{\sqrt{t}} = 18t^{-\frac{1}{2}}$$

So, to get a general expression for the height, integrating the above equation with respect to t.

$$h = \int 18t^{-\frac{1}{2}}dt = 18(2t^{\frac{1}{2}}) + c = 36\sqrt{t} + c$$

Given that when t = 0, the height h = 5 cm.

$$5 = 0 + c \Rightarrow c = 5$$

$$h = 36\sqrt{t} + 5.$$

(i) To find the height of the tree after 4 years.

When t = 4 years,

$$h = 36\sqrt{t} + 5 \Rightarrow h = 36\sqrt{4} + 5 = 77$$

The height of the tree after 4 years is 77 cm

(ii) When h = 149cm

$$h = 36\sqrt{t} + 5 \Rightarrow 149 = 36\sqrt{t} + 5$$

$$\sqrt{t} = \frac{149-5}{36} = 4 \Rightarrow t = 16$$

Thus after 16 years the height of the tree will be 149 cm.

Example 11.14

At a particular moment, a student needs to stop his speedy bike to avoid a collision with the barrier ahead at a distance 40 metres away from him. Immediately he slows (retardation) the bike under braking at a rate of 8 metre/second^2. If the bike is moving at a speed of 24m/s, when the brakes are applied, would it stop before collision?

Solution

Let a be the acceleration, v be the velocity of the car, and s be the distance.

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Stated in calculus terminology, velocity, $v = \dfrac{ds}{dt}$, is the rate of change of position with time,

and acceleration, $a = \dfrac{dv}{dt}$, is rate of change of velocity with time.

The acceleration to be negative because if you take the direction of movement to be positive, then for a bike that is slowing down, its acceleration vector will be oriented in the opposite direction of its motion (retardation).

Given that the retardation of the car is 8 meter/second^2.

$$\text{Therefore, } a = \frac{dv}{dt} = -8 \text{ meter/second}^2.$$

$$\text{Therefore, } v = \int a\,dt = \int -8dt = -8t + c_1$$

$$v = -8t + c_1.$$

When the brakes are applied,

$$t = 0, \text{ and } v = 24\text{m/s}.$$

$$\text{So, } 24 = -8(0) + c_1 \Rightarrow c_1 = 24$$

$$\text{Therefore, } v = -8t + 24.$$

$$\text{That is, } \frac{ds}{dt} = -8t + 24.$$

It is required to find the distance, not the velocity, so need more integration in order.

$$s = \int v\,dt = \int (-8t + 24)dt$$

$$s = -4t^2 + 24t + c_2$$

To determine c_2, the stopping distance s is measured from where, and when, the brakes are applied so that at $t = 0,\ s = 0$.

$$s = -4t^2 + 24t + c_2 \Rightarrow 0 = -4(0)^2 + 24(0) + c_2 \Rightarrow c_2 = 0$$

$$s = -4t^2 + 24t$$

The stopping distance s could be evaluated if we knew the braking time. The time can be determined from the speed statement.

The bike stops when $v = 0$, $\Rightarrow v = -8t + 24 \Rightarrow 0 = -8t + 24 \Rightarrow t = 3$.

When $t = 3$, we get

$$s = -4t^2 + 24t \Rightarrow s = -4(3)^2 + 24(3)$$

$$s = 36 \text{ metres} < 40 \text{ metres}$$

The bike stops at a distance 4 metres to the barrier.

EXERCISE 11.4

(1) If $f'(x) = 4x - 5$ and $f(2) = 1$, find $f(x)$.

(2) If $f'(x) = 9x^2 - 6x$ and $f(0) = -3$, find $f(x)$.

(3) If $f''(x) = 12x - 6$ and $f(1) = 30$, $f'(1) = 5$ find $f(x)$.

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(4) A ball is thrown vertically upward from the ground with an initial velocity of 39.2 m/sec. If the only force considered is that attributed to the acceleration due to gravity, find

(i) how long will it take for the ball to strike the ground?

(ii) the speed with which will it strike the ground? and

(iii) how high the ball will rise?

(5) A wound is healing in such a way that t days since Sunday the area of the wound has been decreasing at a rate of $-\frac{6}{(t+2)^2}$ cm^2 per day where $0<t\leq 8$. If on Monday the area of the wound was 1.4 cm^2

(i) What was the area of the wound on Sunday?

(ii) What is the anticipated area of the wound on Thursday if it continues to heal at the same rate?

11.7 Methods of Integration

Integration is not as easy as differentiation. This is first due to its nature. Finding a derivative of a given function is facilitated by the fact that the differentiation itself has a constructive character. A derivative is simply defined as

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Suppose we are asked to find the derivative of $\log x$, we know in all details how to proceed in order to obtain the result.

When we are asked to find the integral of $\log x$, we have no constructive method to find integral or even how to start.

In the case of differentiation we use the laws of differentiation of several functions in order to find derivatives of their various combinations, like their sum, product, quotient, composition of functions etc.

There are very few such rules available in the theory of integration and their application is rather restricted. But the significance of these methods of integration is very great.

In every case one must learn to select the most appropriate method and use it in the most convenient form. This skill can only be acquired after long practice.

Already we have seen two important properties of integration. The following are the four important methods of integrations.

(1) Integration by decomposition into sum or difference.

(2) Integration by substitution.

(3) Integration by parts

(4) Integration by successive reduction.

BDV3YH

Here we discuss only the first three methods of integration and the other will be studied in higher classes.

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11.7.1 Decomposition method

Sometimes it is very difficult to integrate the given function directly. But it can be integrated after decomposing it into a sum or difference of number of functions whose integrals are already known.

For example $\left(1-x^3\right)^2$, $\dfrac{x^2-x+1}{x^3}$, $\cos 5x \sin 3x$, $\cos^3 x$, $\dfrac{e^{2x}-1}{e^x}$, do not have direct formulae to integrate. But these functions can be decomposed into a sum or difference of functions, whose individual integrals are known. In most of the cases the given integrand will be any one of the algebraic, trigonometric or exponential forms, and sometimes combinations of these functions.

Example 11.15

Integrate the following with respect to x:

(i) $\left(1-x^3\right)^2$ (ii) $\dfrac{x^2-x+1}{x^3}$

Solution

(i)
$$\begin{aligned}\int\left(1-x^3\right)^2 dx &= \int(1-2x^3+x^6)dx\\ &= \int dx-2\int x^3dx+\int x^6dx\\ &= x-\frac{x^4}{2}+\frac{x^7}{7}+c.\end{aligned}$$

(ii)
$$\begin{aligned}\int\frac{x^2-x+1}{x^3}dx &= \int\left(\frac{x^2}{x^3}-\frac{x}{x^3}+\frac{1}{x^3}\right)dx\\ &= \int\frac{1}{x}dx-\int\frac{1}{x^2}dx+\int\frac{1}{x^3}dx.\end{aligned}$$

$$\int\frac{x^2-x+1}{x^3}dx = \log|x|+\frac{1}{x}-\frac{1}{2x^2}+c.$$

Example 11.16

Integrate the following with respect to x:

(i) $\cos 5x \sin 3x$ (ii) $\cos^3 x$.

Solution

(i)
$$\begin{aligned}\int\cos 5x\sin 3x\,dx &= \frac{1}{2}\int 2\cos 5x\sin 3x\,dx\\ &= \frac{1}{2}\int\left(\sin 8x-\sin 2x\right)dx\end{aligned}$$

$$\int\cos 5x\sin 3x\,dx = \frac{1}{2}\left(-\frac{\cos 8x}{8}+\frac{\cos 2x}{2}\right)+c.$$

(ii)
$$\begin{aligned}\int\cos^3 x\,dx &= \frac{1}{4}\int\left(3\cos x+\cos 3x\right)dx\\ &= \frac{1}{4}\left(3\sin x+\frac{\sin 3x}{3}\right)+c\end{aligned}$$

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Example 11.17

Integrate the following with respect to x:

(i) $\dfrac{e^{2x}-1}{e^{x}}$ (ii) $e^{3x}\left(e^{2x}-1\right)$.

Solution

(i)
$$\int \frac{e^{2x}-1}{e^{x}}dx = \int\left(\frac{e^{2x}}{e^{x}}-\frac{1}{e^{x}}\right)dx$$
$$= \int\left(e^{x}-e^{-x}\right)dx = e^{x}+e^{-x}+c.$$

(ii)
$$\int e^{3x}\left(e^{2x}-1\right)dx = \int\left(e^{5x}-e^{3x}\right)dx = \frac{e^{5x}}{5}-\frac{e^{3x}}{3}+c.$$

Example 11.18

Evaluate : $\displaystyle\int \frac{1}{\sin^2 x\cos^2 x}dx.$

Solution

$$\int \frac{1}{\sin^2 x\cos^2 x}dx = \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}dx$$
$$= \int \frac{1}{\cos^2 x}dx+\int \frac{1}{\sin^2 x}dx$$
$$= \int \sec^2 x dx+\int \operatorname{cosec}^2 x dx$$
$$= \tan x-\cot x+c.$$

Example 11.19

Evaluate : $\displaystyle\int \frac{\sin x}{1+\sin x}dx.$

Solution

$$\int \frac{\sin x}{1+\sin x}dx = \int\left(\frac{\sin x}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right)dx$$
$$= \int \frac{\sin x-\sin^2 x}{1-\sin^2 x}dx = \int \frac{\sin x-\sin^2 x}{\cos^2 x}dx = \int \frac{\sin x}{\cos^2 x}dx-\int \frac{\sin^2 x}{\cos^2 x}dx$$
$$= \int \tan x\sec x dx-\int \tan^2 x dx$$
$$= \int \tan x\sec x dx-\int (\sec^2 x-1)dx$$
$$= \sec x-\tan x+x+c.$$

Example 11.20

Evaluate : $\displaystyle\int \sqrt{1+\cos 2x}\, dx.$

Solution

$$\int \sqrt{1+\cos 2x}\, dx = \int \sqrt{2\cos^2 x}\, dx = \sqrt{2}\int \cos x dx = \sqrt{2}\sin x+c$$

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Example 11.21

Evaluate : $\int \frac{(x-1)^2}{x^3+x}\,dx.$

Solution

$$\begin{aligned}\int \frac{(x-1)^2}{x^3+x}\,dx &= \int \frac{x^2+1-2x}{x(x^2+1)}\,dx \\ &= \int \left(\frac{(x^2+1)}{x(x^2+1)} - \frac{2x}{x(x^2+1)}\right) dx \\ &= \int \frac{1}{x}\,dx - 2\int \frac{1}{1+x^2}\,dx \\ &= \log|x| - 2\tan^{-1} x + c.\end{aligned}$$

Example 11.22

Evaluate : $\int (\tan x + \cot x)^2\,dx.$

Solution

$$\begin{aligned}\int (\tan x + \cot x)^2\,dx &= \int [\tan^2 x + 2\tan x \cot x + \cot^2 x]\,dx \\ &= \int [(\sec^2 x - 1) + 2 + (\operatorname{cosec}^2 x - 1)]\,dx \\ &= \int (\sec^2 x + \operatorname{cosec}^2 x)\,dx \\ &= \tan x + (-\cot x) + c \\ &= \tan x - \cot x + c.\end{aligned}$$

Example 11.23

Evaluate : $\int \frac{1-\cos x}{1+\cos x}\,dx.$

Solution

$$\begin{aligned}\int \frac{1-\cos x}{1+\cos x}\,dx &= \int \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}}\,dx = \int \tan^2 \frac{x}{2}\,dx \\ &= \int \left(\sec^2 \frac{x}{2} - 1\right) dx = \frac{\tan \frac{x}{2}}{\frac{1}{2}} - x + c \\ &= 2\tan \frac{x}{2} - x + c.\end{aligned}$$

Example 11.24

Evaluate : $\int \sqrt{1+\sin 2x}\;dx.$

Solution

$$\int \sqrt{1+\sin 2x}\;dx = \int \sqrt{(\cos^2 x + \sin^2 x) + (2\sin x \cos x)}\;dx$$

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$$= \int \sqrt{(\cos x + \sin x)^2}\, dx = \int (\cos x + \sin x) dx$$

$$= \sin x - \cos x + c$$

Example 11.25

Evaluate : $\int \frac{x^3+2}{x-1} dx.$

Solution

$$\int \frac{x^3+2}{x-1} dx = \int \frac{x^3-1+3}{x-1} dx = \int \left(\frac{x^3-1}{x-1} + \frac{3}{x-1} \right) dx$$

$$= \int \left[\frac{(x-1)(x^2+x+1)}{x-1} + \frac{3}{x-1} \right] dx$$

$$= \int \left(x^2 + x + 1 + \frac{3}{x-1} \right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + x + 3\log|(x-1)| + c.$$

Example 11.26

Evaluate : (i) $\int a^x e^x dx$ (ii) $\int e^{x\log 2} e^x\, dx.$

Solution

(i) $$\int a^x e^x dx = \int (ae)^x dx = \frac{(ae)^x}{\log(ae)} + c$$

(ii) $$\int e^{x\log 2} e^x dx = \int e^{\log 2^x} e^x dx = \int 2^x e^x dx$$

$$= \int (2e)^x dx = \frac{(2e)^x}{\log(2e)} + c.$$

Example 11.27

Evaluate : $\int (x-3)\sqrt{x+2}\, dx.$

Solution

$$\int (x-3)\sqrt{x+2}\, dx = \int (x+2-5)\sqrt{x+2}\, dx$$

$$= \int (x+2)\sqrt{x+2}\, dx - 5\int \sqrt{x+2}\, dx$$

$$= \int (x+2)^{\frac{3}{2}} dx - 5\int (x+2)^{\frac{1}{2}} dx$$

$$= \frac{(x+2)^{\frac{5}{2}}}{\frac{5}{2}} - 5\frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{10}{3}(x+2)^{\frac{3}{2}} + c.$$

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Example 11.28

Evaluate : $\int \frac{1}{\sqrt{x+1}+\sqrt{x}}dx.$

Solution

$$\int \frac{1}{\sqrt{x+1}+\sqrt{x}}dx = \int \frac{1}{\sqrt{x+1}+\sqrt{x}}\left[\frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\right]dx$$

$$= \int \frac{\sqrt{x+1}-\sqrt{x}}{\left(\sqrt{x+1}^2\right)-\left(\sqrt{x}\right)^2}dx$$

$$= \int \frac{\sqrt{x+1}-\sqrt{x}}{x+1-x}dx = \int\left(\sqrt{x+1}-\sqrt{x}\right)dx$$

$$= \int \sqrt{x+1}dx - \int \sqrt{x}dx = \int (x+1)^{\frac{1}{2}}dx - \int x^{\frac{1}{2}}dx$$

$$= \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{3}\left[(x+1)^{\frac{3}{2}} - x^{\frac{3}{2}}\right] + c.$$

11.7.2 Decomposition by Partial Fractions

One of the important methods to evaluate integration is partial fractions. If the integrand is in the form of an algebraic fraction and the integral cannot be evaluated by simple methods, then the fraction need to be expressed in **partial fractions** before integration takes place. We will assume that we have a rational function $\frac{p(x)}{q(x)}$, $\left(q(x) \neq 0\right)$ in which degree of $p(x)$ < degree of $q(x)$. If this is not the case, we can always perform long division.

Example 11.29

Evaluate : (i) $\int \frac{3x+7}{x^2-3x+2}dx$ (ii) $\int \frac{x+3}{(x+2)^2(x+1)}dx.$

Solution

(i) $$\int \frac{3x+7}{x^2-3x+2}dx = \int \frac{13}{x-2}dx - \int \frac{10}{x-1}dx \quad \left\{\begin{array}{l}\text{Resolving into}\\ \text{partial fractions}\end{array}\right.$$

$$= 13\log|x-2| - 10\log|x-1| + c$$

(ii) $$\int \frac{x+3}{(x+2)^2(x+1)}dx = \int \frac{-2}{x+2}dx - \int \frac{1}{(x+2)^2}dx + \int \frac{2}{x+1}dx \quad \left\{\begin{array}{l}\text{Resolving into}\\ \text{partial fractions}\end{array}\right.$$

$$= -2\int \frac{1}{x+2}dx - \int \frac{1}{(x+2)^2}dx + 2\int \frac{1}{x+1}dx$$

$$= -2\log|x+2| - \int (x+2)^{-2}dx + 2\log|x+1| + c$$

$$= -2\log|x+2| + \frac{1}{x+2} + 2\log|x+1| + c.$$

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EXERCISE 11.5

Integrate the following functions with respect to x :

(1) $\dfrac{x^3+4x^2-3x+2}{x^2}$ (2) $\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^2$ (3) $(2x-5)(36+4x)$

(4) $\cot^2 x+\tan^2 x$ (5) $\dfrac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$ (6) $\dfrac{\cos 2x}{\sin^2 x\cos^2 x}$

(7) $\dfrac{3+4\cos x}{\sin^2 x}$ (8) $\dfrac{\sin^2 x}{1+\cos x}$ (9) $\dfrac{\sin 4x}{\sin x}$

(10) $\cos 3x\cos 2x$ (11) $\sin^2 5x$ (12) $\dfrac{1+\cos 4x}{\cot x-\tan x}$

(13) $e^{x\log a}e^x$ (14) $(3x+4)\sqrt{3x+7}$ (15) $\dfrac{8^{1+x}+4^{1-x}}{2^x}$

(16) $\dfrac{1}{\sqrt{x+3}-\sqrt{x-4}}$ (17) $\dfrac{x+1}{(x+2)(x+3)}$ (18) $\dfrac{1}{(x-1)(x+2)^2}$

(19) $\dfrac{3x-9}{(x-1)(x+2)(x^2+1)}$ (20) $\dfrac{x^3}{(x-1)(x-2)}$

11.7.3 Method of substitution or change of variable

The method of substitution in integration is similar to finding the derivative of function of function in differentiation. By using a suitable substitution, the variable of integration is changed to new variable of integration which will be integrated in an easy manner.

We know that, if u is a function of x then $\dfrac{du}{dx}=u'$.

Hence we can write $\int f(u)u'dx=\int f(u)du$

Thus, $\int f[g(x)]g'(x)dx=\int f(u)du$, where $u=g(x)$

The success of the above method depends on the selection of suitable substitution either $x=\phi(u)$ or $u=g(x)$.

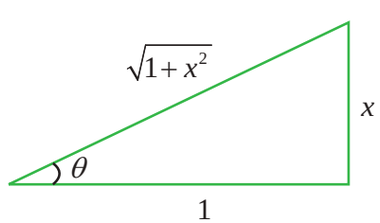
Note 11.2

The substitution for the variable of integration is in trigonometric function, use a rough diagram to find the re -substitution value for it. Suppose the variable of integration x is substituted as $x=\tan\theta$. After integration suppose the solution is $\sec\theta+\operatorname{cosec}\theta$

For example, if $x=\tan\theta$, then from the figure

$\operatorname{cosec}\theta=\left(\dfrac{\sqrt{1+x^2}}{x}\right)$,

$\sec\theta=\left(\dfrac{\sqrt{1+x^2}}{1}\right)$



Then $\sec\theta+\operatorname{cosec}\theta=\sqrt{1+x^2}+\left(\dfrac{\sqrt{1+x^2}}{x}\right)$.

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Example 11.30

Evaluate the following integrals :

(i) $\int 2x\sqrt{1+x^2}\,dx$ (ii) $\int e^{-x^2}x\,dx$ (iii) $\int \frac{\sin x}{1+\cos x}dx$

(iv) $\int \frac{1}{1+x^2}dx$ (v) $\int x(a-x)^8\,dx$

Solution

(i) $\int 2x\sqrt{1+x^2}\,dx$

Putting $1+x^2=u$, then $2x\,dx=du$

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{u}\,du$$

$$= \int u^{\frac{1}{2}}du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}}+c = \frac{2}{3}u^{\frac{3}{2}}+c = \frac{2}{3}(1+x^2)^{\frac{3}{2}}+c\,.$$

(ii) $\int e^{-x^2}x\,dx$

Putting $x^2=u$ then $2x\,dx=du$

Therefore, $$\int e^{-x^2}x\,dx = \int e^{-u}\frac{du}{2}$$

$$= \frac{1}{2}\int e^{-u}du = \frac{1}{2}(-e^{-u})+c = -\frac{1}{2}e^{-u}+c = -\frac{1}{2}e^{-x^2}+c.$$

(iii) $\int \frac{\sin x}{1+\cos x}dx$

Putting $1+\cos x=u$, then $-\sin x\,dx=du$

Therefore, $$\int \frac{\sin x}{1+\cos x}dx = \int \frac{-du}{u} = -\log|u|+c = -\log|1+\cos x|+c.$$

(iv) $\int \frac{1}{1+x^2}dx$

Putting $x=\tan u$, then $dx=\sec^2 u\,du$

$$\int \frac{1}{1+x^2}dx = \int \frac{\sec^2 u}{1+\tan^2 u}du = \int \frac{\sec^2 u}{\sec^2 u}du = \int du = u+c$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1}x+c.$$

(v) $\int x(a-x)^8\,dx$

Putting $u=a-x$, then $du=-dx$

$$\int x(a-x)^8\,dx = \int x(a-x)^8\,dx$$

$$= \int (a-u)(u)^8(-du)$$

$$= \int \left(-a(u)^8+u^9\right)du$$

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$$= \int u^9 du - a\int u^8 du$$

$$= \frac{u^{10}}{10} - a\frac{u^9}{9} + c$$

$$\int x(a-x)^8\, dx = \frac{(a-x)^{10}}{10} - \frac{a(a-x)^9}{9} + c.$$

11.7.4 Important Results

(1) $$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$$

(2) $$\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c, \quad n \neq -1$$

Proof

(1) Let $I = \int \frac{f'(x)}{f(x)} dx$

Putting $f(x) = u$ then $f'(x)dx = du$

Thus, $I = \int \frac{du}{u} = \log|u| + c$

Therefore, $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c.$

(2) Let $I = \int f'(x)[f(x)]^n dx$

Putting $f(x) = u$ then $f'(x)dx = du$

Thus, $I = \int u^n du = \frac{u^{n+1}}{n+1} + c$

Therefore, $\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c.$

Example 11.31

Integrate the following with respect to x.

(i) $\int \tan x\, dx$ (ii) $\int \cot x\, dx$ (iii) $\int \operatorname{cosec} x\, dx$ (iv) $\int \sec x\, dx$

Solution

(i) Let $I = \int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$

Putting $\cos x = u$ then, $-\sin x\, dx = du$

Thus, $I = \int -\frac{1}{u} du = -\log|u| + c = -\log|\cos x| + c = \log|\sec x| + c.$

(ii) Let $I = \int \cot x\, dx = \int \frac{\cos x}{\sin x} dx$

Putting $\sin x = u$ then, $\cos x\, dx = du$

Thus, $I = \int \frac{1}{u} du = \log|u| + c = \log|\sin x| + c.$

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(iii) Let $I = \int \text{cosec}\, x dx = \dfrac{\int \text{cosec}\, x(\text{cosec}\, x - \cot x)}{\text{cosec}\, x - \cot x} dx$

$$= \int \frac{\text{cosec}^2 x - \text{cosec} x \cot x}{\text{cosec} x - \cot x} dx$$

Putting $\text{cosec}\, x - \cot x = u$, then $(\text{cosec}^2 x - \text{cosec}\, x \cot x)\, dx = du$

Thus, $I = \int \frac{1}{u} du = \log|u| + c = \log|\text{cosec} x - \cot x| + c.$

(iv) Let $I = \int \sec x dx = \int \dfrac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \int \dfrac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

Putting $\sec x + \tan x = u$, then $(\sec^2 x + \sec x \tan x)\, dx = du$

Thus, $I = \int \frac{1}{u} du = \log|u| + c = \log|\sec x + \tan x| + c$

Therefore, $\int \sec x dx = \log|\sec x + \tan x| + c.$

Thus the following are the important standard results.

(1)	$\int \tan x\, dx = \log\|\sec x\| + c$
(2)	$\int \cot x\, dx = \log\|\sin x\| + c$
(3)	$\int \text{cosec} x\, dx = \log\|\text{cosec}\, x - \cot x\| + c$
(4)	$\int \sec x\, dx = \log\|\sec x + \tan x\| + c$

Example 11.32

Integrate the following with respect to x.

(i) $\int \dfrac{2x+4}{x^2+4x+6} dx$ (ii) $\int \dfrac{e^x}{e^x - 1} dx$ (iii) $\int \dfrac{1}{x \log x} dx$

(iv) $\int \dfrac{\sin x + \cos x}{\sin x - \cos x} dx$ (v) $\int \dfrac{\cos 2x}{(\sin x + \cos x)^2} dx$

Solution

(i) Let $I = \int \dfrac{2x+4}{x^2+4x+6} dx$

Putting $x^2 + 4x + 6 = u$, then $(2x+4)\, dx = du$

Thus, $I = \int \frac{du}{u} = \log|u| + c = \log|x^2 + 4x + 6| + c$

Therefore, $\int \dfrac{2x+4}{x^2+4x+6} dx = \log|x^2 + 4x + 6| + c.$

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(ii) Let $I = \int \frac{e^x}{e^x - 1} dx.$

Putting $e^x - 1 = u$, then $e^x dx = du$

Thus, $I = \int \frac{du}{u} = \log|u| + c = \log|e^x - 1| + c$

Therefore, $\int \frac{e^x}{e^x - 1} dx = \log|e^x - 1| + c.$

(iii) Let $I = \int \frac{1}{x \log x} dx.$

Putting $\log x = u$, then $\frac{1}{x} dx = du$

Thus, $I = \int \frac{du}{u} = \log|u| + c = \log|\log x| + c$

Therefore, $\int \frac{1}{x \log x} dx = \log|\log x| + c.$

(iv) Let $I = \int \frac{\sin x + \cos x}{\sin x - \cos x} dx.$

Putting $\sin x - \cos x = u$, then $(\cos x + \sin x)\, dx = du$

Thus, $I = \int \frac{du}{u} = \log|u| + c = \log|\sin x - \cos x| + c$

Therefore, $\int \frac{\sin x + \cos x}{\sin x - \cos x} dx = \log|\sin x - \cos x| + c$

(v) Let $I = \int \frac{\cos 2x}{(\sin x + \cos x)^2} dx = \int \frac{\cos 2x}{1 + \sin 2x} dx.$

Putting $1 + \sin 2x = u$, then $2\cos 2x\, dx = du$

Thus, $I = \int \frac{du}{2u} = \frac{1}{2}\log|u| + c = \frac{1}{2}\log|1 + \sin 2x| + c.$

EXERCISE 11.6

Integrate the following with respect to x

(1) $\frac{x}{\sqrt{1+x^2}}$ (2) $\frac{x^2}{1+x^6}$ (3) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$ (4) $\frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}}$

(5) $\frac{\sin\sqrt{x}}{\sqrt{x}}$ (6) $\frac{\cot x}{\log(\sin x)}$ (7) $\frac{\operatorname{cosec} x}{\log\left(\tan\frac{x}{2}\right)}$ (8) $\frac{\sin 2x}{a^2 + b^2 \sin^2 x}$

(9) $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$ (10) $\frac{\sqrt{x}}{1+\sqrt{x}}$ (11) $\frac{1}{x \log x \log(\log x)}$ (12) $\alpha\beta x^{\alpha-1} e^{-\beta x^\alpha}$

(13) $\tan x \sqrt{\sec x}$ (14) $x(1-x)^{17}$ (15) $\sin^5 x \cos^3 x$ (16) $\frac{\cos x}{\cos(x-a)}$

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11.7.5 Integration by parts

Integration by parts method is generally used to find the integral when the integrand is a product of two different types of functions or a single logarithmic function or a single inverse trigonometric function or a function which is not integrable directly. From the formula for derivative of product of two functions we obtain this useful method of integration.

If u and v are two differentiable functions then we have

$$d(uv) = vdu + udv$$

$$udv = d(uv) - vdu$$

Integrating

$$\int udv = \int d(uv) - \int vdu$$

$$\int udv = uv - \int vdu$$

$\int udv$ in terms of another integral $\int vdu$ and does not give a final expression for the integral $\int udv$. It only partially solves the problem of integrating the product $u\,dv$. Hence the term **'Partial Integration'** has been used in many European countries. The term **"Integration by Parts"** is used in many other countries as well as in our own.

The success of this method depends on the proper choice of u

(i) If integrand contains any non integrable functions directly from the formula, like log$x$, $\tan^{-1} x$ etc., we have to take these non integrable functions as u and other as dv.

(ii) If the integrand contains both the integrable function, and one of these is x^n (where n is a positive integer) then take $u = x^n$.

(iii) For other cases the choice of u is ours.

Example 11.33

Evaluate the following integrals

(i) $\int xe^x dx$ (ii) $\int x\cos x\,dx$ (iii) $\int \log x\,dx$ (iv) $\int \sin^{-1} x\,dx$

Solution

(i) Let $I = \int xe^x dx.$

Since x is an algebraic function and e^x is an exponential function,

so take $u = x$ then $du = dx$

$dv = e^x dx \Rightarrow v = e^x$

Applying Integration by parts, we get

$$\int udv = uv - \int vdu$$

$$\Rightarrow \int xe^x dx = xe^x - \int e^x dx$$

That is, $\int xe^x dx = xe^x - e^x + c.$

(ii) Let $I = \int x\cos x\,dx$

Since x is an algebraic function and cos x is a trigonometric function,

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so take $u = x$ then $du = dx$

$$dv = \cos x\,dx \Rightarrow v = \sin x$$

Applying Integration by parts, we get

$$\int u\,dv = uv - \int v\,du$$

$$\Rightarrow \int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$\Rightarrow \int x\cos x\,dx = x\sin x + \cos x + c$$

(iii) Let $I = \int \log x\,dx$

Take $u = \log x$ then $du = \frac{1}{x}dx$

$$dv = dx \Rightarrow v = x$$

Applying Integration by parts, we get

$$\int u\,dv = uv - \int v\,du$$

$$\Rightarrow \int \log x\,dx = x\log x - \int x\frac{1}{x}dx$$

$$\Rightarrow \int \log x\,dx = x\log x - x + c$$

(iv) Let $I = \int \sin^{-1} x\,dx$

$$u = \sin^{-1}(x), dv = dx$$

Then $du = \frac{1}{\sqrt{1-x^2}}, v = x$

$$\int \sin^{-1} x dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$\int \sin^{-1} x dx = x\sin^{-1} x + \frac{1}{2}\int \frac{dt}{\sqrt{t}}, \text{where } t = 1-x^2$$

$$= x\sin^{-1} x + \sqrt{t} + c$$

$$= x\sin^{-1} x + \sqrt{1-x^2} + c$$

Example 11.34

Evaluate : $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

Solution

Let $I = \int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

Putting $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$

Therefore, $I = \int \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\sec^2\theta\,d\theta$

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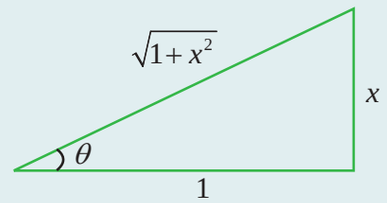
$$= \int \tan^{-1}(\tan 2\theta)\sec^2\theta\, d\theta$$
$$= \int 2\theta \sec^2\theta\, d\theta$$
$$= 2\int (\theta)(\sec^2\theta\, d\theta)$$



Applying integration by parts

$$I = 2\left[\theta\tan\theta - \int \tan\theta d\theta\right] \qquad \tan\theta = x$$
$$= 2(\theta\tan\theta - \log|\sec\theta|) + c \qquad \sec\theta = \sqrt{1+x^2}$$
$$\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx = 2x\tan^{-1}x - 2\log|\sqrt{1+x^2}| + c$$

11.7.6 Bernoulli's formula for Integration by Parts

If u and v are functions of x, then the Bernoulli's rule is

$$\int u dv = uv - u'v_1 + u''v_2 - \cdots$$

where $u', u'', u''', \ldots$ are successive derivatives of u and

$v, v_1, v_2, v_3, \cdots$ are successive integrals of dv

Bernoulli's formula is advantageously applied when $u = x^n$ (n is a positive integer)

For the following problems we have to apply the integration by parts two or more times to find the solution. In this case Bernoulli's formula helps to find the solution easily.

Example 11.35

Integrate the following with respect to x.

(i) x^2e^{5x} (ii) $x^3\cos x$ (iii) x^3e^{-x}

Solution

(i) $\int x^2e^{5x}dx.$

	$dv = e^{5x}dx$
$u = x^2$	$v = \frac{e^{5x}}{5}$
$u' = 2x$	$v_1 = \frac{e^{5x}}{5^2}$
$u'' = 2$	$v_2 = \frac{e^{5x}}{5^3}$
$u''' = 0$	$v_3 = \frac{e^{5x}}{5^4}$

Applying Bernoulli's formula

$$\int u dv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int x^2e^{5x}\,dx = (x^2)\left(\frac{e^{5x}}{5}\right) - (2x)\left(\frac{e^{5x}}{5^2}\right) + (2)\left(\frac{e^{5x}}{5^3}\right) - (0)\left(\frac{e^{5x}}{5^4}\right) + 0 + \cdots + 0 + c$$

$$= \frac{x^2e^{5x}}{5} - \frac{2xe^{5x}}{25} + \frac{2e^{5x}}{125} + c.$$

(ii) $\int x^3\cos x\,dx.$

	$dv = \cos x\,dx$
$u = x^3,$	$v = \sin x$
$u' = 3x,^2$	$v_1 = -\cos x$
$u'' = 6x,$	$v_2 = -\sin x$
$u''' = 6,$	$v_3 = \cos x$

Applying Bernoulli's formula

$$\int u dv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int x^3\cos x\,dx = (x^3)(\sin x) - (3x^2)(-\cos x)$$
$$+ (6x)(-\sin x) - (6)(\cos x) + c$$
$$= x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x + c.$$

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(iii) $\int x^3 e^{-x} dx.$

Applying Bernoulli's formula

$$\int u dv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int x^3 e^{-x}\, dx = \left(x^3\right)\left(-e^{-x}\right) - \left(3x^2\right)\left(e^{-x}\right) + \left(6x\right)\left(-e^{-x}\right) - \left(6\right)\left(e^{-x}\right) + c$$

$$= -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + c.$$

	$dv = e^{-x}dx$
$u = x^3,$	$v = -e^{-x}$
$u' = 3x^2,$	$v_1 = +e^{-x}$
$u'' = 6x,$	$v_2 = -e^{-x}$
$u''' = 6,$	$v_3 = e^{-x}$

EXERCISE 11.7

Integrate the following with respect to x:

(1) (i) $9xe^{3x}$ (ii) $x\sin 3x$ (iii) $25xe^{-5x}$ (iv) $x\sec x\tan x$

(2) (i) $x\log x$ (ii) $27\ x^2e^{3x}$ (iii) $x^2\cos x$ (iv) $x^3\sin x$

(3) (i) $\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}$ (ii) $x^5e^{x^2}$ (iii) $\tan^{-1}\left(\dfrac{8x}{1-16x^2}\right)$ (iv) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

11.7.8 Integrals of the form (i) $\int e^{ax}\sin bx\, dx$ (ii) $\int e^{ax}\cos bx\, dx$

The following examples illustrate that there are some integrals whose integration continues forever. Whenever we integrate function of the form $e^{ax}\cos bx$ or $e^{ax}\sin bx$, we have to apply the Integration by Parts rule twice to get the similar integral on both sides to solve.

Result 11.1

(i) $\displaystyle\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] + c$

(ii) $\displaystyle\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] + c$

Proof : (i)

$$\text{Let } I = \int e^{ax}\sin bx\, dx$$

$$\text{Take } u = \sin bx; du = b\cos bx\, dx$$

$$dv = e^{ax}; v = \frac{e^{ax}}{a},$$

Applying Integration by parts, we get

$$I = \frac{e^{ax}}{a}\sin bx - \int \frac{e^{ax}}{a} b\cos bx\, dx$$

$$I = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\, dx$$

$$\text{Take } u = \cos bx; du = -b\sin bx\, dx,$$

$$dv = e^{ax}; v = \frac{e^{ax}}{a},$$

Again applying integration by parts, we get

$$I = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\left[\frac{e^{ax}}{a}\cos bx + \int \frac{e^{ax}}{a} b\sin bx\, dx\right]$$

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$$I = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\frac{e^{ax}}{a}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\, dx$$

$$I = \frac{e^{ax}}{a}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}I$$

$$\left(1+\frac{b^2}{a^2}\right)I = \frac{ae^{ax}\sin bx - be^{ax}\cos bx}{a^2}$$

$$\left(\frac{a^2+b^2}{a^2}\right)I = \frac{e^{ax}[a\sin bx - b\cos bx]}{a^2}$$

$$\text{Therefore, } I = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]+c$$

$$\text{Therefore, } \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]+c$$

$$\text{Similarly, } \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]+c$$

$$\boxed{\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]+c}$$

$$\boxed{\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]+c}$$

Caution

In applying integration by parts to specific integrals, the pair of choice for u and dv once initially assumed should be maintained for the successive integrals on the right hand side. (See the above two examples). The pair of choice should not be interchanged.

Examples 11.36

Evaluate the following integrals

(i) $\int e^{3x}\cos 2x\, dx$ (ii) $\int e^{-5x}\sin 3x\, dx$

(i) $\int e^{3x}\cos 2x\, dx$

Using the formula

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]+c$$

For $a = 3$ and $b = 2$, we get

$$\int e^{3x}\cos 2x\, dx = \left(\frac{e^{3x}}{3^2+2^2}\right)(3\cos 2x + 2\sin 2x)+c$$

$$= \left(\frac{e^{3x}}{13}\right)(3\cos 2x + 2\sin 2x)+c.$$

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(ii) $\int e^{-5x} \sin 3x \, dx$

Using the formula

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx] + c$$

for $a = -5, b = 3$, we get

$$\int e^{-5x} \sin 3x \, dx = \left(\frac{e^{-5x}}{(-5)^2+3^2}\right)(-5\sin 3x - 3\cos 3x) + c$$

$$\int e^{-5x} \sin 3x \, dx = -\left(\frac{e^{-5x}}{34}\right)(5\sin 3x + 3\cos 3x) + c$$

EXERCISE 11.8

Integrate the following with respect to x

(1) (i) $e^{ax} \cos bx$ (ii) $e^{2x} \sin x$ (iii) $e^{-x} \cos 2x$

(2) (i) $e^{-3x} \sin 2x$ (ii) $e^{-4x} \sin 2x$ (iii) $e^{-3x} \cos x$

Result 11.2

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$$

Proof

Let $I = \int e^x [f(x) + f'(x)] dx$

$= \int e^x f(x) dx + \int e^x f'(x) dx$

Take $u = f(x)$; $du = f'(x)\, dx$, in the first integral

$dv = e^x; v = e^x$,

That is, $I = e^x f(x) - \int e^x f'(x) dx + \int e^x f'(x) dx + c$

Therefore, $I = e^x f(x) + c$.

Examples 11.37

Evaluate the following integrals

(i) $\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$ (ii) $\int e^x (\sin x + \cos x)\, dx$ (iii) $\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$

Solution

(i) Let $I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

Take $f(x) = \frac{1}{x}$, then $f'(x) = \frac{-1}{x^2}$

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This is of the form $\int e^x[f(x)+f'(x)]dx$

$$\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx = e^x\frac{1}{x}+c.$$

(ii) Let $I = \int e^x(\sin x+\cos x)\,dx$

Take $f(x) = \sin x$, then $f'(x)=\cos x$

This is of the form $\int e^x[f(x)+f'(x)]\,dx$

$$\int e^x(\sin x+\cos x)\,dx = e^x\sin x+c.$$

(iii) $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$

$$\text{Let } I = \int e^x\frac{(1-x)^2}{(1+x^2)^2}dx$$

$$= \int e^x\frac{(1+x^2-2x)}{(1+x^2)^2}dx$$

$$= \int e^x\left(\frac{1}{(1+x^2)}-\frac{2x}{(1+x^2)^2}\right)dx$$

$$\text{If } f(x) = \frac{1}{(1+x^2)}, \text{ then } f'(x)=-\frac{2x}{(1+x^2)^2}$$

Using $\int e^x\left(f(x)+f'(x)\right)dx = e^x f(x)+c$

$$\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx = \int e^x\left(\frac{1}{(1+x^2)}-\frac{2x}{(1+x^2)^2}\right)dx = e^x\frac{1}{(1+x^2)}+c.$$

EXERCISE 11.9

Integrate the following with respect to x:

(1) $e^x(\tan x+\log\sec x)$ (2) $e^x\left(\frac{x-1}{2x^2}\right)$ (3) $e^x\sec x(1+\tan x)$

(4) $e^x\left(\frac{2+\sin 2x}{1+\cos 2x}\right)$ (5) $e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)$ (6) $\frac{\log x}{(1+\log x)^2}$

11.7.9 Integration of Rational Algebraic Functions

In this section we are going to discuss how to integrate the rational algebraic functions whose numerator and denominator contains some positive integral powers of x with constant coefficients.

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Type I

Integrals of the form $\int \frac{dx}{a^2 \pm x^2}$, $\int \frac{dx}{x^2 - a^2}$, $\int \frac{dx}{\sqrt{a^2 \pm x^2}}$, $\int \frac{dx}{\sqrt{x^2 - a^2}}$

(i) $$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

(ii) $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

(iii) $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

(iv) $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + c$$

(v) $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + c$$

(vi) $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + c$$

Proof

(i) Let $I = \int \frac{dx}{a^2 - x^2}$.

$$= \int \frac{dx}{(a-x)(a+x)}$$

$$= \frac{1}{2a} \int \left[\frac{1}{a+x} + \frac{1}{a-x} \right] dx \qquad \left\{ \begin{array}{l} \text{Resolving into} \\ \text{partial fractions} \end{array} \right.$$

$$= \frac{1}{2a} \left[\log|a+x| - \log|a-x| \right] + c$$

$$= \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

(ii) Let $I = \int \frac{dx}{x^2 - a^2}$.

$$= \int \frac{dx}{(x-a)(x+a)}$$

$$= \frac{1}{2a} \int \left[\frac{1}{x-a} - \frac{1}{x+a} \right] dx \qquad \left\{ \begin{array}{l} \text{Resolving into} \\ \text{partial fractions} \end{array} \right.$$

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$$= \frac{1}{2a}[\log|x-a| - \log|x+a|] + c$$

$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

Therefore $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

(iii) Let $I = \int \frac{dx}{a^2+x^2}$.

Putting $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$

$dx = a\sec^2\theta$

$$I = \int \frac{a\sec^2\theta}{a^2+a^2\tan^2\theta}d\theta = \int \frac{a\sec^2\theta}{a^2(1+\tan^2\theta)}d\theta = \int \frac{a\sec^2\theta}{a^2\sec^2\theta}d\theta = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Therefore, $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

(iv) Let $I = \int \frac{dx}{\sqrt{a^2-x^2}}$.

Putting $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$dx = a\cos\theta$

$$I = \int \frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}d\theta = \int \frac{a\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}}d\theta = \int \frac{a\cos\theta}{a\cos\theta}d\theta = \int d\theta$$

$$= \theta + c = \sin^{-1}\left(\frac{x}{a}\right) + c$$

Therefore, $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

(v) Let $I = \int \frac{dx}{\sqrt{x^2-a^2}}$

Putting $x = a\sec\theta \Rightarrow \theta = \sec^{-1}\left(\frac{x}{a}\right)$

$dx = a\sec\theta\tan\theta d\theta$

$$I = \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}d\theta = \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2(\sec^2\theta-1)}}d\theta = \int \frac{a\sec\theta\tan\theta}{a\tan\theta}d\theta = \int \sec\theta d\theta$$

$$= \log|\sec\theta + \tan\theta| + c$$

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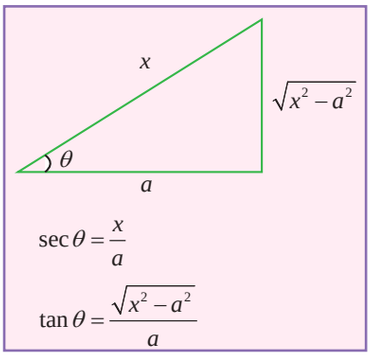
$$= \log\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+c$$

$$= \log\left|x+\sqrt{x^2-a^2}\right|-\log a+c$$

$$= \log\left|x+\sqrt{x^2-a^2}\right|+c_1 \text{ where } c_1 = c-\log a$$

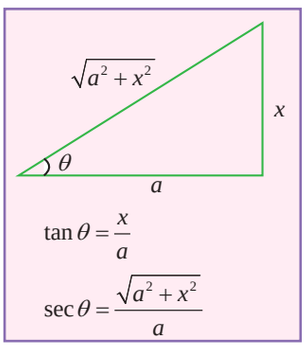
Therefore, $\displaystyle\int\frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right|+c_1$



(vi) Let $I = \displaystyle\int\frac{dx}{\sqrt{a^2+x^2}}$

Putting $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{a}\right)$

$dx = a\sec^2\theta d\theta$



$$I = \int\frac{a\sec^2\theta}{\sqrt{a^2\tan^2\theta+a^2}}d\theta = \int\frac{a\sec^2\theta}{\sqrt{a^2(\tan^2\theta+1)}}d\theta$$

$$= \int\frac{a\sec^2\theta}{a\sec\theta}d\theta = \int\sec\theta d\theta$$

$$= \log|\sec\theta+\tan\theta|+c$$

$$= \log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}+1}\right|+c$$

$$= \log\left|x+\sqrt{x^2+a^2}\right|-\log a+c$$

$$= \log\left|x+\sqrt{x^2+a^2}\right|+c_1 \text{ where } c_1 = c-\log a$$

$$\int\frac{dx}{\sqrt{a^2+x^2}} = \log\left|x+\sqrt{x^2+a^2}\right|+c_1$$

Remark: Remember the following useful substitution of the given integral as a functions of a^2-x^2, a^2+x^2 and x^2-a^2.

Given	Substitution
a^2-x^2	$x = a\sin\theta$
a^2+x^2	$x = a\tan\theta$
x^2-a^2	$x = a\sec\theta$

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Examples 11.38

Evaluate the following integrals

(i) $\int \frac{1}{(x-2)^2+1}dx$ (ii) $\int \frac{x^2}{x^2+5}dx$ (iii) $\int \frac{1}{\sqrt{1+4x^2}}dx$ (iv) $\int \frac{1}{\sqrt{4x^2-25}}dx$

Solution

(i) Let $I = \int \frac{1}{(x-2)^2+1}dx = \int \frac{1}{(x-2)^2+1^2}dx$

Putting $x - 2 = t \Rightarrow dx = dt$

Thus, $I = \int \frac{1}{t^2+1^2}dt = \tan^{-1}(t)+c = \tan^{-1}(x-2)+c$

(ii) Let $I = \int \frac{x^2}{x^2+5}dx$

$$= \int \frac{x^2+5-5}{x^2+5}dx = \int\left(1-\frac{5}{x^2+5}\right)dx = \int dx - \int \frac{5}{x^2+5}dx$$

$$= x - 5\int \frac{1}{x^2+\left(\sqrt{5}\right)^2}dx$$

$$= x - 5\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$$

$$I = x - \sqrt{5}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$$

(iii) Let $I = \int \frac{1}{\sqrt{1+4x^2}}dx = \int \frac{1}{\sqrt{1+(2x)^2}}dx$

Putting $2x = t \Rightarrow 2\,dx = dt \Rightarrow dx = \frac{1}{2}dt$

Thus, $I = \frac{1}{2}\int \frac{1}{\sqrt{1^2+t^2}}dt$

$$I = \frac{1}{2}\log\left|t+\sqrt{t^2+1}\right|+c = \frac{1}{2}\log\left|2x+\sqrt{(2x)^2+1}\right|+c$$

$$I = \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right|+c$$

(iv) Let $I = \int \frac{1}{\sqrt{4x^2-25}}dx = \int \frac{1}{\sqrt{(2x)^2-25}}dx$

Putting $2x = t \Rightarrow 2\,dx = dt \Rightarrow dx = \frac{1}{2}dt$

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$$\text{Thus, } I = \frac{1}{2}\int \frac{1}{\sqrt{t^2-5^2}}\,dt$$

$$= \frac{1}{2}\log\left|t+\sqrt{t^2-5^2}\right|+c$$

$$I = \frac{1}{2}\log\left|2x+\sqrt{4x^2-25}\right|+c$$

Type II

Integrals of the form $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

First we express ax^2+bx+c as sum or difference of two square terms that is, any one of the forms to Type I. The following rule is used to express the expression ax^2+bx+c as a sum or difference of two square terms.

(1) Make the coefficient of x^2 as unity.

(2) Completing the square by adding and subtracting the square of half of the coefficient of x.

$$\text{That is, } ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

Examples 11.39

Evaluate the following integrals

(i) $\int \frac{1}{x^2-2x+5}\,dx$ (ii) $\int \frac{1}{\sqrt{x^2+12x+11}}\,dx$ (iii) $\int \frac{1}{\sqrt{12+4x-x^2}}\,dx$

Solution

(i)
$$\int \frac{1}{x^2-2x+5}\,dx = \int \frac{1}{x^2-2(1)x+(1)^2+4}\,dx$$

$$= \int \frac{1}{(x-1)^2+2^2}\,dx$$

$$\int \frac{1}{x^2-2x+5}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+c$$

(ii)
$$\int \frac{1}{\sqrt{x^2+12x+11}}\,dx = \int \frac{1}{\sqrt{(x+6)^2-25}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+6)^2-5^2}}\,dx$$

$$= \log\left|x+6+\sqrt{(x+6)^2-5^2}\right|+c$$

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Therefore, $\int \frac{1}{\sqrt{x^2+12x+11}}dx = \log\left|x+6+\sqrt{x^2+12x+11}\right|+c$

(iii)
$$\int \frac{1}{\sqrt{12+4x-x^2}}dx = \int \frac{1}{\sqrt{12-(x^2-4x)}}dx$$
$$= \int \frac{1}{\sqrt{12-\{(x-2)^2-4\}}}dx$$
$$= \int \frac{1}{\sqrt{4^2-(x-2)^2}}dx$$
$$= \sin^{-1}\left(\frac{(x-2)}{4}\right)+c$$

EXERCISE 11.10

Find the integrals of the following :

(1) (i) $\frac{1}{4-x^2}$ (ii) $\frac{1}{25-4x^2}$ (iii) $\frac{1}{9x^2-4}$

(2) (i) $\frac{1}{6x-7-x^2}$ (ii) $\frac{1}{(x+1)^2-25}$ (iii) $\frac{1}{\sqrt{x^2+4x+2}}$

(3) (i) $\frac{1}{\sqrt{(2+x)^2-1}}$ (ii) $\frac{1}{\sqrt{x^2-4x+5}}$ (iii) $\frac{1}{\sqrt{9+8x-x^2}}$

Type III

Integrals of the form $\int \frac{px+q}{ax^2+bx+c}dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

To evaluate the above integrals, first we write

$$px+q = A\frac{d}{dx}(ax^2+bx+c)+B$$
$$px+q = A(2ax+b)+B$$

Calculate the values of A and B, by equating the coefficients of like powers of x on both sides

(i) The given first integral can be written as

$$\int \frac{px+q}{ax^2+bx+c}dx = \int \frac{A(2ax+b)+B}{ax^2+bx+c}dx$$
$$= A\int \frac{2ax+b}{ax^2+bx+c}dx + B\int \frac{1}{ax^2+bx+c}dx$$

(The first integral is of the form $\int \frac{f'(x)}{f(x)}dx$)

$$= A\log|ax^2+bx+c| + B\int \frac{1}{ax^2+bx+c}dx$$

The second term on the right hand side can be evaluated using the previous types.

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(ii) The given second integral can be written as

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx = \int \frac{A(2ax+b)+B}{\sqrt{ax^2+bx+c}}dx$$

$$= A\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx + B\int \frac{1}{\sqrt{ax^2+bx+c}}dx$$

(The first integral is of the form $\int f'(x)[f(x)]^n dx$)

$$= A\left(2\sqrt{ax^2+bx+c}\right) + B\int \frac{1}{\sqrt{ax^2+bx+c}}dx$$

The second term on the right hand side can be evaluated using the previous types.

Examples 11.40

Evaluate the following integrals

(i) $\int \frac{3x+5}{x^2+4x+7}dx$ (ii) $\int \frac{x+1}{x^2-3x+1}dx$ (iii) $\int \frac{2x+3}{\sqrt{x^2+x+1}}dx$ (iv) $\int \frac{5x-7}{\sqrt{3x-x^2-2}}dx$

Solution

(i) Let $I = \int \frac{3x+5}{x^2+4x+7}dx$

$$3x+5 = A\frac{d}{dx}(x^2+4x+7)+B$$

$$3x+5 = A(2x+4)+B$$

Comparing the coefficients of like terms, we get

$$2A = 3 \Rightarrow A = \frac{3}{2}; 4A+B=5 \Rightarrow B=-1$$

$$I = \int \frac{\frac{3}{2}(2x+4)-1}{x^2+4x+7}dx$$

$$I = \frac{3}{2}\int \frac{2x+4}{x^2+4x+7}dx - \int \frac{1}{x^2+4x+7}dx$$

$$= \frac{3}{2}\log\left|x^2+4x+7\right| - \int \frac{1}{(x+2)^2+\left(\sqrt{3}\right)^2}dx$$

$$= \frac{3}{2}\log\left|x^2+4x+7\right| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right)+c$$

(ii) Let $I = \int \frac{x+1}{x^2-3x+1}dx$

$$x+1 = A\frac{d}{dx}(x^2-3x+1)+B$$

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$$x + 1 = A(2x-3)+B$$

Comparing the coefficients of like terms, we get

$$2A = 1 \Rightarrow A=\frac{1}{2}; -3A+B=1 \Rightarrow B=\frac{5}{2}$$

$$I = \int \frac{\frac{1}{2}(2x-3)+\frac{5}{2}}{x^2-3x+1}dx$$

$$I = \frac{1}{2}\int \frac{2x-3}{x^2-3x+1}dx+\frac{5}{2}\int \frac{1}{x^2-3x+1}dx$$

$$= \frac{1}{2}\log\left|x^2-3x+1\right|+\frac{5}{2}\int \frac{1}{\left(x-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}dx$$

$$= \frac{1}{2}\log\left|x^2-3x+1\right|+\frac{5}{2}\frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\log\left|\frac{x-\frac{3}{2}-\frac{\sqrt{5}}{2}}{x-\frac{3}{2}+\frac{\sqrt{5}}{2}}\right|+c$$

$$= \frac{1}{2}\log\left|x^2-3x+1\right|+\frac{\sqrt{5}}{2}\log\left|\frac{2x-3-\sqrt{5}}{2x-3+\sqrt{5}}\right|+c$$

(iii) Let $I = \int \frac{2x+3}{\sqrt{x^2+x+1}}dx$

$$2x + 3 = A\frac{d}{dx}(x^2+x+1)+B$$

$$2x + 3 = A(2x + 1) + B$$

Comparing the coefficients of like terms, we get

$$2A = 2 \Rightarrow A = 1; \quad A + B = 3 \Rightarrow B = 2$$

$$I = \int \frac{(2x+1)+2}{\sqrt{x^2+x+1}}dx$$

$$I = \int \frac{2x+1}{\sqrt{x^2+x+1}}dx+2\int \frac{1}{\sqrt{x^2+x+1}}dx$$

$$= 2\sqrt{x^2+x+1}+2\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}dx$$

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$$= 2\sqrt{x^2+x+1}+2\log\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|+c$$

Therefore, $I = 2\sqrt{x^2+x+1}+2\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+c$

(iv) $\int \frac{5x-7}{\sqrt{3x-x^2-2}}dx$

$$5x-7 = A\frac{d}{dx}(3x-x^2-2)+B$$

$$5x-7 = A(3-2x)+B$$

Comparing the coefficients of like terms, we get

$$-2A = 5 \Rightarrow A=-\frac{5}{2}; 3A+B=-7 \Rightarrow B=\frac{1}{2}$$

$$I = \int \frac{\frac{-5}{2}(3-2x)+\frac{1}{2}}{\sqrt{3x-x^2+2}}dx$$

$$I = -\frac{5}{2}\int \frac{3-2x}{\sqrt{3x-x^2+2}}dx+\frac{1}{2}\int \frac{1}{\sqrt{3x-x^2+2}}dx$$

$$= \left(-\frac{5}{2}\right)2\sqrt{3x-x^2+2}+\frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}dx$$

$$= -5\sqrt{3x-x^2+2}+\frac{1}{2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{17}}{2}}\right)+c$$

Thus, $I = -5\sqrt{3x-x^2+2}+\frac{1}{2}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+c$

EXERCISE 11.11

Integrate the following with respect to x :

(1) (i) $\frac{2x-3}{x^2+4x-12}$ (ii) $\frac{5x-2}{2+2x+x^2}$ (iii) $\frac{3x+1}{2x^2-2x+3}$

(2) (i) $\frac{2x+1}{\sqrt{9+4x-x^2}}$ (ii) $\frac{x+2}{\sqrt{x^2-1}}$ (iii) $\frac{2x+3}{\sqrt{x^2+4x+1}}$

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Type IV

Integrals of the form $\int \sqrt{a^2 \pm x^2}\,dx, \int \sqrt{x^2 - a^2}\,dx$

Result 11.3

(1) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

(2) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$

(3) $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + c$

Proof :

(1) Let $I = \int \sqrt{a^2 - x^2}\,dx$

Take $u = \sqrt{a^2 - x^2}$ then $du = \frac{-2x}{2\sqrt{a^2 - x^2}}\,dx$

$dv = dx \Rightarrow v = x$

Applying Integration by parts, we get

$$\int u\,dv = uv - \int v\,du$$

$$\Rightarrow I = x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}}\,dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}}\,dx$$

$$= x\sqrt{a^2 - x^2} - \int \left(\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} + \frac{(-a^2)}{\sqrt{a^2 - x^2}}\right)dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\,dx + \int \frac{a^2}{\sqrt{a^2 - x^2}}\,dx$$

$$= x\sqrt{a^2 - x^2} - I + a^2 \int \frac{1}{\sqrt{a^2 - x^2}}\,dx$$

$$2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)$$

Therefore, $I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

Similarly we can prove other two results.

Note 11.3

The above problems can also be solved by substituting $x = a\sin\theta$

Examples 11.41

Evaluate the following :

(i) $\int \sqrt{4 - x^2}\,dx$ (ii) $\int \sqrt{25x^2 - 9}\,dx$ (iii) $\int \sqrt{x^2 + x + 1}\,dx$ (iv) $\int \sqrt{(x-3)(5-x)}\,dx$

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Solution

(i) Let $I = \sqrt{4-x^2}\,dx$

$$= \int \sqrt{2^2 - x^2}\,dx$$

$$= \frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right) + c$$

Therefore, $I = \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right) + c$

(ii) Let $I = \int \sqrt{25x^2 - 9}\,dx$

$$= \int \sqrt{(5x)^2 - 3^2}\,dx$$

$$= \frac{1}{5}\left[\frac{5x}{2}\sqrt{(5x)^2 - 3^2} - \frac{3^2}{2}\log\left|5x + \sqrt{(5x)^2 - 3^2}\right|\right] + c$$

Therefore, $I = \frac{1}{5}\left[\frac{5x}{2}\sqrt{25x^2 - 9} - \frac{9}{2}\log\left|5x + \sqrt{25x^2 - 9}\right|\right] + c$

(iii) Let $I = \int \sqrt{x^2 + x + 1}\,dx$

$$= \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{x + \frac{1}{2}}{2}\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\log\left|\left[x + \frac{1}{2} + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right]\right| + c$$

Therefore, $I = \frac{2x+1}{4}\sqrt{x^2 + x + 1} + \frac{3}{8}\log\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right| + c$

(iv) Let $I = \int \sqrt{(x-3)(5-x)}\,dx$

$$= \int \sqrt{8x - x^2 - 15}\,dx$$

$$= \int \sqrt{1^2 - (x-4)^2}\,dx$$

$$= \frac{x-4}{2}\sqrt{1^2 - (x-4)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-4}{1}\right) + c$$

Therefore, $I = \frac{x-4}{2}\sqrt{8x - x^2 - 15} + \frac{1}{2}\sin^{-1}(x-4) + c$

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EXERCISE 11.12

Integrate the following functions with respect to x :

(1) (i) $\sqrt{x^2+2x+10}$ (ii) $\sqrt{x^2-2x-3}$ (iii) $\sqrt{(6-x)(x-4)}$

(2) (i) $\sqrt{9-(2x+5)^2}$ (ii) $\sqrt{81+(2x+1)^2}$ (iii) $\sqrt{(x+1)^2-4}$

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EXERCISE 11.13

Choose the correct or the most suitable answer from given four alternatives.

(1) If $\int f(x)dx = g(x)+c$, then $\int f(x)g'(x)dx$

(1) $\int (f(x))^2 dx$ (2) $\int f(x)g(x)dx$ (3) $\int f'(x)g(x)dx$ (4) $\int (g(x))^2 dx$

(2) If $\int \frac{3^{\frac{1}{x}}}{x^2}dx = k\,(3^{\frac{1}{x}})+c$, then the value of k is

(1) $\log 3$ (2) $-\log 3$ (3) $-\frac{1}{\log 3}$ (4) $\frac{1}{\log 3}$

(3) If $\int f'(x)e^{x^2}dx = (x-1)e^{x^2}+c$, then $f(x)$ is

(1) $2x^3-\frac{x^2}{2}+x+c$ (2) $\frac{x^3}{2}+3x^2+4x+c$ (3) x^3+4x^2+6x+c (4) $\frac{2x^3}{3}-x^2+x+c$

(4) The gradient (slope) of a curve at any point (x, y) is $\frac{x^2-4}{x^2}$. If the curve passes through the point (2, 7), then the equation of the curve is

(1) $y = x+\frac{4}{x}+3$ (2) $y = x+\frac{4}{x}+4$ (3) $y = x^2+3x+4$ (4) $y = x^2-3x+6$

(5) $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$ is

(1) $\cot(xe^x)+c$ (2) $\sec(xe^x)+c$ (3) $\tan(xe^x)+c$ (4) $\cos(xe^x)+c$

(6) $\int \frac{\sqrt{\tan x}}{\sin 2x}dx$ is

(1) $\sqrt{\tan x}+c$ (2) $2\sqrt{\tan x}+c$ (3) $\frac{1}{2}\sqrt{\tan x}+c$ (4) $\frac{1}{4}\sqrt{\tan x}+c$

(7) $\int \sin^3 x dx$ is

(1) $\frac{-3}{4}\cos x-\frac{\cos 3x}{12}+c$ (2) $\frac{3}{4}\cos x+\frac{\cos 3x}{12}+c$

(3) $\frac{-3}{4}\cos x+\frac{\cos 3x}{12}+c$ (4) $\frac{-3}{4}\sin x-\frac{\sin 3x}{12}+c$

(8) $\int \frac{e^{6\log x}-e^{5\log x}}{e^{4\log x}-e^{3\log x}}dx$ is

(1) $x+c$ (2) $\frac{x^3}{3}+c$ (3) $\frac{3}{x^3}+c$ (4) $\frac{1}{x^2}+c$

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(9) $\int \frac{\sec x}{\sqrt{\cos 2x}} dx$ is

(1) $\tan^{-1}(\sin x)+c$ (2) $2\sin^{-1}(\tan x)+c$ (3) $\tan^{-1}(\cos x)+c$ (4) $\sin^{-1}(\tan x)+c$

(10) $\int \tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\, dx$ is

(1) x^2+c (2) $2x^2+c$ (3) $\frac{x^2}{2}+c$ (4) $-\frac{x^2}{2}+c$

(11) $\int 2^{3x+5} dx$ is

(1) $\frac{3(2^{3x+5})}{\log 2}+c$ (2) $\frac{2^{3x+5}}{2\log(3x+5)}+c$ (3) $\frac{2^{3x+5}}{2\log 3}+c$ (4) $\frac{2^{3x+5}}{3\log 2}+c$

(12) $\int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}\, dx$ is

(1) $\frac{1}{2}\sin 2x+c$ (2) $-\frac{1}{2}\sin 2x+c$ (3) $\frac{1}{2}\cos 2x+c$ (4) $-\frac{1}{2}\cos 2x+c$

(13) $\int \frac{e^x(x^2\tan^{-1}x+\tan^{-1}x+1)}{x^2+1}\, dx$ is

(1) $e^x\tan^{-1}(x+1)+c$ (2) $\tan^{-1}(e^x)+c$ (3) $e^x\frac{(\tan^{-1}x)^2}{2}+c$ (4) $e^x\tan^{-1}x+c$

(14) $\int \frac{x^2+\cos^2 x}{x^2+1}\operatorname{cosec}^2 x dx$ is

(1) $\cot x+\sin^{-1}x+c$ (2) $-\cot x+\tan^{-1}x+c$

(3) $-\tan x+\cot^{-1}x+c$ (4) $-\cot x-\tan^{-1}x+c$

(15) $\int x^2\cos x\, dx$ is

(1) $x^2\sin x+2x\cos x-2\sin x+c$ (2) $x^2\sin x-2x\cos x-2\sin x+c$

(3) $-x^2\sin x+2x\cos x+2\sin x+c$ (4) $-x^2\sin x-2x\cos x+2\sin x+c$

(16) $\int \sqrt{\frac{1-x}{1+x}}dx$ is

(1) $\sqrt{1-x^2}+\sin^{-1}x+c$ (2) $\sin^{-1}x-\sqrt{1-x^2}+c$

(3) $\log\left|x+\sqrt{1-x^2}\right|-\sqrt{1-x^2}+c$ (4) $\sqrt{1-x^2}+\log\left|x+\sqrt{1-x^2}\right|+c$

(17) $\int \frac{dx}{e^x-1}$ is

(1) $\log|e^x|-\log|e^x-1|+c$ (2) $\log|e^x|+\log|e^x-1|+c$

(3) $\log|e^x-1|-\log|e^x|+c$ (4) $\log|e^x+1|-\log|e^x|+c$

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(18) $\int e^{-4x}\cos x\,dx$ is

(1) $\frac{e^{-4x}}{17}[4\cos x-\sin x]+c$ (2) $\frac{e^{-4x}}{17}[-4\cos x+\sin x]+c$

(3) $\frac{e^{-4x}}{17}[4\cos x+\sin x]+c$ (4) $\frac{e^{-4x}}{17}[-4\cos x-\sin x]+c$

(19) $\int \frac{\sec^2 x}{\tan^2 x-1}dx$

(1) $2\log\left|\frac{1-\tan x}{1+\tan x}\right|+c$ (2) $\log\left|\frac{1+\tan x}{1-\tan x}\right|+c$

(3) $\frac{1}{2}\log\left|\frac{\tan x+1}{\tan x-1}\right|+c$ (4) $\frac{1}{2}\log\left|\frac{\tan x-1}{\tan x+1}\right|+c$

(20) $\int e^{-7x}\sin 5x\,dx$ is

(1) $\frac{e^{-7x}}{74}[-7\sin 5x-5\cos 5x]+c$ (2) $\frac{e^{-7x}}{74}[7\sin 5x+5\cos 5x]+c$

(3) $\frac{e^{-7x}}{74}[7\sin 5x-5\cos 5x]+c$ (4) $\frac{e^{-7x}}{74}[-7\sin 5x+5\cos 5x]+c$

(21) $\int x^2 e^{\frac{x}{2}}dx$ is

(1) $x^2e^{\frac{x}{2}}-4xe^{\frac{x}{2}}-8e^{\frac{x}{2}}+c$ (2) $2x^2e^{\frac{x}{2}}-8xe^{\frac{x}{2}}-16e^{\frac{x}{2}}+c$

(3) $2x^2e^{\frac{x}{2}}-8xe^{\frac{x}{2}}+16e^{\frac{x}{2}}+c$ (4) $x^2\frac{e^{\frac{x}{2}}}{2}-\frac{xe^{\frac{x}{2}}}{4}+\frac{e^{\frac{x}{2}}}{8}+c$

(22) $\int \frac{x+2}{\sqrt{x^2-1}}dx$ is

(1) $\sqrt{x^2-1}-2\log\left|x+\sqrt{x^2-1}\right|+c$ (2) $\sin^{-1}x-2\log\left|x+\sqrt{x^2-1}\right|+c$

(3) $2\log\left|x+\sqrt{x^2-1}\right|-\sin^{-1}x+c$ (4) $\sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+c$

(23) $\int \frac{1}{x\sqrt{(\log x)^2-5}}dx$ is

(1) $\log\left|x+\sqrt{x^2-5}\right|+c$ (2) $\log\left|\log x+\sqrt{\log x-5}\right|+c$

(3) $\log\left|\log x+\sqrt{(\log x)^2-5}\right|+c$ (4) $\log\left|\log x-\sqrt{(\log x)^2-5}\right|+c$

(24) $\int \sin\sqrt{x}dx$ is

(1) $2\left(-\sqrt{x}\cos\sqrt{x}+\sin\sqrt{x}\right)+c$ (2) $2\left(-\sqrt{x}\cos\sqrt{x}-\sin\sqrt{x}\right)+c$

(3) $2\left(-\sqrt{x}\sin\sqrt{x}-\cos\sqrt{x}\right)+c$ (4) $2\left(-\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}\right)+c$

(25) $\int e^{\sqrt{x}}dx$ is

(1) $2\sqrt{x}(1-e^{\sqrt{x}})+c$ (2) $2\sqrt{x}(e^{\sqrt{x}}-1)+c$

(3) $2e^{\sqrt{x}}(1-\sqrt{x})+c$ (4) $2e^{\sqrt{x}}(\sqrt{x}-1)+c$

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SUMMARY

Derivatives	Antiderivatives
$\frac{d}{dx}(c)=0,$ where c is a constant	$\int 0\,dx = c$, where c is a constant
$\frac{d}{dx}(kx)=k,$ where k is a constant	$\int k\,dx = kx+c$ where c is a constant
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$	$\int x^n dx = \frac{x^{n+1}}{n+1}+c, \quad n \neq -1$ (Power rule)
$\frac{d}{dx}\log x = \left(\frac{1}{x}\right)$	$\int \frac{1}{x}\,dx = \log\lvert x\rvert + c$
$\frac{d}{dx}(-\cos x)=\sin x$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}(\sin x)=\cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dx}(\tan x)=\sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$
$\frac{d}{dx}(-\cot x)=\operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
$\frac{d}{dx}(\sec x)=\sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + c$
$\frac{d}{dx}(-\operatorname{cosec} x)=\operatorname{cosec} x \cot x$	$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$
$\frac{d}{dx}(e^x)=e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}\left(\frac{a^x}{\log a}\right)=a^x$	$\int a^x dx = \frac{a^x}{\log a}+c$
$\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$
$\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$

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(1) If k is any constant, then $\int kf(x)dx = k\int f(x)\,dx$

(2) $\int (f_1(x) \pm f_2(x))\,dx = \int f_1(x)dx \pm \int f_2(x)dx$

If $\int f(x)\,dx = g(x)+c$, then $\int f(ax+b)\,dx = \frac{1}{a}\,g(ax+b)+c$

(1)	$\int \tan x\,dx = \log\lvert\sec x\rvert + c$
(2)	$\int \cot x\,dx = \log\lvert\sin x\rvert + c$
(3)	$\int cosecx\,dx = \log\lvert\text{cosec}\,x - \cot x\rvert + c$
(4)	$\int \sec x\,dx = \log\lvert\sec x + \tan x\rvert + c$

Bernoulli's formula for integration by Parts:

If u and v are functions of x, then the Bernoulli's rule is

$\int udv = uv - u'v_1 + u''v_2 + \ldots$

where $u', u'', u''', \ldots$ are successive derivatives of u and

$v, v_1, v_2, v_2, \ldots$, are successive integrals of dv

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] + c$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] + c$$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert + c$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left\lvert\frac{x-a}{x+a}\right\rvert + c$	$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left\lvert x+\sqrt{x^2-a^2}\right\rvert + c$	$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left\lvert x+\sqrt{x^2+a^2}\right\rvert + c$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left\lvert x+\sqrt{x^2-a^2}\right\rvert + c$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left\lvert x+\sqrt{x^2+a^2}\right\rvert + c$$

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ICT CORNER 11(a)

Integral Calculus

Expected Outcome

Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI standard Integration" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Simple Integration". You can enter any function in the f(x)box. Graph of $f(x)$ appear on left side and the Integrated function will appear on right side. (Note: for x^5 enter x^5) Move the slider "integration constant" to change the constant value in integration.

Step1

Step2

Browse in the link:

XI standard Integration: https://ggbm.at/c63hdegc

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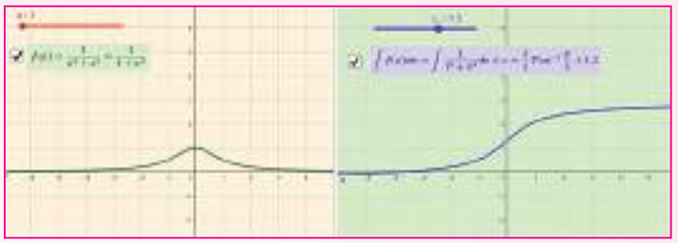
ICT CORNER 11(b)

Integral Calculus

Expected Outcome

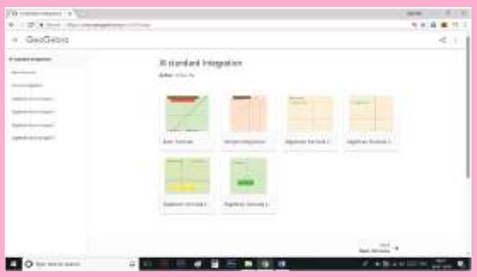


Step 1

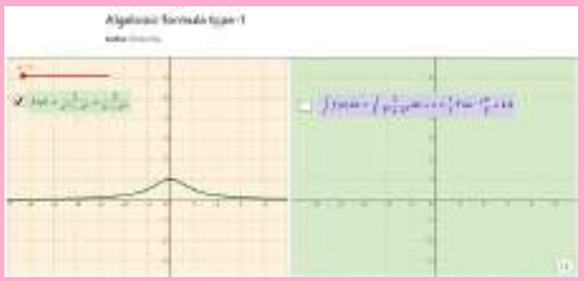
Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI standard Integration" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Algebraic type-1". The graph of the function given on left side and the Integration of the function appear on right side. Click on both to see the graph. You can move the slider "a" to change the value. Algebraic types are grouped as 4 types open other three Algebraic types and observe.



Step1



Step2

Browse in the link:

XI standard Integration: https://ggbm.at/c63hdegc

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Chapter 12 Introduction to Probability Theory

BEDV2S

The most important questions of life are, indeed, for the most part, really only problems of probability

Pierre - Simon Laplace

12.1 Introduction

Laplace
1749-1827

A gambler's dispute in 1654 led to the creation of a mathematical theory of probability by two famous French mathematicians, Blaise Pascal and Pierre de Fermat. The fundamental principles of probability theory were formulated by Pascal and Fermat for the first time. After an extensive research, Laplace published his monumental work in 1812, and laid the foundation to Probability theory. In statistics, the Bayesian interpretation of probability was developed mainly by Laplace.

The topic of probability is seen in many facets of the modern world. From its origin as a method of studying games, probability has involved in a powerful and widely applicable branch of mathematics. The uses of probability range from the determination of life insurance premium, to the prediction of election outcomes, the description of the behaviour of molecules in a gas. Its utility is one good reason why the study of probability has found in the way into a school textbook.
The interpretation of the word 'probability' involves synonyms such as chance, possible, probably, likely, odds, uncertainty, prevalence, risk, expectancy etc.

Our entire world is filled with uncertainty. We make decisions affected by uncertainty virtually every day. In order to measure uncertainty, we turn to a branch of mathematics called theory of probability. Probability is a measure of the likeliness that an event will occur.

Learning Objectives

On completion of this chapter, the students are expected to

- understand the classical theory of probability and axiomatic approach to probability.
- understand mutually exclusive, mutually inclusive and exhaustive events.
- understand the concepts of conditional probability and independent events.
- apply Bayes' theorem.
- apply probability theory in day-to-day life.

12.2 Basic definitions

Before we study the theory of probability, let us recollect the definition of certain terms already studied in earlier classes, which are frequently used.

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EXPERIMENT

Deterministic Experiment | **Random Experiment**

(Genetic determination)

(Hitting the target)

Definition 12.1

An **experiment** is defined as a process for which its result is well defined.

Definition 12.2

Deterministic experiment is an experiment whose outcomes can be predicted with certain, under ideal conditions.

Definition 12.3

A random experiment (or non-deterministic) is an experiment

(i) whose all possible outcomes are known in advance,

(ii) whose each outcome is not possible to predict in advance, and

(iii) can be repeated under identical conditions.

A die is 'rolled', a fair coin is 'tossed' are examples for random experiments.

Definition 12.4

A **simple event** (or elementary event or sample point) is the most basic possible outcome of a random experiment and it cannot be decomposed further.

Definition 12.5

A sample space is the set of all possible outcomes of a random experiment. Each point in sample space is an elementary event.

Illustration 12.1

(1) (i) If a die is rolled, then the sample space $S = \{1, 2, 3, 4, 5, 6\}$.

(ii) A coin is tossed, then the sample space $S = \{H, T\}$.

(2) (i) Suppose we toss a coin until a head is obtained. One cannot say in advance how many tosses will be required, and so the sample space.

$S = \{H, TH, TTH, TTTH, ...\}$ is an infinite set.

(ii) The sample space associated with the number of passengers waiting to buy train tickets in counters is $S = \{0,1,2,...\}$.

(3) (i) If the experiment consists of choosing a number randomly between 0 and 1, then the sample space is $S = \{ x: 0 < x < 1\}$.

(ii) The sample space for the life length (t in hours) of a tube light is $S = \{ t: 0 < t < 1000\}$.

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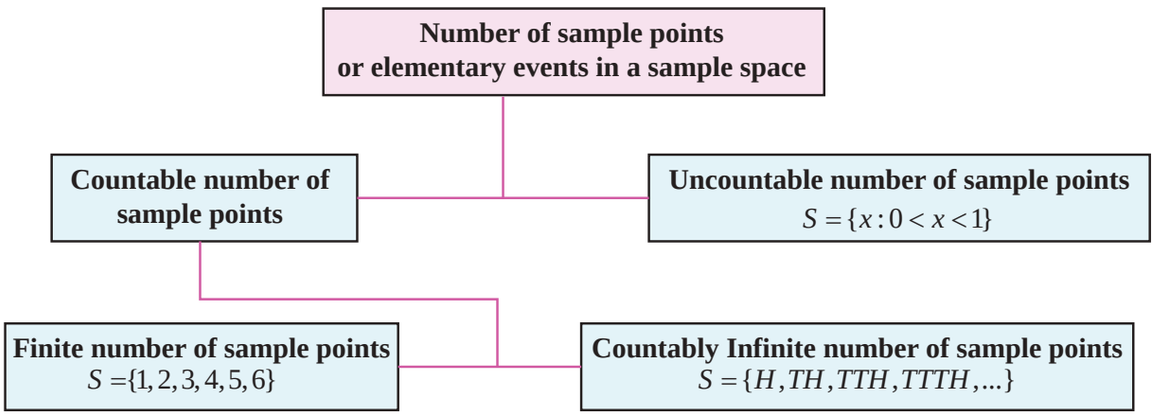
From (2) and (3), one need to distinguish between two types of infinite sets, where one type is significantly 'larger' than the other. In particular, S in (2) is called **countably infinite**, while the S in (3) is called **uncountably infinite**. The fact that one can list the elements of a countably infinite set means that the set can be put in one-to-one correspondence with natural numbers $\mathbb{N}$. On the other hand, you cannot list the elements in uncountable set.

From the above example, one can understand that the sample space may consist of countable or uncountable number of elementary events.



12.3 Finite sample space

In this section we restrict our sample spaces that have at most a finite number of points.

Types of events

Let us now define some of the important types of events, which are used frequently in this chapter.

- Sure event or certain event
- Complementary event
- Mutually inclusive event
- Equally likely events
- Impossible event
- Mutually exclusive events
- Exhaustive events
- Independent events (defined after learning the concepts of probability)

Definition 12.6

When the sample space is finite, any subset of the sample space is an **event**. That is, all elements of the power set $\mathscr{P}(S)$ of the sample space are defined as **events**. An event is a collection of sample points or elementary events.

The sample space S is called **sure event or certain event.** The null set $\varnothing$ in S is called an **impossible event.**

Definition 12.7

For every event A, there corresponds another event $\overline{A}$ is called the **complementary event** to A. It is also called the event 'not A'.

Illustration 12.2

Suppose a sample space S is given by $S = \{1,2,3,4\}$.

Let the set of all possible subsets of S (the power set of S) be $\mathscr{P}(S)$.

$$\mathscr{P}(S) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$$

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(i) All the elements of $\mathscr{P}(S)$ are events.

(ii) $\varnothing$ is an impossible event.

(iii) $\{1\},\{2\},\{3\},\{4\}$ are the simple events or elementary events.

(iv) $\{1, 2, 3, 4\}$ is a sure event or certain event.

Definition 12.8

Two events cannot occur simultaneously are mutually exclusive events. $A_1, A_2, A_3, ..., A_k$ are **mutually exclusive** means that, $A_i \cap A_j = \varnothing$, for $i \neq j$.

Definition 12.9

Two events are **mutually inclusive** when they can both occur simultaneously.

$A_1, A_2, A_3, ..., A_k$ are mutually inclusive means that, $A_i \cap A_j \neq \varnothing$, for $i \neq j$

Illustration 12.3

When we roll a die, the sample space $S = \{1,2,3,4,5,6\}$.

(i) Since $\{1, 3\} \cap \{2, 4, 5, 6\} = \varnothing$, the events $\{1,3\}$ and $\{2, 4,5,6\}$ are mutually exclusive events.

(ii) The events $\{1,6,\},\{2,3,5\}$ are mutually exclusive.

(iii) The events $\{2,3,5\},\{5,6\}$ are mutually inclusive, since $\{2, 3, 5\} \cap \{5, 6\} = \{5\} \neq \varnothing$

Definition 12.10

$A_1, A_2, A_3, ..., A_k$ are called **exhaustive events** if, $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k = S$

Definition 12.11

$A_1, A_2, A_3, ..., A_k$ are called **mutually exclusive and exhaustive events** if,

(i) $A_i \cap A_j = \varnothing$, for $i \neq j$ (ii) $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k = S$

Illustration 12.4

When a die is rolled, sample space $S = \{1,2,3,4,5,6\}$.

Some of the events are $\{2,3\},\{1,3,5\},\{4,6\},\{6\}$ and $\{1,5\}$.

(i) Since $\{2, 3\} \cup \{1, 3, 5\} \cup \{4, 6\} = \{1, 2, 3, 4, 5, 6\} = S$ (sample space), the events $\{2,3\},\{1,3,5\},\{4,6\}$ are exhaustive events.

(ii) Similarly $\{2,3\},\{4,6\}$ and $\{1,5\}$ are also exhaustive events.

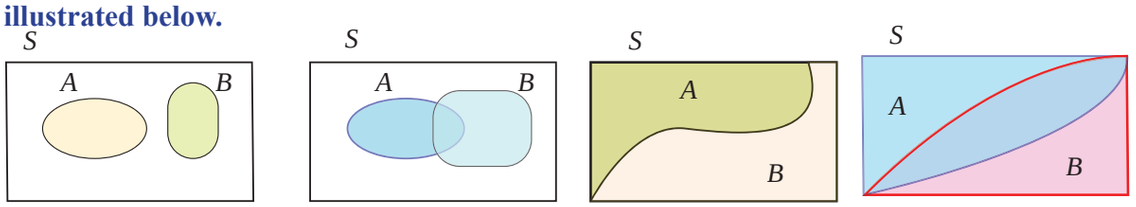
(iii) $\{1,3,5\},\{4,6\},\{6\}$ and $\{1,5\}$ are not exhaustive events.

(Since $\{1, 3, 5\} \cup \{4, 6\} \cup \{6\} \cup \{1, 5\} \neq S$)

(iv) $\{2,3\},\{4,6\}$, and $\{1,5\}$ are mutually exclusive and exhaustive events, since

$\{2,3\} \cap \{4,6\} = \varnothing$, $\{2,3\} \cap \{1,5\} = \varnothing$, $\{4,6\} \cap \{1,5\} = \varnothing$ and $\{2, 3\} \cup \{4, 6\} \cup \{1, 5\} = S$

Types of events associated with sample space are easy to visualize in terms of Venn diagrams, as illustrated below.



A and *B* are Mutually exclusive | *A* and *B* are Mutually inclusive | *A* and *B* are Mutually exclusive and exhaustive | *A* and *B* are Mutually inclusive and exhaustive

Definition 12.12

The events having the same chance of occurrences are called **equally likely** events.

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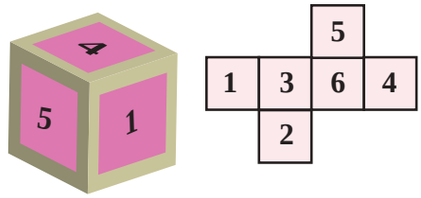
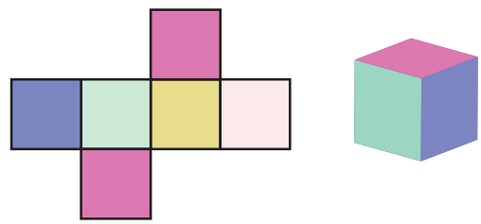
Example for equally likely events: Suppose a fair die is rolled.

Number on the face	1	2	3	4	5	6
Chance of occurrence	1	1	1	1	1	1



Example for not equally likely events: A colour die is shown in figure is rolled.

Colour on the face					
Chance of occurrence	1	1	1	2	1



Similarly, suppose if we toss a coin, the events of getting a head or a tail are equally likely.

Methods to find sample space

Illustration 12.5

Two coins are tossed, the sample space is

(i) $S = \{H,T\} \times \{H,T\} = \{(H,H),(H,T),(T,H),(T,T)\}$ or $\{HH, HT, TH, TT\}$

(ii) If a coin is tossed and a die is rolled simultaneously, then the sample space is

$S = \{H,T\} \times \{1,2,3,4,5,6\} = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$ or

$S = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$.

Also one can interchange the order of outcomes of coin and die. The following table gives the sample spaces for some random experiments.

Random Experiment	Total Number of Outcomes	Sample space
Tossing a fair coin	$2^1 = 2$	$\{H,\ T\}$
Tossing two coins	$2^2 = 4$	$\{HH,\ HT,\ TH,\ TT\}$
Tossing three coins	$2^3 = 8$	$\{HHH,\ HHT,\ HTH,\ THH,\ HTT,\ THT,\ TTH,\ TTT\}$
Rolling fair die	$6^1 = 6$	$\{1, 2, 3, 4, 5, 6\}$
Rolling Two dice or single die two times.	$6^2 = 36$	{(1,1) ,(1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2),(6,3), (6,4), (6,5), (6,6)}
Drawing a card from a pack of 52 playing cards	$52^1 = 52$	Heart ♥ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Diamond ♦ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Spade ♠ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour Club ♣ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour

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Notations

Let A and B be two events.

(i) $A \cup B$ stands for the occurrence of A or B or both.

(ii) $A \cap B$ stands for the simultaneous occurrence of A and B. $A \cap B$ can also be written as AB

(iii) $\overline{A}$ or A' or A^c stands for non-occurrence of A

(iv) $(A \cap \overline{B})$ stands for the occurrence of only A.

Priori : Knowledge which precedes from theoretical deduction or making assumption. Not from experience or observation

12.4 Probability

12.4.1 Classical definition (A priori) of probability (Bernoulli's principle of equally likely)

Earlier classes we have studied the frequency (A posteriori) definition of probability and the problems were solved. Now let us learn the fundamentals of the axiomatic approach to probability theory.

Posteriori : Knowledge which precedes from experience or observation.

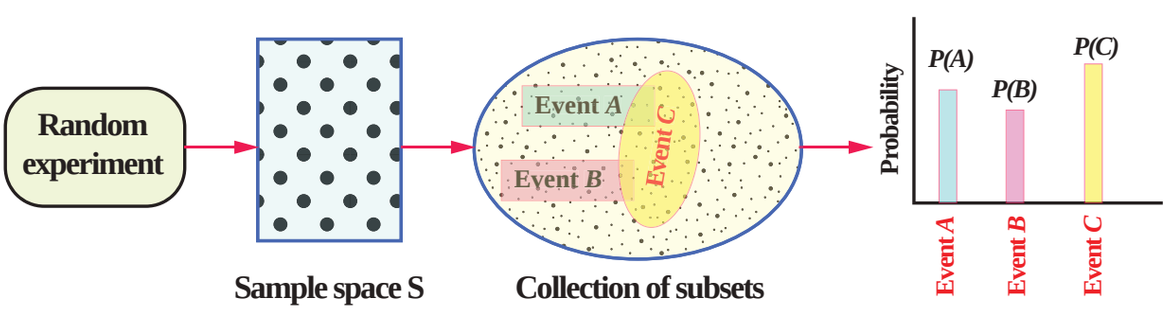
The basic assumption of underlying the classical theory is that the outcomes of a random experiment are equally likely. If there are n exhaustive, mutually exclusive and equally likely outcomes of an experiment and m of them are favorable to an event A, then the mathematical probability of A is defined as the ratio $\frac{m}{n}$. In other words, $P(A) = \frac{m}{n}$.

Definition 12.13

Let S be the sample space associated with a random experiment and A be an event. Let $n(S)$ and $n(A)$ be the number of elements of S and A respectively. Then the probability of the event A is defined as

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to } A}{\text{Exhaustive number of cases in } S}$$

Every probabilistic model involves an underlying process is shown in the following figure.



The classical definition of probability is limited in its application only to situations where there are a finite number of possible outcomes. It mainly considered discrete events and its methods were mainly combinatorial. This renders it inapplicable to some important random experiments, such as 'tossing a coin until a head appears' which give rise to the possibility of infinite set of outcomes. Another limitation of the classical definition was the condition that each possible outcome is 'equally likely'.

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These types of limitations in the classical definition of probability led to the evolution of the modern definition of probability which is based on the concept of sets. It is known an axiomatic approach.

The foundations of the Modern Probability theory were laid by Andrey Nikolayevich Kolmogorov, a Russian mathematician who combined the notion of sample space introduced by Richard von Mises, and measure theory and presented his axiomatic system for probability theory in 1933. We introduce the axiomatic approach proposed by A.N. Kolmogorov. Based on this, it is possible to construct a logically perfect structure of the modern theory of probability theory. The classical theory of probability is a particular case of axiomatic probability. The axioms are a set of rules, which can be used to prove theorems of probability.

A.N. Kolmogorov

12.4.2 Axiomatic approach to Probability

Axioms of probability

Let S be a finite sample space, let $\mathscr{P}(S)$ be the class of events, and let P be a real valued function defined on $\mathscr{P}(S)$. Then $P(A)$ is called probability function of the event A, when the following axioms are hold:

$[P_1]$	For any event A,	$P(A) \geq 0$	(Non-negativity axiom)
$[P_2]$	For any two mutually exclusive events	$P(A \cup B) = P(A) + P(B)$	(Additivity axiom)
$[P_3]$	For the certain event	$P(S) = 1$	(Normalization axiom)

Note 12.1

(i) $0 \leq P(A) \leq 1$

(ii) If $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive events in a sample space S, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_n)$$

Theorems on finite probability spaces (without proof)

When the outcomes are equally likely Theorem 12.1 is applicable, else Theorem 12.2 is applicable.

Theorem 12.1

Let S be a sample space and for any subset A of S, let $P(A) = \dfrac{n(A)}{n(S)}$. Then $P(A)$ satisfies axioms of probability $[P_1], [P_2]$, and $[P_3]$.

Theorem12.2

Let S be a finite sample space say $S = \{a_1, a_2, a_3, \ldots, a_n\}$. A finite probability space is obtained by assigning to each point a_i in S a real number p_i, is called the probability of a_i, satisfying the following properties:

(i) Each $p_i \geq 0$. (ii) The sum of the p_i is 1, that is, $\sum p_i = p_1 + p_2 + p_3 + \cdots + p_1 = 1$.

If the probability $P(A)$, of an event A is defined as the sum of the probabilities of the points in A, then the function $P(A)$ satisfies the axioms of probability $[P_1], [P_2]$, and $[P_3]$.

Note: Sometimes the points in a finite sample space and their assigned probabilities are given in the form of a table as follows:

Outcome	a_1	a_2	a_3	$\ldots$	a_n
Probability	p_1	p_2	p_3	$\ldots$	p_n

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Here is an illustration of how to construct a probability law starting from some common sense assumptions about a model.

Illustration 12.6

(1) Let $S=\{1,2,3\}$. Suppose $\mathscr{P}(S)$ is the power set of S, and $P(A)=\dfrac{n(A)}{n(S)}$.

Then $P(\{1\})=\dfrac{1}{3}$, $P(\{2\})=\dfrac{1}{3}$, and $P(\{3\})=\dfrac{1}{3}$,

satisfies axioms of probability $[P_1],[P_2]$, and $[P_3]$. Here all the outcomes are equally likely.

(2) Let $S=\{1,2,3\}$. Suppose $\mathscr{P}(S)$ is the power set of S,

If the probability $P(A)$, of an event A of S is defined as the sum of the probabilities of the points in A,

then $P(\{1\})=\dfrac{1}{2}$, $P(\{2\})=\dfrac{1}{4}$, $P(\{3\})=\dfrac{1}{4}$,

satisfy the axioms of probability $[P_1],[P_2]$, and $[P_3]$.

(3) Let $S=\{1,2,3\}$ and $\mathscr{P}(S)$ is the power set of S. If the probability $P(A)$, of an event A of S is defined as the sum of the probabilities of the points in A,

then $P(\{1\})=0$, $P(\{2\})=\dfrac{1}{\sqrt{2}}$, and $P(\{3\})=1-\dfrac{1}{\sqrt{2}}$,

satisfy the above axioms $[P_1],[P_2]$, and $[P_3]$.

In (2) and (3), the outcomes are not equally likely.

Note 12.2

Irrational numbers also can act as probabilities.

Classroom Activity: Each student to flip a coin 10 times,

Calculate: $p=\dfrac{\text{Number of times heads occur}}{10}$

Find the cumulative ratio of heads to tosses. As number of tosses increases $p\to\dfrac{1}{2}$

Example 12.1

If an experiment has exactly the three possible mutually exclusive outcomes A, B, and C, check in each case whether the assignment of probability is permissible.

(i) $P(A)=\dfrac{4}{7}$, $P(B)=\dfrac{1}{7}$, $P(C)=\dfrac{2}{7}$.

(ii) $P(A)=\dfrac{2}{5}$, $P(B)=\dfrac{1}{5}$, $P(C)=\dfrac{3}{5}$.

(iii) $P(A)=0.3$, $P(B)=0.9$, $P(C)=-0.2$.

(iv) $P(A)=\dfrac{1}{\sqrt{3}}$, $P(B)=1-\dfrac{1}{\sqrt{3}}$, $P(C)=0$.

(v) $P(A)=0.421$, $P(B)=0.527$ $P(C)=0.042$.

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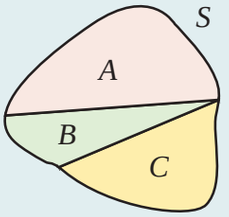
Solution

Since the experiment has exactly the three possible mutually exclusive outcomes A, B and C, they must be exhaustive events.

$$\Rightarrow S = A \cup B \cup C$$



Therefore, by axioms of probability

$P(A) \geq 0, \quad P(B) \geq 0, \quad P(C) \geq 0$ and

$P(A \cup B \cup C) = P(A) + P(B) + P(C) = P(S) = 1$

(i) Given that $P(A) = \frac{4}{7} \geq 0$, $\quad P(B) = \frac{1}{7} \geq 0$, and $\quad P(C) = \frac{2}{7} \geq 0$

Also $P(S) = P(A) + P(B) + P(C) = \frac{4}{7} + \frac{1}{7} + \frac{2}{7} = 1$

Therefore the assignment of probability is permissible.

(ii) Given that $P(A) = \frac{2}{5} \geq 0$, $\quad P(B) = \frac{1}{5} \geq 0$, and $\quad P(C) = \frac{3}{5} \geq 0$

But $P(S) = P(A) + P(B) + P(C) = \frac{2}{5} + \frac{1}{5} + \frac{3}{5} = \frac{6}{5} > 1$

Therefore the assignment is not permissible.

(iii) Since $P(C) = -0.2$ is negative, the assignment is not permissible.

(iv) The assignment is permissible because

$P(A) = \frac{1}{\sqrt{3}} \geq 0$, $\quad P(B) = 1 - \frac{1}{\sqrt{3}} \geq 0$, and $P(C) = 0 \geq 0$

$P(S) = P(A) + P(B) + P(C) = \frac{1}{\sqrt{3}} + 1 - \frac{1}{\sqrt{3}} + 0 = 1.$

(v) Even though $P(A) = 0.421 \geq 0$, $P(B) = 0.527 \geq 0$, and $P(C) = 0.042 \geq 0$, the sum of the probability

$P(S) = P(A) + P(B) + P(C) = 0.421 + 0.527 + 0.042 = 0.990 < 1$.

Therefore, the assignment is not permissible.

Example 12.2

An integer is chosen at random from the first ten positive integers. Find the probability that it is (i) an even number (ii) multiple of three.

Solution

The sample space is

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $\quad n(S) = 10$

Let A be the event of choosing an even number and

B be the event of choosing an integer multiple of three.

$A = \{2, 4, 6, 8, 10\}, \quad n(A) = 5,$

$B = \{3, 6, 9\}, \quad n(B) = 3$

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$$P\text{ (choosing an even integer)} = P(A) = \frac{n(A)}{n(S)} = \frac{5}{10} = \frac{1}{2}.$$

$$P\text{ (choosing an integer multiple of three)} = P(B) = \frac{n(B)}{n(S)} = \frac{3}{10}.$$

Example 12.3

Three coins are tossed simultaneously, what is the probability of getting (i) exactly one head (ii) at least one head (iii) at most one head?

Solution:

Notice that three coins are tossed simultaneously = one coin is tossed three times.

The sample space $S = \{H,T\} \times \{H,T\} \times \{H,T\}$

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\},\ n(S) = 8$$

Let A be the event of getting exactly one head, B be the event of getting atleast one head and C be the event of getting at most one head.

$$A = \{HTT, THT, TTH\};\ n(A) = 3$$

$$B = \{HTT, THT, TTH, HHT, HTH, THH, HHH\};\ n(B) = 7$$

$$C = \{TTT, HTT, THT, TTH\};\ n(C) = 4.$$

Therefore the required probabilities are

(i) $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{3}{8}$

(ii) $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{7}{8}$

(iii) $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{4}{8} = \dfrac{1}{2}.$

Note 12.3

When the number of elements in a sample space is considerably small we can solve by finger-counting the elements in the events. But when the number of elements is too large to count then combinatorics helps us to solve the problems.

For the following problem, combinatorics is used to find the number of elements in the sample space and the events.

Example 12.4

Suppose ten coins are tossed. Find the probability to get (i) exactly two heads (ii) at most two heads (iii) at least two heads

Solution

Ten coins are tossed simultaneously one time = one coin is tossed 10 times

Let S the sample space,

That is $$S = \overbrace{\{H,T\} \times \{H,T\} \times \{H,T\} \times \cdots \times \{H,T\}}^{10\text{ times}}$$

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Let A be the event of getting exactly two heads,

B be the event of getting at most two heads, and

C be the event of getting at least two heads.

When ten coins are tossed, the number of elements in sample space is $2^n = 2^{10} = 1024$

$$n(S) = 1024$$

$$n(A) = {}^{10}C_2 = 45$$

$$n(B) = {}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 = 1 + 10 + 45 = 56$$

$$n(C) = {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 + \cdots + {}^{10}C_{10}$$

$$= n(S) - ({}^{10}C_0 + {}^{10}C_1) = 1024 - 11 = 1013$$

The required probabilities are

(i) $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{45}{1024}$

(ii) $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{56}{1024} = \dfrac{7}{128}$

(iii) $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{1013}{1024}$.

Example 12.5

Suppose a fair die is rolled. Find the probability of getting

(i) an even number (ii) multiple of three.

Solution

Let S be the sample space,

A be the event of getting an even number,

B be the event of getting multiple of three.

Therefore,

$S = \{1, 2, 3, 4, 5, 6\}$. $\Rightarrow n(S) = 6$

$A = \{2, 4, 6\}$ $\Rightarrow n(A) = 3$

$B = \{3, 6\}$ $\Rightarrow n(B) = 2$

The required probabilities are

(i) P (getting an even number) = $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{3}{6} = \dfrac{1}{2}$

(ii) P (getting multiple of three) = $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{2}{6} = \dfrac{1}{3}$.

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Example 12.6

When a pair of fair dice is rolled, what are the probabilities of getting the sum

(i) 7 (ii) 7 or 9 (iii) 7 or 12?

Solution

The sample space $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$

$$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

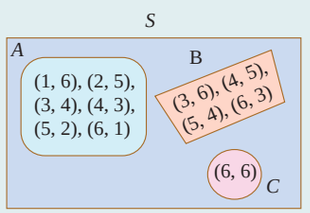
Number of possible outcomes $= 6^2 = 36 = n(S)$

Let A be the event of getting sum 7, B be the event of getting the sum 9 and C be the event of getting sum 12. Then

$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} \Rightarrow n(A) = 6$

$B = \{(3,6), (4,5), (5,4), (6,3)\} \Rightarrow n(B) = 4$

$C = \{(6,6)\} \Rightarrow n(C) = 1$



(i) P (getting sum 7) $= P(A)$

$$= \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(ii) P (getting sum 7 or 9) $= P(A \text{ or } B) = P(A \cup B)$

$$= P(A) + P(B)$$

(Since A and B are mutually exclusive that is, $A \cap B = \varnothing$)

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} = \frac{6}{36} + \frac{4}{36} = \frac{5}{18}$$

(iii) P (getting sum 7 or 12) $= P(A \text{ or } C) = P(A \cup C)$

$$= P(A) + P(C) \quad \text{(since } A \text{ and } C \text{ are mutually exclusive)}$$

$$= \frac{n(A)}{n(S)} + \frac{n(C)}{n(S)} = \frac{6}{36} + \frac{1}{36} = \frac{7}{36}.$$

Example 12.7

Three candidates X, Y, and Z are going to play in a chess competition to win FIDE (World Chess Federation) cup this year. X is thrice as likely to win as Y and Y is twice as likely as to win Z. Find the respective probability of X, Y and Z to win the cup.

Solution

Let A, B, C be the event of winning FIDE cup respectively by X, Y, and Z this year.

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Given that X is thrice as likely to win as Y.

$$A : B \;::\; 3 : 1. \qquad (1)$$

Y is twice as likely as to win Z

$$B : C \;::\; 2 : 1 \qquad (2)$$

From (1) and (2)

$$A : B : C \;::\; 6 : 2 : 1$$

$A = 6k$, $B = 2k$, $C = k$, where k is proportional constant.

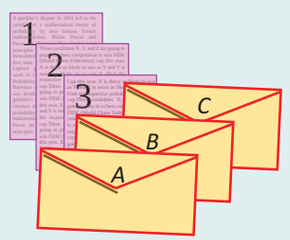
Probability to win the cup by X is $\quad P(A) = \dfrac{6k}{9k} = \dfrac{2}{3}$

Probability to win the cup by Y is $\quad P(B) = \dfrac{2k}{9k} = \dfrac{2}{9}$ and

Probability to win the cup by Z is $\quad P(C) = \dfrac{k}{9k} = \dfrac{1}{9}$.

Example 12.8

Three letters are written to three different persons and addresses on three envelopes are also written. Without looking at the addresses, what is the probability that (i) exactly one letter goes to the right envelopes (ii) none of the letters go into the right envelopes?



Solution

Let A, B, and C denote the envelopes and 1, 2, and 3 denote the corresponding letters.

The different combination of letters put into the envelopes are shown in the table.

Let c_i denote the outcomes of the events.

Let X be the event of putting the letters into the exactly only one right envelopes.

Let Y be the event of putting none of the letters into the right envelope.

Outcomes

Envelope	c_1	c_2	c_3	c_4	c_5	c_6
A	1	1	2	2	3	3
B	2	3	1	3	1	2
C	3	2	3	1	2	1

$S = \{c_1, c_2, c_3, c_4, c_5, c_6\}$, $n(S) = 6$

$$X = \{c_2, c_3, c_6\}, \qquad n(X) = 3$$

$$Y = \{c_4, c_5\} \qquad n(Y) = 2$$

$$P(X) = \frac{3}{6} = \frac{1}{2} \qquad P(Y) = \frac{2}{6} = \frac{1}{3}.$$

Example 12.9

Let the matrix $M = \begin{bmatrix} x & y \\ z & 1 \end{bmatrix}$. If x, y and z are chosen at random from the set $\{1, 2, 3\}$, and repetition is allowed $(i.e., x = y = z)$, what is the probability that the given matrix M is a singular matrix?

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Solution

If the given matrix M is singular, then

$$\begin{vmatrix} x & y \\ z & 1 \end{vmatrix} = 0.$$

That is, $x - yz = 0$.

Hence the possible ways of selecting (x, y, z) are

$\{(1,1,1),(2,1,2),(2,2,1),(3,1,3),(3,3,1)\} = A(say)$

The number of favourable cases $\quad n(A) = 5$

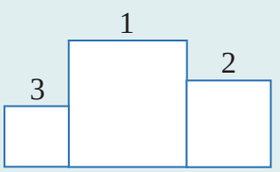
The total number of cases are $\quad n(S) = 3^3 = 27$

The probability of the given matrix is a singular matrix is

$$p = \frac{n(A)}{n(S)} = \frac{5}{27}.$$

Example 12.10

For a sports meet, a winners' stand comprising of three wooden blocks is in the form as shown in figure. There are six different colours available to choose from and three of the wooden blocks is to be painted such that no two of them has the same colour. Find the probability that the smallest block is to be painted in red, where red is one of the six colours.



Solution

Let S be the sample space and A be the event that the smallest block is to be painted in red.

$$n(S) = {}^6P_3 = 6 \times 5 \times 4 = 120$$

$$n(A) = 5 \times 4 = 20$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{20}{120} = \frac{1}{6}.$$

	6	6	3
$n(S)$	6	5	4
$n(A)$	5	4	Red

12.4.3 ODDS

The word **odds** is frequently used in probability and statistics. Odds relate the chances in favour of an event A to the chances against it. Suppose 'a' represents the number of ways that an event can occur and 'b' represents the number of ways that the event can fail to occur.

The odds of an event A are $a : b$ in favour of an event and

$$P(A) = \frac{a}{a+b}.$$

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Further, it may be noted that the odds are $a : b$ in favour of an event is the same as to say that the odds are $b : a$ against the event.

If the probability of an event is p, then the odds in favour of its occurrence are p to $(1-p)$ and the odds against its occurrence are $(1-p)$ to p.

Illustration 12.7

(i) Suppose a die is rolled.

Let S be the sample space and A be the event of getting 5.

$n(S)=6$, $n(A)=1$ and $n(\bar{A})=5$.

It can also be interpreted as

Odds in favour of A is $1:5$ or $\frac{1}{5}$, odds against A is $5:1$ or $\frac{5}{1}$,

and $P(A)=\frac{n(A)}{n(A)+n(\bar{A})}=\frac{1}{5+1}=\frac{1}{6}=\frac{n(A)}{n(S)}$.

(ii) Suppose B is an event such that odds in **favour** of B is 3:5, then $P(B)=\frac{3}{8}$

(iii) Suppose C is an event such that odds **against** C is 4:11, then $P(C)=\frac{11}{15}$.

Example 12.11

A man has 2 ten rupee notes, 4 hundred rupee notes and 6 five hundred rupee notes in his pocket. If 2 notes are taken at random, what are the odds in favour of both notes being of hundred rupee denomination and also its probability?

Solution

Let S be the sample space and A be the event of taking 2 hundred rupee note.

Therefore, $n(S)=12c_2=66$, $n(A)=4c_2=6$ and $n(\bar{A})=66-6=60$

Therefore, odds in favour of A is 6: 60

That is, odds in favour of A is 1: 10, and $P(A)=\frac{1}{11}$.

EXERCISE 12.1

(1) An experiment has the four possible mutually exclusive and exhaustive outcomes A, B, C, and D. Check whether the following assignments of probability are permissible.

(i) $P(A) = 0.15$, $P(B) = 0.30$, $P(C) = 0.43$, $P(D) = 0.12$

(ii) $P(A) = 0.22$, $P(B) = 0.38$, $P(C) = 0.16$, $P(D) = 0.34$

(iii) $P(A) = \frac{2}{5}$, $P(B) = \frac{3}{5}$, $P(C) = -\frac{1}{5}$, $P(D) = \frac{1}{5}$

(2) If two coins are tossed simultaneously, then find the probability of getting

(i) one head and one tail (ii) atmost two tails

(3) Five mangoes and 4 apples are in a box. If two fruits are chosen at random, find the probability that (i) one is a mango and the other is an apple (ii) both are of the same variety.

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(4) What is the chance that
(i) non-leap year (ii) leap year should have fifty three Sundays?

(5) Eight coins are tossed once, find the probability of getting
(i) exactly two tails (ii) atleast two tails (iii) atmost two tails

(6) An integer is chosen at random from the first 100 positive integers. What is the probability that the integer chosen is a prime or multiple of 8?

(7) A bag contains 7 red and 4 black balls, 3 balls are drawn at random.
Find the probability that (i) all are red (ii) one red and 2 black.

(8) A single card is drawn from a pack of 52 cards. What is the probability that
(i) the card is an ace or a king (ii) the card will be 6 or smaller
(iii) the card is either a queen or 9?

(9) A cricket club has 16 members, of whom only 5 can bowl. What is the probability that in a team of 11 members at least 3 bowlers are selected?

(10) (i) The odds that the event A occurs is 5 to 7, find $P(A)$.
(ii) Suppose $P(B) = \frac{2}{5}$. Express the odds that the event B occurs.

12.5 Some basic Theorems on Probability

The problems solved in the last sections are related to mutually exclusive events. So we have used the formula $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$. But when the events are mutually inclusive, the additivity axioms counts $(A \cap B)$ twice. We have a separate formula for the events when they are mutually inclusive.

In the development of probability theory, all the results are derived directly or indirectly using only the axioms of probability. Here we derive some of the basic important theorems on probability.

Theorem 12.3

The probability of the impossible event is zero. That is,

$$\boxed{P(\varnothing) = 0}$$

Proof

Impossible event contains no sample point.

Therefore, $S \cup \varnothing = S$

$$P(S \cup \varnothing) = P(S)$$

$$P(S) + P(\varnothing) = P(S) \quad \text{(since } S \text{ and } \varnothing \text{ are mutually exclusive)}$$

$$P(\varnothing) = 0$$

Example 12.12

Find the probability of getting the number 7, when a usual die is rolled.

Solution

The event of getting 7 is an impossible event. Therefore, $P(\text{getting } 7) = 0$

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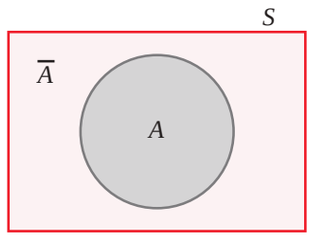
Theorem 12.4

If $\overline{A}$ is the complementary event of A, then

$$\boxed{P(\overline{A}) = 1 - P(A)}$$



Proof

Let S be a sample space, we have

$$A \cup \overline{A} = S$$

$$P(A \cup \overline{A}) = P(S)$$

$$P(A) + P(\overline{A}) = P(S) \text{ (since } A \text{ and } \overline{A} \text{ are mutually exclusive)}$$

$$= 1$$

$$P(\overline{A}) = 1 - P(A) \text{ or } P(A) = 1 - P(\overline{A})$$

Example 12.13

Nine coins are tossed once, find the probability to get at least two heads.

Solution

Let S be the sample space and A be the event of getting at least two heads.

Therefore, the event $\overline{A}$ denotes, getting at most one head.

$$n(S) = 2^9 = 512, \quad n(\overline{A}) = 9C_0 + 9C_1 = 1 + 9 = 10$$

$$P(\overline{A}) = \frac{10}{512} = \frac{5}{256}$$

$$P(A) = 1 - P(\overline{A}) = 1 - \frac{5}{256} = \frac{251}{256}$$

It is easier to calculate $\sum_{r=0}^{1} 9C_r$ than $\sum_{r=2}^{9} 9C_r$

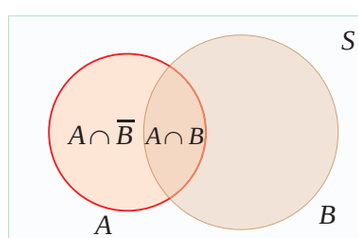
Theorem 12.5

If A and B are any two events and $\overline{B}$ is the complementary events of B, then

$$\boxed{P(A \cap \overline{B}) = P(A) - P(A \cap B)}$$

Proof

Clearly from the figure,



$$(A \cap \overline{B}) \cup (A \cap B) = A$$

$$P\left[(A \cap \overline{B}) \cup (A \cap B)\right] = P(A)$$

$$P(A \cap \overline{B}) + P(A \cap B) = P(A)$$

(since $(A \cap \overline{B})$ and $(A \cap B)$ are mutually exclusive)

$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

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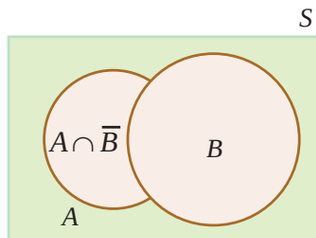
Theorem 12.6 (Addition theorem on probability)

If A and B are any two events, then

$$\boxed{P(A\cup B)=P(A)+P(B)-P(A\cap B)}$$



Proof

From the diagram,

$$A\cup B = (A\cap\overline{B})\cup B$$

$$P(A\cup B) = P\left[(A\cap\overline{B})\cup B\right]$$

$$= P(A\cap\overline{B})+P(B) \quad \text{(since } (A\cap\overline{B}) \text{ and } B \text{ are mutually exclusive)}$$

$$= [P(A)-P(A\cap B)]+P(B)$$

Therefore, $P(A\cup B) = P(A)+P(B)-P(A\cap B)$ ■

Note 12.4

The above theorem can be extended to any 3 events.

(i) $P(A\cup B\cup C) = \{P(A)+P(B)+P(C)\}$
$\quad - \{P(A\cap B)+P(B\cap C)+P(C\cap A)\}+P(A\cap B\cap C)\}$

(ii) $P(A\cup B\cup C) = 1-P(\overline{A\cup B\cup C}) = 1-P(\overline{A}\cap\overline{B}\cap\overline{C})$

Example 12.14

Given that $P(A) = 0.52$, $P(B) = 0.43$, and $P(A\cap B) = 0.24$, find

(i) $P(A\cap\overline{B})$ (ii) $P(A\cup B)$ (iii) $P(\overline{A}\cap\overline{B})$ (iv) $P(\overline{A}\cup\overline{B})$.

Solution

(i) $P(A\cap\overline{B}) = P(A)-P(A\cap B)$

$= 0.52-0.24 = 0.28$

$P(A\cap\overline{B}) = 0.28.$

(ii) $P(A\cup B) = P(A)+P(B)-P(A\cap B)$

$= 0.52+0.43-0.24$

$P(A\cup B) = 0.71.$

(iii) $P(\overline{A}\cap\overline{B}) = P(\overline{A\cup B})$ (By de Morgan's law)

$= 1-P(A\cup B)$

$= 1-0.71 = 0.29.$

(iv) $P(\overline{A}\cup\overline{B}) = P(\overline{A\cap B})$ (By de Morgan's law)

$= 1-P(A\cap B) = 1-0.24$

$= 0.76.$

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Example 12.15

The probability that a girl, preparing for competitive examination will get a State Government service is 0.12, the probability that she will get a Central Government job is 0.25, and the probability that she will get both is 0.07. Find the probability that (*i*) she will get atleast one of the two jobs (*ii*) she will get only one of the two jobs.

Solution

Let I be the event of getting State Government service and C be the event of getting Central Government job.

Given that $P(I) = 0.12$, $P(C) = 0.25$, and $P(I \cap C) = 0.07$

(*i*) P (at least one of the two jobs) $= P(I \text{ or } C) = P(I \cup C)$

$$= P(I) + P(C) - P(I \cap C)$$

$$= 0.12 + 0.25 - 0.07 = 0.30$$

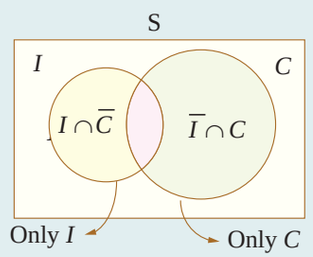
(*ii*) P(only one of the two jobs) $=$ P[only I or only C].

$$= P(I \cap \overline{C}) + P(\overline{I} \cap C)$$

$$= \{P(I) - P(I \cap C)\} + \{P(C) - P(I \cap C)\}$$

$$= \{0.12 - 0.07\} + \{0.25 - 0.07\}$$

$$= 0.23.$$



EXERCISE 12.2

(1) If A and B are mutually exclusive events $P(A) = \frac{3}{8}$ and $P(B) = \frac{1}{8}$, then

find (i) $P(\overline{A})$ (ii) $P(A \cup B)$ (iii) $P(\overline{A} \cap B)$ (iv) $P(\overline{A} \cup \overline{B})$

(2) If A and B are two events associated with a random experiment for which

$P(A) = 0.35$, $P(A \text{ or } B) = 0.85$, and $P(A \text{ and } B) = 0.15$.

Find (i) $P(\text{only } B)$ (ii) $P(\overline{B})$ (iii) $P(\text{only } A)$

(3) A die is thrown twice. Let A be the event, 'First die shows 5' and B be the event, 'second die shows 5'. Find $P(A \cup B)$.

(4) The probability of an event A occurring is 0.5 and B occurring is 0.3. If A and B are mutually exclusive events, then find the probability of

(i) $P(A \cup B)$ (ii) $P\left(A \cap \overline{B}\right)$ (iii) $P\left(\overline{A} \cap B\right)$.

(5) A town has 2 fire engines operating independently. The probability that a fire engine is available when needed is 0.96.

(i) What is the probability that a fire engine is available when needed?

(ii) What is the probability that neither is available when needed?

(6) The probability that a new railway bridge will get an award for its design is 0.48, the probability that it will get an award for the efficient use of materials is 0.36, and that it will get both awards is 0.2. What is the probability, that (i) it will get at least one of the two awards (ii) it will get only one of the awards.

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12.6 Conditional Probability

Illustration 12.8

Consider the following example to understand the concept of conditional probability.

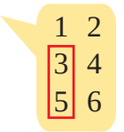
Suppose a fair die is rolled once, then the sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Now we ask two questions

Q_1: What is the probability of getting an odd number which is greater than 2?

Q_2: If the die shows an odd number, then what is the probability that it is greater than 2?

Case 1

The event of getting an odd number which is greater than 2 is $\{3, 5\}$.

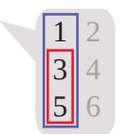


Let P_1 be the probability of getting an odd number which is greater than 2

$$P_1 = \frac{n(\{3,5\})}{n(\{1,\ 2,\ 3,\ 4,\ 5,\ 6\})} = \frac{2}{6} = \frac{1}{3}.$$

Case 2

'If the die shows an odd number' means we restrict our sample space S to a subset containing only odd number.



That is, $S_1 = \{1, 3, 5\}$. Then our interest is to find the probability of the event getting an odd number greater than 2. Let it be P_2

$$P_2 = \frac{n(\{3,5\})}{n(\{\ 1,\ 3,\ 5\})} = \frac{2}{3}$$

In the above two cases the favourable events are the same, but the number of exhaustive outcomes are different. In case 2, we observe that we have first imposed a condition on sample space, then asked to find the probability. This type of probability is called conditional probability.

This can be written by using sample space as

$$P_2 = \frac{\dfrac{n(\{3,5\})}{n(\{1,2,3,4,5,6\})}}{\dfrac{n(\{1,3,5\})}{n(\{1,2,3,4,5,6\})}} = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3}$$

Important note: Sample space is same for probability and conditional probability.

Definition 12.14

The **conditional probability** of an event B, assuming that the event A has already happened is denoted by $P(B/A)$ and is defined as

$$\boxed{P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided } P(A) \neq 0}$$

Similarly,

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad \text{provided } P(B) \neq 0}$$

Example 12.16

If $P(A) = 0.6$, $P(B) = 0.5$, and $P(A \cap B) = 0.2$

Find (i) $P(A/B)$ (ii) $P(\overline{A}/B)$ (iii) $P(A/\overline{B})$.

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Solution

Given that $P(A)=0.6$, $P(B)=0.5$, and $P(A\cap B)=0.2$

(i) $$P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{0.2}{0.5} = \frac{2}{5}$$

(ii) $$P(\overline{A}/B) = \frac{P(\overline{A}\cap B)}{P(B)}$$

$$= \frac{P(B)-P(A\cap B)}{P(B)}$$

$$= \frac{0.5-0.2}{0.5} = \frac{0.3}{0.5} = \frac{3}{5}.$$

(iii) $$P(A/\overline{B}) = \frac{P(A\cap \overline{B})}{P(\overline{B})}$$

$$= \frac{P(A)-P(A\cap B)}{1-P(B)}$$

$$= \frac{0.6-0.2}{1-0.5} = \frac{0.4}{0.5} = \frac{4}{5}.$$

Note 12.5

$$P(A/B)+P(\overline{A}/B)=1$$

Example 12.17

A die is rolled. If it shows an odd number, then find the probability of getting 5.

Solution

Sample space S = {1, 2, 3, 4, 5, 6}.

Let A be the event of die shows an odd number.

Let B be the event of getting 5.

Then, A={1, 3, 5}, B={5}, and $A\cap B$ = {5}.

Therefore, $P(A)=\frac{3}{6}$ and $P(A\cap B)=\frac{1}{6}$

$$P(\text{getting } 5/\text{die shows an odd number}) = P(B/A)$$

$$= \frac{P(A\cap B)}{P(A)} = \frac{\frac{1}{6}}{\frac{3}{6}}$$

$$P(B/A)=\frac{1}{3}.$$

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Rewriting the definition of conditional probability, we get the 'Multiplication theorem on probability'.

Theorem 12.7

(Multiplication theorem on probability)

The probability of the simultaneous happening of two events A and B is given by

$$\boxed{\begin{array}{c} P(A\cap B)=P(A/B)P(B) \\ \text{or} \\ P(A\cap B)=P(B/A)P(A) \end{array}}$$

12.6.1 Independent Events

Events are said to be independent if occurrence or non-occurrence of any one of the event does not affect the probability of occurrence or non-occurrence of the other events.

Definition 12.15

Two events A and B are said to be independent if and only if

$$\boxed{P(A\cap B)=P(A)\cdot P(B)}$$

Note 12.6

(1) This definition is exactly equivalent to

$$P(A/B) = P(A) \quad \text{if } P(B)>0$$
$$P(B/A) = P(B) \quad \text{if } P(A)>0$$

(2) The events $A_1, A_2, A_3, \ldots, A_n$ are mutually independent if
$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = P(A_1)\cdot P(A_2)\cdot \cdots \cdot P(A_n)$.

Theorem 12.8

If A and B are independent then

(i) $\overline{A}$ and $\overline{B}$ are independent.

(ii) A and $\overline{B}$ are independent.

(iii) $\overline{A}$ and B are also independent.

Proof

(i) To prove $\overline{A}$ and $\overline{B}$ are independent:
Since A and B are independent

$$P(A\cap B) = P(A)\cdot(PB)$$

To prove $\overline{A}$ and $\overline{B}$ are independent, we have to prove

$$P(\overline{A}\cap\overline{B}) = P(\overline{A})\cdot P(\overline{B}).$$

By de Morgan's law

$$P(\overline{A}\cap\overline{B}) = P\left(\overline{A\cup B}\right)$$
$$= 1-P\left(A\cup B\right)$$

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$$= 1-\{P(A)+P(B)-P(A\cap B)\}$$

$$= 1-P(A)-P(B)+P(A)\cdot P(B)$$

$$= (1-P(A))(1-P(B))$$

$$= P(\overline{A})\cdot P(\overline{B})$$

Thus $\overline{A}$ and $\overline{B}$ are independent .

Similarly one can prove (ii) and (iii).

Example 12.18

Two cards are drawn from a pack of 52 cards in succession. Find the probability that both are Jack when the first drawn card is (i) replaced (ii) not replaced

Solution

Let A be the event of drawing a Jack in the first draw,

B be the event of drawing a Jack in the second draw.

Case (i)

Card is replaced

$$n(A) = 4 \quad \text{(Jack)}$$

$$n(B) = 4 \quad \text{(Jack)}$$

$$\text{and} \quad n(S) = 52 \quad \text{(Total)}$$

Clearly the event A will not affect the probability of the occurrence of event B and therefore A and B are independent.

$$P(A\cap B) = P(A)\,.\,P(B)$$

$$P(A) = \frac{4}{52}\ ,\ P(B)=\frac{4}{52}$$

$$P(A\cap B) = P(A)\ P(B)$$

$$= \frac{4}{52}\cdot\frac{4}{52}$$

$$= \frac{1}{169}\ .$$

Case (ii)

Card is not replaced

In the first draw, there are 4 Jacks and 52 cards in total. Since the Jack, drawn at the first draw is not replaced, in the second draw there are only 3 Jacks and 51 cards in total. Therefore the first event A affects the probability of the occurrence of the second event B.

Thus A and B are not independent. That is, they are dependent events.

$$\text{Therefore, } P(A\cap B) = P(A)\,.\,P(B/A)$$

$$P(A) = \frac{4}{52}$$

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$$P(B/A) = \frac{3}{51}$$

$$P(A \cap B) = P(A) \,.\, P(B/A)$$

$$= \frac{4}{52} \cdot \frac{3}{51}$$

$$= \frac{1}{221} \,.$$

Example 12.19

A coin is tossed twice. Events E and F are defined as follows

E= Head on first toss, F = Head on second toss. Find

(i) $P(E \cup F)$ (ii) $P(E / F)$

(iii) $P(\overline{E} / F)$. (iv) Are the events E and F independent?

Solution

The sample space is $S = \{H, T\} \times \{H, T\}$

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

$$\text{and } E = \{(H, H), (H, T)\}$$

$$F = \{(H, H), (T, H)\}$$

$$E \cup F = \{(H, H), (H, T), (T, H)\}$$

$$E \cap F = \{(H, H)\}$$

(i) $$P(E \cup F) = P(E) + P(F) - P(E \cap F) \quad \text{or} \left(= \frac{n(E \cup F)}{n(S)} \right)$$

$$= \frac{2}{4} + \frac{2}{4} - \frac{1}{4} = \frac{3}{4} \,.$$

(ii) $$P(E / F) = \frac{P(E \cap F)}{P(F)} = \frac{(1/4)}{(2/4)} = \frac{1}{2}$$

(iii) $$P(\overline{E} / F) = \frac{P(\overline{E} \cap F)}{P(F)}$$

$$= \frac{P(F) - P(E \cap F)}{P(F)}$$

$$= \frac{(2/4) - (1/4)}{(2/4)}$$

$$= \frac{1}{2}$$

(iv) Are the events E and F independent?

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We have $P(E \cap F) = \frac{1}{4}$

$$P(E) = \frac{2}{4}, \quad P(F) = \frac{2}{4}$$

$$P(E)\,P(F) = \frac{2}{4}.\frac{2}{4} = \frac{1}{4}$$

$$\Rightarrow P(E \cap F) = P(E)\,.\,P(F)$$

Therefore E and F are independent events.

Note 12.7

Independent events is a property of probability but mutual exclusiveness is a set-theoretic property. Therefore independent events can be identified by their probabilities and mutually exclusive events can be identified by their events.

Theorem 12.9

Suppose A and B are two events, such that $P(A) \neq 0$, $P(B) \neq 0$.

(1) If A and B are mutually exclusive, they cannot be independent.

(2) If A and B are independent they cannot be mutually exclusive. (Without proof)

Example 12.20

If A and B are two independent events such that $P(A) = 0.4$ and $P(A \cup B) = 0.9$. Find $P(B)$.

Solution

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A).P(B) \qquad \text{(since } A \text{ and } B \text{ are independent)}$$

That is, $0.9 = 0.4 + P(B) - (0.4)\,P(B)$

$$0.9 - 0.4 = (1 - 0.4)\,P(B)$$

Therefore, $P(B) = \frac{5}{6}$.

Example 12.21

An anti-aircraft gun can take a maximum of four shots at an enemy plane moving away from it. The probability of hitting the plane in the first, second, third, and fourth shot are respectively 0.2, 0.4, 0.2 and 0.1. Find the probability that the gun hits the plane.

Solution

Let H_1, H_2, H_3 and H_4 be the events of hitting the plane by the anti-aircraft gun in the first second, third and fourth shot respectively.

Let H be the event that anti-aircraft gun hits the plane. Therefore $\bar{H}$ is the event that the plane is not shot down. Given that

$$P(H_1) = 0.2 \qquad \Rightarrow P(\bar{H}_1) = 1 - P(H_1) = 0.8$$

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$P(H_2) = 0.4 \quad \Rightarrow P(\bar{H}_2) = 1 - P(H_2) = 0.6$

$P(H_3) = 0.2 \quad \Rightarrow P(\bar{H}_3) = 1 - P(H_3) = 0.8$

$P(H_4) = 0.1 \quad \Rightarrow P(\bar{H}_4) = 1 - P(H_4) = 0.9$

The probability that the gun hits the plane is

$$P(H) = 1 - P(\bar{H}) = 1 - P(\overline{H_1 \cup H_2 \cup H_3 \cup H_4})$$

$$= 1 - P\left(\bar{H}_1 \cap \bar{H}_2 \cap \bar{H}_3 \cap \bar{H}_4\right)$$

$$= 1 - P(\bar{H}_1)P(\bar{H}_2)P(\bar{H}_3)P(\bar{H}_4)$$

$$= 1 - (0.8)(0.6)(0.8)(0.9) = 1 - 0.3456$$

$$P(H) = 0.6544$$

Example 12.22

X speaks truth in 70 percent of cases, and Y in 90 percent of cases. What is the probability that they likely to contradict each other in stating the same fact?

Solution

Let A be the event of X speaks the truth, B be the event of Y speaks the truth

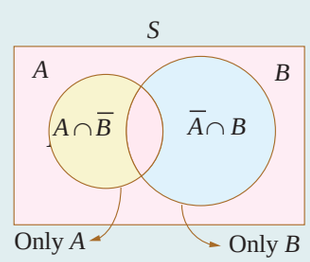
$\therefore \bar{A}$ is the event of X not speaking the truth and $\bar{B}$ is the event of Y not speaking the truth.

Let C be the event that they will contradict each other.

Given that

$P(A) = 0.70 \quad \Rightarrow P(\bar{A}) = 1 - P(A) = 0.30$

$P(B) = 0.90 \quad \Rightarrow P(\bar{B}) = 1 - P(B) = 0.10$



C = (A speaks truth and B does not speak truth or B speaks truth and A does not speak truth)

$C = [(A \cap \bar{B}) \cup (\bar{A} \cap B)]$ (see figure)

since $(A \cap \bar{B})$ and $(\bar{A} \cap B)$ are mutually exclusive,

$$P(C) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= P(A)\,P(\bar{B}) + P(\bar{A})\,P(B)$$

(Since A, B are independent event, A, $\bar{B}$ are also independent events)

$$= (0.70)\ (0.10) + (0.30)\ (0.90)$$

$$= 0.070 + 0.270 = 0.34$$

$$P(C) = 0.34.$$

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Example 12.23

A main road in a City has 4 crossroads with traffic lights. Each traffic light opens or closes the traffic with the probability of 0.4 and 0.6 respectively. Determine the probability of

(i) a car crossing the first crossroad without stopping
(ii) a car crossing first two crossroads without stopping
(iii) a car crossing all the crossroads, stopping at third cross.
(iv) a car crossing all the crossroads, stopping at exactly one cross.

Solution:

Let A_i be the event that the traffic light opens at i th cross, for i= 1, 2, 3, 4.

Let B_i be the event that the traffic light closes at i th cross, for i= 1, 2, 3, 4.

The traffic lights are all independent.

Therefore A_i and B_i are all independent events, for i= 1, 2, 3, 4.

Given that

$$P(A_i) = 0.4,\ i = 1,\ 2,\ 3,\ 4$$

$$P(B_i) = 0.6,\ i = 1,\ 2,\ 3,\ 4$$

(i) Probability of car crossing the first crossroad without stopping,

$$P(A_1) = 0.4.$$

(ii) Probability of car crossing first two crossroads without stopping,

$$P(A_1 \cap A_2) = P(A_1A_2) = (0.4)(0.4) = 0.16$$

(iii) Probability of car crossing all the crossroads, stopping at third cross

$$P(A_1 \cap A_2 \cap B_3 \cap A_4) = P(A_1A_2B_3A_4) = (0.4)(0.4)(0.6)(0.4) = 0.0384$$

(iv) Probability of car crossing all the crossroads, stopping at exactly one of the crossroads is

$$P(B_1A_2A_3A_4 \cup A_1B_2A_3A_4 \cup A_1A_2B_3A_4 \cup A_1A_2A_3B_4)$$

$$= P(B_1A_2A_3A_4) + P(A_1B_2A_3A_4) + P(A_1A_2B_3A_4) + P(A_1A_2A_3B_4)$$

$$= 4(0.4)(0.4)(0.6)(0.4) = 4(0.0384) = 0.1536$$

EXERCISE 12.3

(1) Can two events be mutually exclusive and independent simultaneously?

(2) If A and B are two events such that $P(A \cup B) = 0.7$, $P(A \cap B) = 0.2$, and $P(B) = 0.5$, then show that A and B are independent.

(3) If A and B are two independent events such that $P(A \cup B) = 0.6$, $P(A) = 0.2$, find $P(B)$.

(4) If $P(A) = 0.5$, $P(B) = 0.8$ and $P(B/A) = 0.8$, find $P(A / B)$ and $P(A \cup B)$.

(5) If for two events A and B, $P(A) = \frac{3}{4}$, $P(B) = \frac{2}{5}$ and $A \cup B = S$ (sample space), find the conditional probability $P(A / B)$.

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(6) A problem in Mathematics is given to three students whose chances of solving it are $\frac{1}{3}, \frac{1}{4}$, and $\frac{1}{5}$ (i) What is the probability that the problem is solved? (ii) What is the probability that exactly one of them will solve it?

(7) The probability that a car being filled with petrol will also need an oil change is 0.30; the probability that it needs a new oil filter is 0.40; and the probability that both the oil and filter need changing is 0.15.

(i) If the oil had to be changed, what is the probability that a new oil filter is needed?

(ii) If a new oil filter is needed, what is the probability that the oil has to be changed?

(8) One bag contains 5 white and 3 black balls. Another bag contains 4 white and 6 black balls. If one ball is drawn from each bag, find the probability that (i) both are white (ii) both are black (iii) one white and one black

(9) Two thirds of students in a class are boys and rest girls. It is known that the probability of a girl getting a first grade is 0.85 and that of boys is 0.70. Find the probability that a student chosen at random will get first grade marks.

(10) Given $P(A) = 0.4$ and $P(A \cup B) = 0.7$. Find $P(B)$ if

(i) A and B are mutually exclusive (ii) A and B are independent events

(iii) $P(A / B) = 0.4$ (iv) $P(B / A) = 0.5$

(11) A year is selected at random. What is the probability that

(i) it contains 53 Sundays (ii) it is a leap year which contains 53 Sundays

(12) Suppose the chances of hitting a target by a person X is 3 times in 4 shots, by Y is 4 times in 5 shots, and by Z is 2 times in 3 shots. They fire simultaneously exactly one time. What is the probability that the target is damaged by exactly 2 hits?

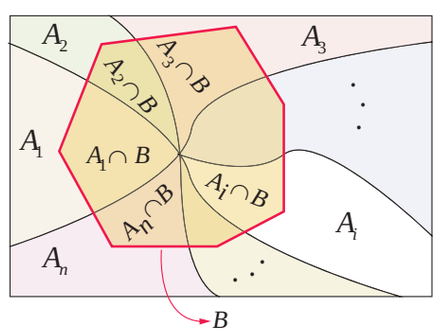
12.7 Total Probability of an event

Theorem 12.10 (Total Probability of an event)

If $A_1, A_2, A_3, ..., A_n$ are mutually exclusive and exhaustive events and B is any event in S then $P(B)$ is called the total probability of event B and

$$\boxed{P(B) = P(A_1).P(B/A_1) + P(A_2).P(B/A_2) + \cdots + P(A_n).P(B/A_n) = \sum_{i=1}^{n} P(A_i).P(B/A_i)}$$

Proof



Since B is any event in S, from the figure shown here

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \cdots (A_n \cap B).$$

Since $A_1, A_2, A_3 ... A_n$ are mutually exclusive,

$(A_1 \cap B), (A_2 \cap B), (A_3 \cap B), \cdots, (A_n \cap B)$ are also mutually exclusive.

Therefore,

$$P(B) = P\left[(A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \cdots \cup (A_n \cap B)\right]$$

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) + \cdots + P(A_n \cap B)$$

$$P(B) = P(A_1).P(B/A_1) + P(A_2).P(B/A_2) + \cdots + P(A_n).P(B/A_n) = \sum_{i=1}^{n} P(A_i).P(B/A_i).$$

The following problems are solved using the law of total probability of an event.

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Example 12.24

Urn-I contains 8 red and 4 blue balls and urn-II contains 5 red and 10 blue balls. One urn is chosen at random and two balls are drawn from it. Find the probability that both balls are red.

Solution

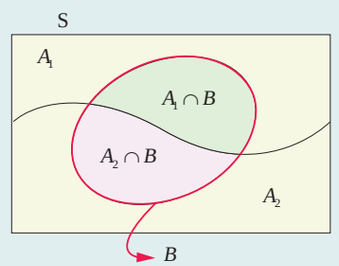
	Red balls	Blue balls	Total
Urn-I	8	4	12
Urn-II	5	10	15
Total	13	14	27

Let A_1 be the event of selecting urn-I and A_2 be the event of selecting urn-II.

Let B be the event of selecting 2 red balls.

We have to find the total probability of event B. That is, $P(B)$.

Clearly A_1 and A_2 are mutually exclusive and exhaustive events.



We have

$$P(A_1) = \frac{1}{2}, \quad P(B/A_1) = \frac{8c_2}{12c_2} = \frac{14}{33}$$

$$P(A_2) = \frac{1}{2}, \quad P(B/A_2) = \frac{5c_2}{15c_2} = \frac{2}{21}$$

We know

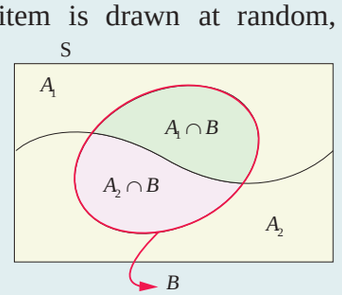
$$P(B) = P(A_1) \,.\, P(B/A_1) + P(A_2) \,.\, P(B/A_2)$$

$$P(B) = \frac{1}{2}.\frac{14}{33} + \frac{1}{2}.\frac{2}{21} = \frac{20}{77}.$$

Example 12.25

A factory has two machines I and II. Machine-I produces 40% of items of the output and Machine-II produces 60% of the items. Further 4% of items produced by Machine-I are defective and 5% produced by Machine-II are defective. If an item is drawn at random, find the probability that it is a defective item.

Solution



Let A_1 be the event that the items are produced by Machine-I, A_2 be the event that items are produced by Machine-II. Let B be the event of drawing a defective item.

We have to find the total probability of event B. That is, $P(B)$.

Clearly A_1 and A_2 are mutually exclusive and exhaustive events.

Therefore, $P(B) = P(A_1) \,.\, P(B/A_1) + P(A_2) \,.\, P(B/A_2)$

We have $\quad P(A_1) = 0.40, \quad P(B/A_1) = 0.04$

$P(A_2) = 0.60, \quad P(B/A_2) = 0.05$

$$P(B) = P(A_1) \,.\, P(B/A_1) + P(A_2) \,.\, P(B/A_2)$$

$$= (0.40)(0.04) + (0.60)(0.05)$$

$$= 0.046.$$

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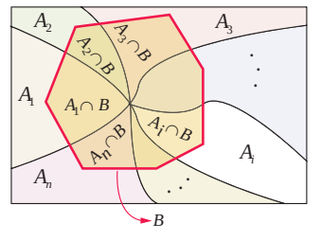
12.8 Bayes' Theorem

1702-1761

Thomas Bayes was an English statistician, philosopher and Presbyterian minister who is known for formulating a specific case of a theorem. Bayesian methods stem from the principle of linking prior (before conducting experiment) probability and conditional probability (likelihood) to posterior (after conducting experiment) probability via Bayes' rule. Bayesian probability is the name given to several related interpretations of probability as an amount of epistemic confidence – the strength of beliefs, hypotheses etc., rather than a frequency.



Theorem 12.11 (Bayes' Theorem)

If $A_1, A_2, A_3, ..., A_n$ are mutually exclusive and exhaustive events such that $P(A_i) > 0$, $i = 1, 2, 3, \ldots . n$ and B is any event in which $P(B) > 0$, then

$$\boxed{P(A_i / B) = \frac{P(A_i)\, P(B / A_i)}{P(A_1)\, P(B / A_1) + P(A_2)\, P(B / A_2) + \cdots + P(A_n)\, P(B / A_n)}}$$

Proof

By the law of total probability of B we have

$$P(B) = P(A_1) \cdot P(B / A_1) + P(A_2) \cdot P(B / A_2) + \cdots + P(A_n) \cdot P(B / A_n)$$

and by multiplication theorem $P(A_i \cap B) = P(B/A_i) P(A_i)$

By the definition of conditional probability,

$$P(A_i / B) = \frac{P(A_i \cap B)}{P(B)}$$

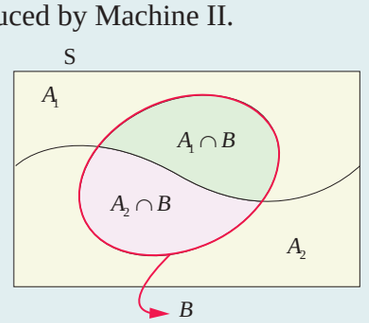
$$P(A_i / B) = \frac{P(B/A_i) P(A_i)}{P(A_1) \cdot P(B / A_1) + P(A_2) \cdot P(B / A_2) + \cdots + P(A_n) \cdot P(B / A_n)} \quad \text{(using formulae)}$$

The above formula gives the relationship between $P(A_i / B)$ and $P(B / A_i)$

Example 12.26

A factory has two machines I and II. Machine I produces 40% of items of the output and Machine II produces 60% of the items. Further 4% of items produced by Machine I are defective and 5% produced by Machine II are defective. An item is drawn at random. If the drawn item is defective, find the probability that it was produced by Machine II. (See the previous example, compare the questions).

Solution



Let A_1 be the event that the items are produced by Machine-I, A_2 be the event that items are produced by Machine-II. Let B be the event of drawing a defective item. Now we are asked to find the conditional probability $P(A_2/B)$. Since A_1, A_2 are mutually exclusive and exhaustive events, by Bayes' theorem,

$$P(A_2 / B) = \frac{P(A_2)\, P(B / A_2)}{P(A_1)\, P(B / A_1) + P(A_2)\, P(B / A_2)}$$

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We have,

$$P(A_1) = 0.40, \quad P(B/A_1) = 0.04$$

$$P(A_2) = 0.60, \quad P(B/A_2) = 0.05$$

$$P(A_2/B) = \frac{P(A_2)\,P(B/A_2)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)}$$

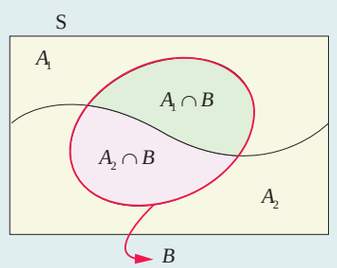
$$P(A_2/B) = \frac{(0.60)(0.05)}{(0.40)(0.04)+(0.60)(0.05)} = \frac{15}{23}.$$

Example 12.27

A construction company employs 2 executive engineers. Engineer-1 does the work for 60% of jobs of the company. Engineer-2 does the work for 40% of jobs of the company. It is known from the past experience that the probability of an error when engineer-1 does the work is 0.03, whereas the probability of an error in the work of engineer-2 is 0.04. Suppose a serious error occurs in the work, which engineer would you guess did the work?

Solution

Let A_1 and A_2 be the events of job done by engineer-1 and engineer-2 of the company respectively. Let B be the event that the error occurs in the work.



We have to find the conditional probability $P(A_1/B)$ and $P(A_2/B)$ to compare their errors in their work.

From the given information, we have

$$P(A_1) = 0.60, \quad P(B/A_1) = 0.03$$

$$P(A_2) = 0.40, \quad P(B/A_2) = 0.04$$

A_1 and A_2 are mutually exclusive and exhaustive events.

Applying Bayes' theorem,

$$P(A_1/B) = \frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)}$$

$$= \frac{(0.60)(0.03)}{(0.60)(0.03)+(0.40)(0.04)}$$

$$P(A_1/B) = \frac{9}{17}.$$

$$P(A_2/B) = \frac{P(A_2)\,P(B/A_2)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)}$$

$$P(A_2/B) = \frac{(0.40)(0.04)}{(0.60)(0.03)+(0.40)(0.04)}$$

$$P(A_2/B) = \frac{8}{17}.$$

Since $P(A_1/B) > P(A_2/B)$, the chance of error done by engineer-1 is greater than the chance of error done by engineer-2. Therefore one may guess that the serious error would have been be done by engineer-1.

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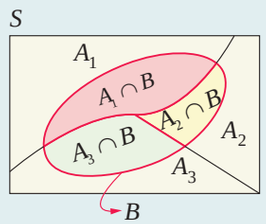
Example 12.28

The chances of X, Y and Z becoming managers of a certain company are 4 : 2 : 3. The probabilities that bonus scheme will be introduced if X, Y and Z become managers are 0.3, 0.5 and 0.4 respectively. If the bonus scheme has been introduced, what is the probability that Z was appointed as the manager?

Solution



Let A_1, A_2 and A_3 be the events of X, Y and Z becoming managers of the company respectively. Let B be the event that the bonus scheme will be introduced.

We have to find the conditional probability $P(A_3 / B)$.

Since A_1, A_2 and A_3 are mutually exclusive and exhaustive events, applying Bayes' theorem

$$P(A_3 / B) = \frac{P(A_3)\,P(B / A_3)}{P(A_1)\,P(B / A_1) + P(A_2)\,P(B / A_2) + P(A_3)\,P(B / A_3)}$$

We have

$$P(A_1) = \frac{4}{9}, \quad P(B / A_1) = 0.3$$

$$P(A_2) = \frac{2}{9}, \quad P(B / A_2) = 0.5$$

$$P(A_3) = \frac{3}{9}, \quad P(B / A_3) = 0.4$$

$$P(A_3 / B) = \frac{P(A_3)\,P(B / A_3)}{P(A_1)\,P(B / A_1) + P(A_2)\,P(B / A_2) + P(A_3)\,P(B / A_3)}$$

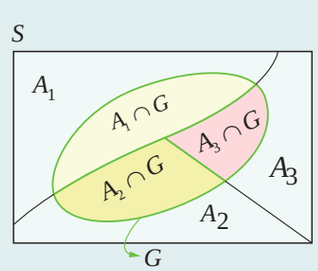
$$P(A_3 / B) = \frac{\left(\frac{3}{9}\right)(0.4)}{\left(\frac{4}{9}\right)(0.3) + \left(\frac{2}{9}\right)(0.5) + \left(\frac{3}{9}\right)(0.4)}$$

$$= \frac{12}{34} = \frac{6}{17}.$$

Example 12.29

A consulting firm rents car from three agencies such that 50% from agency L, 30% from agency M and 20% from agency N. If 90% of the cars from L, 70% of cars from M and 60% of the cars from N are in good conditions (i) what is the probability that the firm will get a car in good condition? (ii) if a car is in good condition, what is probability that it has come from agency N?

Solution



Let A_1, A_2, and A_3 be the events that the cars are rented from the agencies L, M and N respectively.

Let G be the event of getting a car in good condition.

We have to find

(i) the total probability of event G that is, $P(G)$

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(ii) find the conditional probability A_3 given G that is, $P(A_3/G)$

We have

$$P(A_1) = 0.50, \quad P(G/A_1) = 0.90$$

$$P(A_2) = 0.30, \quad P(G/A_2) = 0.70$$

$$P(A_3) = 0.20, \quad P(G/A_3) = 0.60.$$

(i) Since A_1, A_2 and A_3 are mutually exclusive and exhaustive events and G is an event in S, then the total probability of event G is P(G).

$$P(G) = P(A_1)\,P(G/A_1) + P(A_2)\,P(G/A_2) + P(A_3)\,P(G/A_3)$$

$$P(G) = (0.50)(0.90) + (0.30)(0.70) + (0.20)\,(0.60)$$

$$P(G) = 0.78.$$

(ii) The conditional probability A_3 given G is $P(A_3/G)$

By Bayes' theorem,

$$P(A_3/G) = \frac{P(A_3)\,P(G/A_3)}{P(A_1)\,P(G/A_1) + P(A_2)\,P(G/A_2) + P(A_3)\,P(G/A_3)}$$

$$P(A_3/G) = \frac{(0.20)(0.60)}{(0.50)(0.90) + (0.30)(0.70) + (0.20)\,(0.60)}$$

$$= \frac{2}{13}.$$

EXERCISE 12.4

(1) A factory has two Machines-I and II. Machine-I produces 60% of items and Machine-II produces 40% of the items of the total output. Further 2% of the items produced by Machine-I are defective whereas 4% produced by Machine-II are defective. If an item is drawn at random what is the probability that it is defective?

(2) There are two identical urns containing respectively 6 black and 4 red balls, 2 black and 2 red balls. An urn is chosen at random and a ball is drawn from it. (i) find the probability that the ball is black (ii) if the ball is black, what is the probability that it is from the first urn?

(3) A firm manufactures PVC pipes in three plants viz, X, Y and Z. The daily production volumes from the three firms X, Y and Z are respectively 2000 units, 3000 units and 5000 units. It is known from the past experience that 3% of the output from plant X, 4% from plant Y and 2% from plant Z are defective. A pipe is selected at random from a day's total production,

(i) find the probability that the selected pipe is a defective one.

(ii) if the selected pipe is a defective, then what is the probability that it was produced by plant Y ?

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(4) The chances of A, B and C becoming manager of a certain company are 5 : 3 : 2. The probabilities that the office canteen will be improved if A, B, and C become managers are 0.4, 0.5 and 0.3 respectively. If the office canteen has been improved, what is the probability that B was appointed as the manager?

(5) An advertising executive is studying television viewing habits of married men and women during prime time hours. Based on the past viewing records he has determined that during prime time wives are watching television 60% of the time. It has also been determined that when the wife is watching television, 40% of the time the husband is also watching. When the wife is not watching the television, 30% of the time the husband is watching the television. Find the probability that (i) the husband is watching the television during the prime time of television (ii) if the husband is watching the television, the wife is also watching the television.

BF6I7Q

EXERCISE 12.5

Choose the correct or most suitable answer from the given four alternatives

(1) Four persons are selected at random from a group of 3 men, 2 women and 4 children. The probability that exactly two of them are children is

(1) $\frac{3}{4}$ (2) $\frac{10}{23}$ (3) $\frac{1}{2}$ (4) $\frac{10}{21}$

(2) A number is selected from the set $\{1, 2, 3, ..., 20\}$. The probability that the selected number is divisible by 3 or 4 is

(1) $\frac{2}{5}$ (2) $\frac{1}{8}$ (3) $\frac{1}{2}$ (4) $\frac{2}{3}$

(3) A, B, and C try to hit a target simultaneously but independently. Their respective probabilities of hitting the target are $\frac{3}{4}, \frac{1}{2}, \frac{5}{8}$. The probability that the target is hit by A or B but not by C is

(1) $\frac{21}{64}$ (2) $\frac{7}{32}$ (3) $\frac{9}{64}$ (4) $\frac{7}{8}$

(4) If A and B are any two events, then the probability that exactly one of them occur is

(1) $P(A \cup \overline{B}) + P(\overline{A} \cup B)$ (2) $P(A \cap \overline{B}) + P(\overline{A} \cap B)$

(3) $P(A) + P(B) - P(A \cap B)$ (4) $P(A) + P(B) + 2P(A \cap B)$

(5) Let A and B be two events such that $P\left(\overline{A \cup B}\right) = \frac{1}{6}$, $P\left(A \cap B\right) = \frac{1}{4}$ and $P\left(\overline{A}\right) = \frac{1}{4}$. Then the events A and B are

(1) Equally likely but not independent (2) Independent but not equally likely

(3) Independent and equally likely (4) Mutually inclusive and dependent

(6) Two items are chosen from a lot containing twelve items of which four are defective, then the probability that at least one of the item is defective

(1) $\frac{19}{33}$ (2) $\frac{17}{33}$ (3) $\frac{23}{33}$ (4) $\frac{13}{33}$

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(7) A man has 3 fifty rupee notes, 4 hundred rupees notes and 6 five hundred rupees notes in his pocket. If 2 notes are taken at random, what are the odds in favour of both notes being of hundred rupee denomination?

(1) 1:12 (2) 12:1 (3) 13:1 (4) 1:13

(8) A letter is taken at random from the letters of the word 'ASSISTANT' and another letter is taken at random from the letters of the word 'STATISTICS'. The probability that the selected letters are the same is

(1) $\frac{7}{45}$ (2) $\frac{17}{90}$ (3) $\frac{29}{90}$ (4) $\frac{19}{90}$

(9) A matrix is chosen at random from a set of all matrices of order 2, with elements 0 or 1 only. The probability that the determinant of the matrix chosen is non zero will be

(1) $\frac{3}{16}$ (2) $\frac{3}{8}$ (3) $\frac{1}{4}$ (4) $\frac{5}{8}$

(10) A bag contains 5 white and 3 black balls. Five balls are drawn successively without replacement. The probability that they are alternately of different colours is

(1) $\frac{3}{14}$ (2) $\frac{5}{14}$ (3) $\frac{1}{14}$ (4) $\frac{9}{14}$

(11) If A and B are two events such that $A \subset B$ and $P(B) \neq 0$, then which of the following is correct?

(1) $P(A/B) = \frac{P(A)}{P(B)}$ (2) $P(A/B) < P(A)$

(3) $P(A/B) \geq P(A)$ (4) $P(A/B) > P(B)$

(12) A bag contains 6 green, 2 white, and 7 black balls. If two balls are drawn simultaneously, then the probability that both are different colours is

(1) $\frac{68}{105}$ (2) $\frac{71}{105}$ (3) $\frac{64}{105}$ (4) $\frac{73}{105}$

(13) If X and Y be two events such that $P(X/Y) = \frac{1}{2}$, $P(Y/X) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$, then $P(X \cup Y)$ is

(1) $\frac{1}{3}$ (2) $\frac{2}{5}$ (3) $\frac{1}{6}$ (4) $\frac{2}{3}$

(14) An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. The probability that the second ball drawn is red will be

(1) $\frac{5}{12}$ (2) $\frac{1}{2}$ (3) $\frac{7}{12}$ (4) $\frac{1}{4}$

(15) A number x is chosen at random from the first 100 natural numbers. Let A be the event of numbers which satisfies $\frac{(x-10)(x-50)}{x-30} \geq 0$, then $P(A)$ is

(1) 0.20 (2) 0.51 (3) 0.71 (4) 0.70

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(16) If two events A and B are independent such that $P(A)=0.35$ and $P(A\cup B)=0.6$, then $P(B)$ is

(1) $\frac{5}{13}$ (2) $\frac{1}{13}$ (3) $\frac{4}{13}$ (4) $\frac{7}{13}$

(17) If two events A and B are such that $P(\overline{A})=\frac{3}{10}$ and $P(A\cap\overline{B})=\frac{1}{2}$, then $P(A\cap B)$ is

(1) $\frac{1}{2}$ (2) $\frac{1}{3}$ (3) $\frac{1}{4}$ (4) $\frac{1}{5}$

(18) If A and B are two events such that $P(A)=0.4, P(B)=0.8$ and $P(B/A)=0.6$, then $P(\overline{A}\cap B)$ is

(1) 0.96 (2) 0.24 (3) 0.56 (4) 0.66

(19) There are three events A, B and C of which one and only one can happen. If the odds are 7 to 4 against A and 5 to 3 against B, then odds against C is

(1) 23: 65 (2) 65: 23 (3) 23: 88 (4) 88: 23

(20) If a and b are chosen randomly from the set $\{1,2,3,4\}$ with replacement, then the probability of the real roots of the equation $x^2+ax+b=0$ is

(1) $\frac{3}{16}$ (2) $\frac{5}{16}$ (3) $\frac{7}{16}$ (4) $\frac{11}{16}$

(21) It is given that the events A and B are such that $P(A)=\frac{1}{4}$, $P(A/B)=\frac{1}{2}$ and $P(B/A)=\frac{2}{3}$. Then $P(B)$ is

(1) $\frac{1}{6}$ (2) $\frac{1}{3}$ (3) $\frac{2}{3}$ (4) $\frac{1}{2}$

(22) In a certain college 4% of the boys and 1% of the girls are taller than 1.8 meter. Further 60% of the students are girls. If a student is selected at random and is taller than 1.8 meters, then the probability that the student is a girl is

(1) $\frac{2}{11}$ (2) $\frac{3}{11}$ (3) $\frac{5}{11}$ (4) $\frac{7}{11}$

(23) Ten coins are tossed. The probability of getting at least 8 heads is

(1) $\frac{7}{64}$ (2) $\frac{7}{32}$ (3) $\frac{7}{16}$ (4) $\frac{7}{128}$

(24) The probability of two events A and B are 0.3 and 0.6 respectively. The probability that both A and B occur simultaneously is 0.18. The probability that neither A nor B occurs is

(1) 0.1 (2) 0.72 (3) 0.42 (4) 0.28

(25) If m is a number such that $m \le 5$, then the probability that quadratic equation $2x^2+2mx+m+1=0$ has real roots is

(1) $\frac{1}{5}$ (2) $\frac{2}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

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SUMMARY

Let S be the sample space associated with a random experiment and A be an event.

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to } A}{\text{Exhaustive Number of cases in } S}$$

Axioms of probability

Given a finite sample space S and an event A in S, we define $P(A)$, the probability of A, satisfies the following three axioms.

(1) $P(A) \geq 0$

(2) If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

(3) $P(S) = 1$

The probability of the impossible event is zero. That is $\boxed{P(\varnothing) = 0}$

If A and B are any two events and $\overline{B}$ is the complementary events of B, then

$$\boxed{P(A \cap \overline{B}) = P(A) - P(A \cap B)}$$

If A and B are any two events, then

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

The conditional probability of an event B, assuming that the event A has already happened is denoted by $P(B/A)$ and is defined as

$$\boxed{P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{provided } P(A) \neq 0}$$

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided } P(B) \neq 0}$$

The probability of the simultaneous happening of two events A and B is given by

$$\boxed{P(A \cap B) = P(A/B)P(B) \text{ or } P(A \cap B) = P(B/A)P(A)}$$

Two events A and B are said to be **independent** if and only if

$$\boxed{P(A \cap B) = P(A) \,.\, P(B)}$$

Total Probability

If $A_1, A_2, A_3, \ldots A_n$ are mutually exclusive and exhaustive events and B is any event in S then $P(B)$ is called the total probability of event B and

$$\boxed{P(B) = P(A_1).P(B/A_1) + P(A_2).P(B/A_2) + \cdots + P(A_n).P(B/A_n) = \sum_{i=1}^{n} P(A_i).P(B/A_i)}$$

Bayes' Theorem

If $A_1, A_2, A_3, \ldots A_n$ are mutually exclusive and exhaustive events such that $P(A_i) > 0$, $i = 1, 2, 3, \ldots . n$ and B is any event in with $P(B) > 0$, then

$$\boxed{P(A_i/B) = \frac{P(A_i)\,P(B/A_i)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + \ldots + P(A_n)\,P(B/A_n)}}$$

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ICT CORNER 12(a)

Probability

Expected Outcome

Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI standard Probability" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Sets and Probability". You can change the question by clicking "New Problem". Work out the probabilities and to check your answer, click on the respective check boxes to see the answer.

Step1

Step2

Browse in the link:

XI standard Probability: https://ggbm.at/zbpcj934

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ICT CORNER 12(b)

Introduction to Probability Theory

Expected Outcome

Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI standard Probability" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Bayes Theorem". An example is given. Work out the probabilities step by step as given.

To check your answer, click on the respective check boxes to see the answer.

Step1

Step2

Browse in the link:

XI standard Probability: https://ggbm.at/zbpcj934

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ANSWERS

Exercise 7.1

(1) (i) $\frac{1}{2}\begin{bmatrix}1 & 9 & 25\\ 0 & 4 & 16\end{bmatrix}$ (ii) $\frac{1}{4}\begin{bmatrix}1 & 5 & 9 & 13\\ 2 & 2 & 6 & 10\\ 5 & 1 & 3 & 7\end{bmatrix}$ (2) $\pm\sqrt{2}, -3, \frac{1}{2}, 1-\pi$

(3) 5 (4) $A=\frac{1}{3}\begin{bmatrix}-15 & 10 & -8\\ 10 & -5 & 5\end{bmatrix}$, $B=\frac{1}{3}\begin{bmatrix}-12 & 2 & -16\\ 8 & -4 & 13\end{bmatrix}$

(5) $A^4=\begin{bmatrix}1 & 4a\\ 0 & 1\end{bmatrix}$ (6) (ii) $\alpha = 2n\pi \pm \frac{\pi}{3}, n\in\mathbb{Z}$ (7) $x=1$

(9) $k=2$ (12) $-I$ (14) $A=\begin{bmatrix}1 & -2\\ 2 & 0\end{bmatrix}$ (16) 3×4

(18) $A=\begin{bmatrix}1 & 3\\ 2 & 12\\ -5 & 0\end{bmatrix}$ (19) $x=-2,\ y=-1$ (20) (i) $x=3^{\frac{1}{3}}$ (ii) $p=-2, q=0, r=-3$

(21) $A=\begin{bmatrix}0 & -1 & -2\\ 1 & 0 & -1\\ 2 & 1 & 0\end{bmatrix}$, skew-symmetric (24) Pack I - ₹ 180, Pack II - ₹ 340, Pack III - ₹ 480

Exercise 7.2

(10) 0 (13) 0 (15) (i) 0 (ii) 0 (16) 4

(17) –81 (18) 0 (19) –1, 2 (21) 7

Exercise 7.3

(3) $x=0$ (multiplicity 2), $x=-(a+b+c)$ (5) $x=0$ (multiplicity 2), $x=-12$

Exercise 7.4

(1) 2.5 sq.units (2) $k=-1,7$ (3) (i) singular (ii) non-singular (iii) singular

(4) (i) $a=-\frac{6}{7}$ (ii) $b=\frac{49}{8}$ (5) $\frac{1}{2}$ (6) 6

Exercise 7.5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
2	1	1	2	2	2	4	4	2	4	2	4	3	2	4
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
3	3	4	1	3	2	3	3	1	2					

Exercise 8.1

(7) Other sides $\vec{b}-\vec{a}, -\vec{a}, \vec{a}-\vec{b}$ and other diagonal $\vec{b}-2\vec{a}$

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Exercise 8.2

(1) (i) Not direction cosines (ii) direction cosines (iii) Not direction cosines

(2) (i) $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$ (ii) $\left(\frac{3}{\sqrt{19}}, \frac{-1}{\sqrt{19}}, \frac{3}{\sqrt{19}}\right)$ (iii) (0, 0, 1)

(3) (i) $\left(\frac{3}{\sqrt{89}}, \frac{-4}{\sqrt{89}}, \frac{8}{\sqrt{89}}\right)$, $(3, -4, 8)$ (ii) $\left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$, $(3,1,1)$

(iii) (0, 1, 0) and (0, 1, 0) (iv) $\left(\frac{5}{\sqrt{2338}}, \frac{-3}{\sqrt{2338}}, \frac{-48}{\sqrt{2338}}\right)$ and $(5, -3-48)$

(v) $\left(\frac{3}{\sqrt{34}}, \frac{4}{\sqrt{34}}, \frac{-3}{\sqrt{34}}\right)$ and $(3, 4, -3)$ (vi) $\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$ and $(1, 0, -1)$

(4) $\left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, $\left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

(5) $a = \pm\frac{1}{2}$ (6) $a = -1, b = 2, c = -1$, or $a = 1, b = -2, c = 1$ (8) $\lambda = \frac{2}{3}$

(11) (i) $\sqrt{41}$, $\left(\frac{2}{\sqrt{41}}, \frac{1}{\sqrt{41}}, \frac{-6}{\sqrt{41}}\right)$ (ii) $\sqrt{1123}$, $\left(\frac{-15}{\sqrt{1123}}, \frac{27}{\sqrt{1123}}, \frac{13}{\sqrt{1123}}\right)$

(12) $\sqrt{44} + \sqrt{218} + \sqrt{110}$ (13) $\frac{1}{\sqrt{398}}(17\hat{i} - 3\hat{j} - 10\hat{k})$

(14) yes (16) $m = \pm\frac{1}{\sqrt{3}}$

Exercise 8.3

(1) (i) 9 (ii) 4 (2) (i) $\lambda = \frac{5}{2}$ (ii) $\lambda = -2$ (3) $\theta = \frac{\pi}{4}$

(4) (i) $\theta = \cos^{-1}\left(\frac{-9}{49}\right)$ (ii) $\theta = \frac{2\pi}{3}$ (5) $\theta = \frac{2\pi}{3}$ (8) -55

(11) $5\sqrt{2}$ (12) $\frac{41}{7}$ (13) 5 (14) -42

Exercise 8.4

(1) $\sqrt{507}$ (3) $\frac{\pm 10\sqrt{3}}{\sqrt{35}}(5\hat{i} - 3\hat{j} + \hat{k})$ (4) $\pm\frac{(-\hat{i} + 2\hat{j} - \hat{k})}{\sqrt{6}}$

(5) $8\sqrt{3}$ sq. units (6) $\frac{1}{2}\sqrt{165}$ sq. units (10) $\frac{\pi}{3}$

Exercise 8.5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
3	3	4	2	2	3	4	4	2	3	2	1	1	1	3
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
4	1	4	4	1	3	2	4	3	2					

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Exercise 9.1

(1) $\simeq 0.\overline{3}$ (2) $\simeq 0.25$ (3) $\frac{1}{2\sqrt{3}} \simeq 0.288$ (4) $\simeq 0.25$

(5) $\simeq 1$ (6) $\simeq 0$ (7) 1 (8) 3

(9) 2 (10) 3 (11) does not exist (12) does not exist

(13) 0 (14) 1 (15) does not exist

(16) except at $x_0 = 4$ (17) except at $x_0 = \pi$ (19) $f(8^-) = f(8^+) = 25$

(20) No (21) $f(2)$ cannot be concluded (22) 6, 6 (23) does not exist

Exercise 9.2

(1) 32 (2) $\frac{m}{n}$ (3) 108 (4) $\frac{1}{2\sqrt{x}}$

(5) $\frac{1}{6}$ (6) $-\frac{1}{4}$ (7) 3 (8) 4

(9) $\frac{1}{2}$ (10) $-\frac{1}{4}$ (11) $-\frac{3}{4}\sqrt[3]{4}$ (12) 0

(13) $f(x) \to -\infty$ as $x \to 0$ (limit does not exist) (14) $\frac{1}{4}$ (15) $\frac{1}{4a\sqrt{a-b}}$

Exercise 9.3

(1) (i) $f(-2) \to \infty$ as $x \to -2^-$, $f(-2) \to -\infty$ as $x \to -2^+$

(ii) $f\left(\frac{\pi}{2}\right) \to \infty$ as $x \to \frac{\pi}{2}^-$, $f\left(\frac{\pi}{2}\right) \to -\infty$ as $x \to \frac{\pi}{2}^+$

(2) $f(3) \to -\infty$ as $x \to 3^-$, $f(3) \to \infty$ as $x \to 3^+$ (3) $f(x) \to 0$ as $x \to \infty$

(4) 0 (5) $f(x) \to \infty$ as $x \to \infty$ (6) -1 (7) $\frac{1}{4}$

(9) $\frac{1}{\alpha}$ (10) 30

Exercise 9.4

(1) e^7 (2) $e^{\frac{1}{3}}$ (3) 1 (4) $\frac{1}{e^8}$ (5) e^3

(6) $\frac{1}{8}$ (7) $\frac{\alpha}{\beta}$ (8) $\frac{2}{5}$ (9) $\begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m > n \\ f(\alpha) \to \infty \text{ as } \alpha \to 0 & \text{if } m < n \end{cases}$

(10) $2\cos a$ (11) $\frac{b}{a}$ (12) $\frac{2}{3}$ (13) $\frac{1}{2}$ (14) 2

(15) $\log\frac{2}{3}$ (16) $\log 9$ (17) $\frac{1}{2}$ (18) $\log 3 - 1$ (19) a

(20) $-\frac{3}{2}$ (21) e^2 (22) $\frac{1}{4\sqrt{2}}$ (23) 1 (24) e^2

(25) 2 (26) $a - b$ (27) $\frac{1}{2}$ (28) $\frac{1}{2}$

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Exercise 9.5

(2) (i) continuous for all $x \in \mathbb{R}$ (ii) continuous in $\mathbb{R}$

(iii) continuous for all $x \in \mathbb{R} - (2n+1)\frac{\pi}{2}, n \in z$ (iv) continuous for all $x \in \mathbb{R}$

(v) continuous for $(0, \infty)$ (vi) continuous for all $x \in \mathbb{R} - \{0\}$

(vii) continuous for all $x \in \mathbb{R} - \{-4\}$ (viii) continuous for all $x \in \mathbb{R}$

(ix) continuous for all $x \in \mathbb{R} - \{-1\}$ (x) continuous for all $x \in \mathbb{R} - \frac{n\pi}{2}, n \in z$

(3) (i) not continuous at $x = 3$ (ii) continuous for all $x \in \mathbb{R}$

(iii) continuous for all $x \in \mathbb{R}$ (iv) continuous for all $x \in \left[0, \frac{\pi}{2}\right)$

(4) (i) continuous at $x_0 = 1$ (ii) not continuous at $x_0 = 3$

(6) $\alpha = 4$ (8) 6

(9) (i) not continuous at $x = 1$ (ii) not continuous at $x = 0$

(10) continuous at $x = 0, 1, 3$

(11) (i) removable discontinuity at $x = -2$, $g(x) = \begin{cases} \dfrac{x^2 - 2x - 8}{x+2} & \text{if } x \neq -2 \\ -6 & \text{if } x = -2 \end{cases}$

(ii) removable discontinuity at $x = -4$, $g(x) = \begin{cases} \dfrac{x^3 + 64}{x+4} & \text{if } x \neq -4 \\ 48 & \text{if } x = -4 \end{cases}$

(iii) removable discontinuity at $x = 9$, $g(x) = \begin{cases} \dfrac{3 - \sqrt{x}}{9 - x} & \text{if } x \neq 9 \\ \dfrac{1}{6} & \text{if } x = 9 \end{cases}$

(12) -2 (13) $f(0) = 0$ (14) $f(1) = \frac{2}{3}$

Exercise 9.6

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
2	3	4	1	1	4	2	2	2	3	4	3	4	3	1
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
1	1	1	4	2	2	2	2	2	4					

Exercise 10.1

(1) (i) 0 (ii) -4 (iii) $-2x$

(2) (i) $f'(1^-) = -1$, $f'(1^+) = 1$, not differentiable (ii) $f'(x) \to -\infty$ as $x \to 1^-$, not differentiable

(iii) $f'(1^-) = 1$, $f'(1^+) = 2$, not differentiable

(3) (i) differentiable (ii) not differentiable

(iii) not differentiable (iv) not differentiable

(5) at $x = -1$ and $x = 8$ are cusps

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at $x=4$ it is not continuous, at $x=11$, tangent is perpendicular

(6) does not exist

(7) (i) not differentiable at $x=n\pi, n\in z$ (ii) not differentiable at $x=(2n+1)\frac{\pi}{2}, n\in z$

Exercise 10.2

(1) $1-3\cos x$ (2) $\cos x-\sin x$ (3) $x\cos x+\sin x$

(4) $-\sin x-2\sec^2 x$ (5) $3t^2\cos t-t^3\sin t$ (6) $4\sec t\tan t+\sec^2 t$

(7) $e^x(\cos x+\sin x)$ (8) $\dfrac{x\sec^2 x-\tan x}{x^2}$ (9) $\dfrac{1}{1+\cos x}$

(10) $\dfrac{(1-x)\cos x+(1+x)\sin x}{(\sin x+\cos x)^2}$ (11) $\cos x+\sin x$ (12) $\dfrac{x\cos x-2\sin x}{x^3}$

(13) $\tan\theta\sec\theta+\cos\theta+\sin\theta$ (14) $-\dfrac{(1+\cos^2 x)}{\sin^3 x}$ (15) $x\cos 2x+\sin x\cos x$

(16) $e^{-x}\left[\dfrac{1}{x}-\log x\right]$ (17) $e^{-3x}\left[-3(x^2+5)\log(1+x)+\dfrac{x^2+5}{1+x}+2x\log(1+x)\right]$

(18) $\dfrac{\pi}{180}\cos\dfrac{\pi}{180}x$ (19) $\dfrac{\log_{10} e}{x}$

Exercise 10.3

(1) $5(2x+4)(x^2+4x+6)^4$ (2) $3\sec^2 3x$ (3) $-\sec^2 x\sin(\tan x)$

(4) $x^2(1+x^3)^{-\frac{2}{3}}$ (5) $\dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$ (6) $e^x\cos(e^x)$

(7) $7(3x^2+4)(x^3+4x)^6$ (8) $\dfrac{3}{2}\left(t-\dfrac{1}{t}\right)^{\frac{1}{2}}\left(1+\dfrac{1}{t^2}\right)$ (9) $\dfrac{1}{3}\sec^2 t(1+\tan t)^{-\frac{2}{3}}$

(10) $-3x^2\sin(a^3+x^3)$ (11) $-my$ (12) $20\sec 5x\tan 5x$

(13) $\dfrac{8(2x-5)^3}{(8x^2-5)^4}[-4x^2+30x-5]$ (14) $\dfrac{8x^3+14x}{3(x^2+2)^{\frac{2}{3}}}$ (15) $e^{-x^2}[1-2x^2]$

(16) $-\dfrac{3t^2}{2(t^3+1)^{\frac{3}{4}}(t^3-1)^{\frac{5}{4}}}$ (17) $\dfrac{14-3x}{2(7-3x)\sqrt{7-3x}}$ (18) $-\sin x\sec^2(\cos x)$

(19) $\sin x(1+\sec^2 x)$ (20) $\dfrac{5^{-\frac{1}{x}}(\log 5)}{x^2}$ (21) $\dfrac{\sec^2 x}{\sqrt{1+2\tan x}}$

(22) $3\sin x\cos x(\sin x-\cos x)$ (23) $-k\sin kx\sin(2\cos kx)$ (24) $-6\sin 2x(1+\cos^2 x)^5$

(25) $\dfrac{3e^{3x}+2e^{4x}}{(1+e^x)^2}$ (26) $\dfrac{2\sqrt{x}+1}{4\sqrt{x}\sqrt{x+\sqrt{x}}}$ (27) $e^{x\cos x}[\cos x-x\sin x]$

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(28) $\dfrac{4\sqrt{x}\sqrt{x+\sqrt{x}}+2\sqrt{x}+1}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$ (29) $\dfrac{\cos\left(\tan\sqrt{\sin x}\right)\sec^2\left(\sqrt{\sin x}\right)\cos x}{2\sqrt{\sin x}}$ (30) $\dfrac{-2}{1+x^2}$

Exercise 10.4

(1) $x^{\cos x}\left(\dfrac{\cos x}{x}-\sin x\log x\right)$ (2) $x^{\log x}\left(\dfrac{2\log x}{x}\right)+(\log x)^x\left[\dfrac{1}{\log x}+\log(\log x)\right]$

(3) $\dfrac{y(2x-1)}{x(1+2y)}$ (4) $\dfrac{y(x\log y-y)}{x(y\log x-x)}$ (5) $(\cos x)^{\log x}\left[\dfrac{\log(\cos x)}{x}-\tan x\log x\right]$

(6) $-\dfrac{b^2x}{a^2y}$ (7) $\dfrac{x\sqrt{x^2+y^2}+y}{x-y\sqrt{x^2+y^2}}$ (8) $\dfrac{1-\sec^2(x+y)-\sec^2(x-y)}{\sec^2(x+y)-\sec^2(x-y)}$

(10) $\dfrac{1}{2}$ (11) $\dfrac{6}{1+9x^2}$ (12) 1 (13) $-\tan t$

(14) $\tan t$ (15) $\dfrac{t^2-1}{2t}$ (16) $\dfrac{2}{1+x^2}$ (17) $\dfrac{3}{\sqrt{1-x^2}}$

(18) 1 (19) $\cos x^2$ (20) 2 (21) $\dfrac{1}{2}$

(22) -1 (23) $\dfrac{x}{(1-x^2)^{\frac{3}{2}}}$

Exercise 10.5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
2	4	3	3	1	4	3	2	1	4	3	3	2	2	4
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
2	3	1	4	2	4	1	1	3	2					

Exercise 11.1

(1) (i) $\dfrac{x^{12}}{12}+c$ (ii) $-\dfrac{1}{6x^6}+c$ (iii) $\dfrac{3}{7}x^{\frac{7}{3}}+c$ (iv) $\dfrac{8}{13}x^{\frac{13}{8}}+c$

(2) (i) $-\cot x+c$ (ii) $\sec x+c$ (iii) $-\operatorname{cosec} x+c$ (iv) $\tan x+c$

(3) (i) 12^3x+c (ii) $\log|x|+c$ (iii) e^x+c

(4) (i) $\tan^{-1}x+c$ (ii) $\sin^{-1}x+c$

Exercise 11.2

(1) (i) $\dfrac{(x+5)^7}{7}+c$ (ii) $\dfrac{1}{9(2-3x)^3}+c$ (iii) $\dfrac{2}{9}(3x+2)^{\frac{3}{2}}+c$

(2) (i) $\dfrac{-\cos 3x}{3}+c$ (ii) $-\dfrac{\sin(5-11x)+c}{11}$ (iii) $-\dfrac{\cot(5x-7)}{5}+c$

(3) (i) $\dfrac{1}{3}e^{3x-6}+c$ (ii) $-\dfrac{e^{8-7x}}{7}+c$ (iii) $-\dfrac{1}{4}\log|6-4x|+c$

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(4) (i) $5\tan\frac{x}{5}+c$ (ii) $-\frac{1}{5}\operatorname{cosec}(5x+3)+c$ (iii) $-\frac{1}{15}\sec(2-15x)+c$

(5) (i) $\frac{1}{4}\sin^{-1}(4x)+c$ (ii) $\frac{1}{9}\sin^{-1}(9x)+c$ (iii) $\frac{1}{6}\tan^{-1}(6x)+c$

Exercise 11.3

(1) $\frac{(x+4)^6}{6}+\frac{1}{3(2-5x)^3}+\frac{\cot(3x-1)}{3}+c$ (2) $-2\sin(5-2x)+3e^{3x-6}-6\log|6-4x|+c$

(3) $5\tan\frac{x}{5}+9\sin 2x+2\sec(5x+3)+c$ (4) $2\sin^{-1}(4x)+9\sin^{-1}(3x)-3\tan^{-1}(5x)+c$

(5) $2\tan^{-1}(3x+2)+3\sin^{-1}(3-4x)+c$ (6) $\sin\left(\frac{x}{3}-4\right)+\log|7x+9|+5e^{\frac{x}{5}+3}+c$

Exercise 11.4

(1) $2x^2-5x+3$ (2) $3(x^3-x^2-1)$ (3) $2x^3-3x^2+5x+26$

(4) (i) 8 seconds (ii) 39.2 m/sec (iii) 78.4 m/sec

(5) (i) 2.4 sq.cm (ii) 0.4 sq.cm

Exercise 11.5

(1) $\frac{x^2}{2}+4x-3\log|x|-\frac{2}{x}+c$ (2) $\frac{x^2}{2}+\log|x|+2x+c$ (3) $\frac{8x^3}{3}+26x^2-180x+c$

(4) $\tan x-\cot x-2x+c$ (5) $2[\sin x+x\cos\alpha]+c$ (6) $-2\operatorname{cosec}2x+c$

(7) $-3\cot x-4\operatorname{cosec}x+c$ (8) $x-\sin x+c$ (9) $2\left[\frac{\sin 3x}{3}+\sin x\right]+c$

(10) $\frac{1}{2}\left[\frac{\sin 5x}{5}+\sin x\right]+c$ (11) $\frac{1}{2}\left[x-\frac{\sin 10x}{10}\right]+c$ (12) $-\frac{1}{8}\cos 4x+c$

(13) $\frac{(ae)^x}{\log(ae)}+c$ (14) $\frac{2}{15}(3x+7)^{\frac{5}{2}}-\frac{2}{3}(3x+7)^{\frac{3}{2}}+c$

(15) $\frac{2^{2x+2}}{\log 2}-\frac{2^{2-3x}}{3\log 2}+c$ (16) $\frac{2}{21}[(x+3)^{\frac{3}{2}}+(x-4)^{\frac{3}{2}}]+c$

(17) $2\log|x+3|-\log|x+2|+c$ (18) $\frac{1}{9}\log|x-1|-\frac{1}{9}\log|x+2|+\frac{1}{3(x+2)}+c$

(19) $\log\left|\frac{x+2}{x-1}\right|+3\tan^{-1}x+c$ (20) $\frac{x^2}{2}+3x-\log|x-1|+8\log|x-2|+c$

Exercise 11.6

(1) $\sqrt{1+x^2}+c$ (2) $\frac{1}{3}\tan^{-1}(x^3)+c$ (3) $\log|e^x+e^{-x}|+c$

(4) $\log|10^x+x^{10}|+c$ (5) $-2\cos\sqrt{x}+c$ (6) $\log|\log(\sin x)|+c$

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(7) $\log\left|\log\left(\tan\frac{x}{2}\right)\right|+c$ (8) $\frac{1}{b^2}\log|a^2+b^2\sin^2 x|+c$ (9) $\frac{(\sin^{-1}x)^2}{2}+c$

(10) $(1+\sqrt{x})^2-4(1+\sqrt{x})+2\log\left|1+\sqrt{x}\right|+c$ (11) $\log|\log(\log x)|+c$

(12) $-e^{-\beta x^{\alpha}}+c$ (13) $2\sqrt{\sec x}+c$ (14) $\frac{(1-x)^{19}}{19}-\frac{(1-x)^{18}}{18}+c$

(15) $\frac{\sin^6 x}{6}-\frac{\sin^8 x}{8}+c$ (16) $(x-a)\cos a-\sin a\log|\sec(x-a)|+c$

Exercise 11.7

(1) (i) $e^{3x}[3x-1]+c$ (ii) $-\frac{x\cos 3x}{3}+\frac{\sin 3x}{9}+c$

(iii) $-e^{-5x}[5x+1]+c$ (iv) $x\sec x-\log|\sec x+\tan x|+c$

(2) (i) $\frac{x^2\log|x|}{2}-\frac{x^2}{4}+c$ (ii) $e^{3x}[9x^2-6x+2]+c$

(iii) $x^2\sin x+2x\cos x-2\sin x+c$ (iv) $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+c$

(3) (i) $-\sin^{-1}x\sqrt{1-x^2}+x+c$ (ii) $\frac{1}{2}e^{x^2}[x^4-2x^2+2]+c$

(iii) $\frac{1}{2}\left[4x\tan^{-1}4x-\log\left|\sqrt{1+16x^2}\right|\right]+c$ (iv) $2\left[x\tan^{-1}x-\log\left|\sqrt{1+x^2}\right|\right]+c$

Exercise 11.8

(1) (i) $\frac{e^{ax}}{a^2+b^2}[a\cos bx+b\sin bx]+c$ (ii) $\frac{e^{2x}}{5}[2\sin x-\cos x]+c$

(iii) $\frac{e^{-x}}{5}[2\sin 2x-\cos 2x]+c$

(2) (i) $-\frac{e^{-3x}}{13}[3\sin 2x+2\cos 2x]+c$ (ii) $-\frac{e^{-4x}}{10}[2\sin 2x+\cos 2x]+c$

(iii) $\frac{e^{-3x}}{10}[\sin x-3\cos x]+c$

Exercise 11.9

(1) $e^x\log|\sec x|+c$ (2) $\frac{e^x}{2x}+c$ (3) $e^x\sec x+c$

(4) $e^x\tan x+c$ (5) $xe^{\tan^{-1}x}+c$ (6) $\frac{x}{1+\log|x|}+c$

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Exercise 11.10

(1) (i) $\frac{1}{4}\log\left|\frac{2+x}{2-x}\right|+c$ (ii) $\frac{1}{20}\log\left|\frac{5+2x}{5-2x}\right|+c$ (iii) $\frac{1}{12}\log\left|\frac{3x-2}{3x+2}\right|+c$

(2) (i) $\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}-3+x}{\sqrt{2}+3-x}\right|+c$ (ii) $\frac{1}{10}\log\left|\frac{x-4}{x+6}\right|+c$ (iii) $\log\left|x+2+\sqrt{x^2+4x+2}\right|+c$

(3) (i) $\log\left|x+2+\sqrt{(x+2)^2-1}\right|+c$ (ii) $\log\left|x-2+\sqrt{x^2-4x+5}\right|+c$ (iii) $\sin^{-1}\left(\frac{x-4}{5}\right)+c$

Exercise 11.11

(1) (i) $\log\ |x^2+4x-12|-\frac{7}{8}\log\left|\frac{x-2}{x+6}\right|+c$ (ii) $\frac{5}{2}\log|x^2+2x+2|-7\tan^{-1}(x+1)+c$

(iii) $\frac{3}{4}\log|2x^2-2x+3|+\frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c$

(2) (i) $5\sin^{-1}\frac{x-2}{\sqrt{13}}-2\sqrt{9+4x-x^2}+c$ (ii) $\sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+c$

(iii) $2\sqrt{x^2+4x+1}-\log\left|x+2+\sqrt{x^2+4x+1}\right|+c$

Exercise 11.12

(1) (i) $\frac{x+1}{2}\sqrt{x^2+2x+10}+\frac{9}{2}\log\left|x+1+\sqrt{x^2+2x+10}\right|+c$

(ii) $\frac{x-1}{2}\sqrt{x^2-2x-3}-2\log\left|x-1+\sqrt{x^2-2x-3}\right|+c$

(iii) $\frac{x-5}{2}\sqrt{10x-x^2-24}+\frac{1}{2}\sin^{-1}(x-5)+c$

(2) (i) $\frac{1}{4}\left[(2x+5)\sqrt{9-(2x+5)^2}+9\sin^{-1}\left(\frac{2x+5}{3}\right)\right]+c$

(ii) $\frac{1}{4}\left[(2x+1)\sqrt{81+(2x+1)^2}+81\log\left|2x+1+\sqrt{81+(2x+1)^2}\right|\right]+c$

(iii) $\frac{x+1}{2}\sqrt{(x+1)^2-4}+2\log\left|x+1+\sqrt{(x+1)^2-4}\right|+c$

Exercise 11.13

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
1	3	4	1	3	1	3	2	4	3	4	2	4	4	1
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
1	3	2	4	1	3	4	3	1	4					

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Exercise 12.1

(1) (i) permissible (ii) not permissible (iii) not permissible

(2) (i) $\frac{1}{2}$ (ii) 1 (3) (i) $\frac{5}{9}$ (ii) $\frac{4}{9}$ (4) (i) $\frac{1}{7}$ (ii) $\frac{2}{7}$ (5)(i) $\frac{7}{64}$ (ii) $\frac{247}{256}$ (iii) $\frac{37}{256}$

(6) $\frac{37}{100}$ (7) (i) $\frac{7}{33}$ (ii) $\frac{14}{55}$ (8) (i) $\frac{2}{13}$ (ii) $\frac{5}{13}$ (iii) $\frac{2}{13}$

(9) $\frac{627}{728}$ (10) (i) $\frac{5}{12}$ (ii) 2 to 3

Exercise 12.2

(1) (i) $\frac{5}{8}$ (ii) $\frac{1}{2}$ (iii)) $\frac{1}{8}$ (iv) 1 (2) (i) 0.50 (ii) 0.35 (iii) 0.20

(3) $\frac{11}{36}$ (4) (i) 0.8 (ii) 0.5 (iii) 0.3

(5) (i) 0.9984 (ii) 0.0016 (6) (i) 0.64 (ii) 0.44

Exercise 12.3

(1) No (3) 0.5 (4) (i) 0.5 (ii) 0.9

(5) $\frac{3}{8}$ (6) (i) $\frac{3}{5}$ (ii) $\frac{13}{30}$ (7) (i) 0.5 (ii) 0.375

(8) (i) $\frac{1}{4}$ (ii) $\frac{9}{40}$ (iii) $\frac{21}{40}$ (9) 0.75

(10) (i) 0.3 (ii) 0.5 (iii) 0.5 (iv) 0.5 (11) (i) $\frac{5}{28}$ (ii) $\frac{1}{14}$ (12) $\frac{13}{30}$

Exercise 12.4

(1) 0.028(2) (i) $\frac{11}{20}$ (ii) $\frac{6}{11}$ (3) (i) $\frac{7}{250}$ (ii) $\frac{3}{7}$

(4) $\frac{15}{41}$ (5) (i) $\frac{9}{25}$ (ii) $\frac{2}{3}$

Exercise 12.5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
4	3	1	2	2	1	1	4	2	3	3	1	4	2	3
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
1	4	3	2	3	2	2	4	4	3					

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GLOSSARY

CHAPTER 7
MATRICES and DETERMINANTS

matrix	அணி
order	வரிசை
row matrix	நிரை அணி
column matrix	நிரல் அணி
zero matrix	பூஜ்ஜிய அணி
null matrix	வெற்று அணி
square matrix	சதுர அணி
diagonal matrix	மூலைவிட்ட அணி
unit matrix	அலகு அணி
triangular matrix	முக்கோண வடிவ அணி
upper triangular matrix	மேல் முக்கோண வடிவ அணி
lower triangular matrix	கீழ் முக்கோண வடிவ அணி
principal diagonal	முதன்மை மூல விட்டம்
scalar matrix	திசையிலி அணி
conformable	உகந்த
commutative property	பரிமாற்றுப் பண்பு
associative property	சேர்ப்புப் பண்பு
identity property	சமனிப் பண்பு
inverse property	எதிர்மறைப் பண்பு
distributive property	பங்கீட்டுப் பண்பு
symmetric	சமச்சீர்
skew-symmetric	எதிர் சமச்சீர்
determinant	அணிக்கோவை
singular matrix	பூஜ்ஜியக் கோவை அணி
non-singular matrix	பூஜ்ஜியமற்ற கோவை அணி

CHAPTER 8
VECTOR ALGEBRA

vector	வெக்டர்
initial point	தொடக்கப் புள்ளி
terminal point	முடிவுப் புள்ளி
support of the vector	வெக்டரின் தாங்கி
free vector	கட்டிலா வெக்டர்
localised vector	அறுதியிட்ட வெக்டர்
co-initial vectors	ஒரே தொடக்கப்புள்ளி வெக்டர்கள்
co-terminal vectors	ஒரே முடிவுப்புள்ளி வெக்டர்கள்
collinear vectors	ஒரே கோடமை வெக்டர்கள்
parallel vectors	இணை வெக்டர்கள்
coplanar vectors	ஒரு தள வெக்டர்கள்
equal vectors	சம வெக்டர்கள்
zero vector	பூஜ்ஜிய வெக்டர்
unit vector	அலகு வெக்டர்
like vectors	ஒரே திசை வெக்டர்கள்
unlike vectors	எதிர் திசை வெக்டர்கள்
scalar multiplication	திசையிலிப் பெருக்கம்
position vector	நிலை வெக்டர்
section formula	பிரிவு சூத்திரம்
resolution of vector	வெக்டரைக் கூறுகளாகப் பிரித்தல்
direction cosines	திசைக் கொசைன்கள்
direction ratios	திசை விகிதங்கள்
scalar product	திசையிலிப் பெருக்கம்
vector product	வெக்டர் பெருக்கம்

CHAPTER 9
LIMITS AND CONTINUITY

calculus	நுண் கணிதம்
limit	எல்லை
one sided limit	ஒருபுற எல்லை
left hand limit	இடப்புற எல்லை
right hand limit	வலப்புற எல்லை
infinite limit	முடிவிலா எல்லை
limit at infinity	முடிவிலியில் எல்லை
vertical asymptote	செங்குத்துத் தொலைத் தொடுகோடு

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horizontal asymptote	கிடைமட்டத் தொலைத் தொடுகோடு
Sandwich theorem	இடையீட்டுத் தேற்றம்
continuity	தொடர்ச்சித் தன்மை
discontinuity	தொடர்ச்சியின்மைத் தன்மை
removable discontinuity	நீக்கக் கூடிய தொடர்ச்சியின்மை
jump discontinuity	துள்ளல் தொடர்ச்சியின்மை

CHAPTER 10 DIFFERENTIAL CALCULUS - DIFFERENTIABILITY AND METHODS OF DIFFERENTIATION

analytic equation	பகுமுறை சமன்பாடு
derivative	வகைக்கெழு
velocity	திசைவேகம்
acceleration	முடுக்கம்
jerk	குலுக்கம்
tangency	தொடுகோட்டுப் பண்பு
difference quotient	வித்தியாசங்களின் விகிதம்
secant line	வெட்டுக்கோடு
tangent line	தொடுகோடு
slope of the curve	வளைவரை சாய்வு
rectilinear motion	நேர்கோட்டியக்கம்
position function	நிலைச்சார்பு
differentiable	வகைமை / வகையிடத்தக்க
differentiation	வகையிடல்
left hand derivative	இடப்புற வகையிடல்
right hand derivative	வலப்புற வகையிடல்
non-differentiability	வகையிடத் தக்கதல்லாத
quotient rule	வகுத்தல் விதி
chain rule	பிணைப்பு விதி
composite functions rule	சார்புகளின் சேர்ப்பு விதி
function of a function rule	சார்பின் சார்பு விதி
intermediate argument	இடைநிலை மாறி
implicit differentiation	உள்ளார்ந்த சார்பு வகையிடல்
explicit differentiation	வெளிப்படைச் சார்பு வகையிடல்
parametric differentiation	துணையலகு சார்பு வகையிடல்
higher order derivative	உயர்வரிசை வகையிடல்

CHAPTER 11 INTEGRAL CALCULUS

integrand	தொகைச் சார்பு
integrator	தொகை மாறி
anti-derivative	எதிர்மறை வகையிடல்
indefinite integral	எல்லை வரையறுக்கப்படாத தொகை

CHAPTER 12 INTRODUCTION TO PROBABILITY THEORY

experiment	சோதனை
deterministic experiment	நிர்ணயிக்கப்பட்ட சோதனை
random experiment	சமவாய்ப்பு சோதனை
event	நிகழ்ச்சி
sure event or certain event	நிச்சய நிகழ்ச்சி
impossible event	இயலா நிகழ்ச்சி
exhaustive event	யாவுமளாவிய நிகழ்ச்சி
sample space	கூறுவெளி
mutually exclusive events	ஒன்றையொன்று விலக்கும் நிகழ்ச்சிகள்
equally likely events	சமவாய்ப்பு நிகழ்ச்சிகள்
conditional probability	சார்புநிலை நிகழ்தகவு
independent events	சார்பிலா நிகழ்ச்சிகள்

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