JUPITER ACADEMY

Mathematics JEE Solutions 2022

Mathematics

1. $\dfrac{a+b}{7}=\dfrac{b+c}{8}=\dfrac{c+a}{9}$, then (a, b, c) are the sides of ΔABC, then find $\dfrac{R}{r}$ where R is circumradius and r is inradius

(1) $\dfrac{5}{2}$ (2) 3

(3) 1 (4) $\dfrac{1}{2}$

Sol. Answer (1)

$$\frac{a+b}{7}=\frac{b+c}{8}=\frac{c+a}{9}=k \text{ (say)}$$

$$\Rightarrow a=4k,\ b=3k,\ c=5k \Rightarrow s=6k$$

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}=6k^2$$

$$r=\frac{\Delta}{s}=\frac{6k^2}{6k}=k \text{ and } R=\frac{abc}{4\Delta}=\frac{60k^3}{24k^2}=\frac{5}{2}k$$

Here $\dfrac{R}{r}=\dfrac{5}{2}$

2. Circle C touch the lines $L_1=4x-3y+k_1=0$, $L_2=4x-3y+k_2=0$, $k_1, k_2\in R$. If a lines passing the centre of circle intersect at L_1 at $(-1, 2)$ and $L_2(3, -6)$ then equation of circle is

(1) $x^2 + y^2 - 2x + 4y - 11 = 0$

(2) $x^2 + y^2 + 2x - 4y - 11 = 0$

(3) $x^2 + y^2 - 2x + 6y - 11 = 0$

(4) $x^2 + y^2 - 2x - 4y + 11 = 0$

Sol. Answer (1)

$\because$ (-1, 2) lies on line $L_1 = 0$

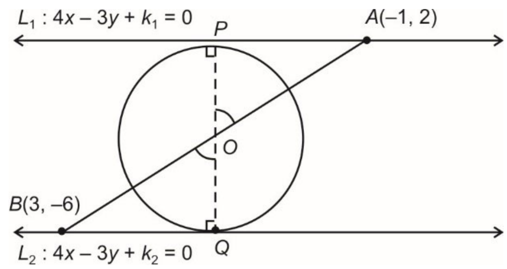
$\therefore$ $K_1 = 10$

and (3, -6) lies on line $L_2 = 0$

$\therefore$ $K_2 = -30$

Distance between lines L_1 and L_2

$$=\frac{|K_1-K_2|}{\sqrt{4^2+(-3)^2}}=8$$



$\therefore$ radius of circle = 4 units

Centre of circle = mid point of AB = (1, -2)

$\therefore$ equation of circle is $(x-1)^2+(y+2)^2=4^2$

$\therefore$ $x^2+y^2-2x+4y-11=0$

3. If $g:(0,\infty)\to R$ is differentiable function

$$\int\left[x\frac{\cos x-\sin x}{e^x+1}+g(x)\frac{e^x+1-xe^x}{(e^x+1)^2}\right]dx=\frac{xg(x)}{e^x+1}+c$$

for all $x > 0$, here c = constant

(1) g is decreasing in $\left(0,\dfrac{\pi}{4}\right)$

(2) g is increasing in $\left(0,\dfrac{\pi}{4}\right)$

(3) $g + g'$ is increasing in $\left(0,\dfrac{\pi}{2}\right)$

(4) $g - g'$ is decreasing in $\left(0,\dfrac{\pi}{2}\right)$

Sol. Answer (2)

$g:(0,\infty)\to R$ is differentiable

$$\int\left[x\frac{\cos x-\sin x}{e^x+1}+g(x)\cdot\frac{e^x+1-xe^x}{(e^x+1)^2}\right]dx=\frac{xg(x)}{e^x+1}+c$$

Differentiating to x :

$$x\cdot\frac{\cos x-\sin x}{e^x+1}+(x)\cdot\frac{e^x+1-xe^x}{(e^x+1)^2}=\frac{(g(x)+xg'(x)(e^x+1)-x(x)e^x}{(e^x+1)^2}$$

$$\Rightarrow x\cdot\frac{(\cos x-\sin x)}{e^x+1}=\frac{xg'(x)}{(e^x+1)}$$

$$\Rightarrow g'(x)=\cos x-\sin x$$

$$\Rightarrow g'(x)>0\ \ \forall\ \ x\in\left(0,\frac{\pi}{4}\right)$$

$\Rightarrow g(x)$ increase in $\left(0,\frac{\pi}{4}\right)$

4. f(x) be a polynomial function such that $f(x)$ + $f'(x)$ + $f''(x)$ = x^5 + 64, value of $\lim_{x\to1}\frac{f(x)}{x-1}$

(1) –15 (2) 15

(3) 60 (4) –60

Sol. Answer (1)

Let $p(x)=a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$

$$p'(x)=5a_5x^4+4a_4x^3+3a_3x^2+2a_2x+a_1$$

$$p''(x)=20a_5x^3+12a_4x^2+6a_3x+2a_2$$

$$\Rightarrow\quad p(x)+p'(x)+p''(x)=a_5x^5+x^4(a_4+5a_5)+x^3(a_3+4a_4+20a_5)+x^2(a_2+3a_3+12a_4)+x(a_1+2a_2+6a_3)+(a_0+a_1+2a_2)$$

But this is equal to x^5 + 64

$\Rightarrow$ $a_5=1\Rightarrow a_4=-5\Rightarrow a_3=0\Rightarrow a_2=60$ $\Rightarrow a_1=-120$ and $a_0=64$

$$\Rightarrow\quad p(x)=x^5-5x^4+60x^2-120x+64$$

$$\lim_{x\to1}\frac{p(x)}{x-1}=p'(1)=5(1)^4-20(1)+120-120=-15$$

5. If y = $y(x)$ be the solution of given equation $y^2\,dx+(x^2-xy+y^2)\,dy=0$ and this curve also passes through (1, 1). Line intersect at $y=\sqrt{3}x$ at $\left(\alpha,\sqrt{3}\alpha\right)$ then find the value of $\log_e\left(\sqrt{3}x\right)$

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{4}$

(3) $\frac{\pi}{6}$ (4) $\frac{\pi}{12}$

Sol. Answer (4)

$$\frac{dy}{dx}=\frac{-y^2}{x^2+y^2-xy}$$

Put $y=vx\Rightarrow\frac{dy}{dx}=v+\frac{xdv}{dx}$

$$\Rightarrow v+\frac{xdv}{dx}=\frac{-v^2}{1+v^2-v}$$

$$\frac{xdv}{dx}=\frac{-v^2}{1+v^2-v}-v=\frac{-v^2-v-v^3+v^2}{1+v^2-v}$$

$$\Rightarrow\frac{v^2-v+1}{-v(v^2+1)}d=\frac{dx}{x}$$

$$\Rightarrow\left(\frac{-1}{v}+\frac{1}{v^2+1}\right)dv=\frac{dx}{x}$$

$\Rightarrow$ –log x + tan^{-1} v = log x + C

$$\Rightarrow -\log\left(\frac{y}{x}\right)+\tan^{-1}\left(\frac{y}{x}\right)=\log x+C$$

Put $\begin{matrix}x=1\\y=1\end{matrix}\Rightarrow\frac{\pi}{4}=C$

$$\Rightarrow\ \tan^{-1}\left(\frac{y}{x}\right)-\log\left(\frac{y}{x}\right)=\log x+\frac{\pi}{4}$$

Put $y=\sqrt{3}x\Rightarrow\frac{\pi}{3}-\log\sqrt{3}=\log x+\frac{\pi}{4}$

$$\Rightarrow\ \log x=\frac{\pi}{12}-\log\sqrt{3}\ \ \Rightarrow\log\sqrt{3}x=\frac{\pi}{12}$$

6. $f:R\to R,g:R\to R$ be two function defined by $f(x)=\log_e(x^2+1)-e^{-x}+1$ and $g(x)=\frac{1-2e^{2x}}{x}$ range of α, if the inequality $f\left(g\frac{(\alpha-1)^2}{3}\right)>f\left(g\left(\alpha-\frac{5}{3}\right)\right)$

(1) $\alpha\in(0,2)$ (2) $\alpha\in(2,3)$

(3) $\alpha\in(3,4)$ (4) $\alpha\in R$

Sol. Answer (2)

$$f(x)=\ln(x^2+1)-e^{-x}+1\ \Rightarrow f'(x)=\frac{2x}{1+x^2}+e^{-x}>0$$

$$g(x)=\frac{1-e^{2x}}{e^x}=e^{-x}-e^x\Rightarrow g'(x)=-e^{-x}-e^{-x}<0$$

$\Rightarrow$ f is $\uparrow$ and $g(x)$ is $\downarrow$

$$\Rightarrow\ f\left(g\left(\frac{(\alpha-1)^2}{3}\right)\right)>f\left(g\left(\alpha-\frac{5}{3}\right)\right)$$

$\Rightarrow g\left(\frac{(\alpha-1)^2}{3}\right) > g\left(\alpha - \frac{5}{3}\right)$

$\Rightarrow \frac{(\alpha-1)^2}{3} < \alpha - \frac{5}{3}$

$\Rightarrow a^2 - 5\alpha + b < 0$

$\Rightarrow (\alpha-2)(\alpha-3) < 0$

Here $\alpha \in (2, 3)$

7. If E_1 and E_2 are two conditional probability event such that $P\left(\frac{E_1}{E_2}\right) = \frac{1}{2}, P\left(\frac{E_2}{E_1}\right) = \frac{3}{4}$, $P(E_1 \cap E_2) = \frac{1}{8}$, then

(1) $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

(2) $P(E_1 \cap E_2') = P(E_1') \cdot P(E_2)$

(3) $P(E_1' \cap E_2') = P(E_1) \cdot P(E_2)$

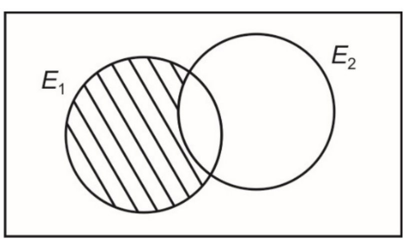
(4) $P(E_1 \cap E_2') = P(E_1) \cdot P(E_2)$

Sol. Answer (4)

$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)} \Rightarrow P(E_2) = \frac{1}{4}$

$P\left(\frac{E_1}{E_2}\right) = \frac{1}{8}, P\left(\frac{E_2}{E_1}\right) = \frac{3}{4}$ and $P(E_1) = \frac{1}{6}$

$\Rightarrow P(E_1 \cup E_2) = \frac{7}{24}$ {by inclusion-exclusion}



$P(E_1 \cap E_2')$

$= P(E_1) - P(E_1 \cap E_2)$

$= \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$

Thus $P(E_1 \cap E_2) = \frac{1}{4} \times \frac{1}{6} = P(E_1)P(E_2)$

8. $\int_0^{\pi} \frac{\sin x \cdot e^{\cos x}}{(1+\cos^2 x)(e^{\cos x} + e^{-\cos x})} dx$

(1) $\frac{\pi}{2}$ (2) π

(3) $\frac{\pi}{4}$ (4) None of these

Sol. Answer (3)

$I = \int_0^{\pi} \frac{\sin x \cdot e^{\cos x}}{(1+\cos^2 x)(e^{\cos x} + e^{-\cos x})} dx$ ...(1)

Use $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$I = \int_0^{\pi} \frac{\sin x \cdot e^{-\cos x}}{(1+\cos^2 x)(e^{-\cos x} + e^{\cos x})} dx$ ...(2)

$(1) + (2) \Rightarrow 2I = \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

$2I = -\left[\frac{-\pi}{4} - \frac{\pi}{4}\right] \Rightarrow I = \frac{\pi}{4}$

9. $f(x) = x^3 + x - 5$

$f(g(x)) = x$, find $g'(63)$

(1) $\frac{1}{49}$ (2) $\frac{1}{48}$

(3) $\frac{1}{17}$ (4) $\frac{1}{16}$

Sol. Answer (1)

$f(x) = x^3 + x - 5$, $f(g(x) = x$

$\Rightarrow g$ is $f^{-1} \Rightarrow g(f(x)) = x$

$\Rightarrow g'(f(x) \cdot f'(x) = 1 \Rightarrow g'(f(x) = \frac{1}{f'(x)}$

$f'(x) = 3x^2 + 1 \Rightarrow g'(f(x) = \frac{1}{3x^2+1}$

Now put $x = 4 \Rightarrow f(4) = 4^3 + 4 - 5 = 63$

$\Rightarrow g'(63) = \frac{1}{3(4)^2+1} = \frac{1}{49}$

10. $y = y(x)$ be the solution $\rightarrow (x + 1)\, y' - y = e^{3x}$ $(x + 1)^2\, y(0) = \frac{1}{3}$, then $x = \frac{-4}{3}$ for the curve $y = y(x)$ is

(1) $\frac{1}{9e^4}$ (2) $\frac{-1}{9e^4}$

(3) $\frac{e^4}{9}$ (4) $\frac{-e^4}{9}$

Sol. Answer (2)

$\frac{dy}{dx} - \left(\frac{1}{x+1}\right)y = e^{3x}(x+1)$

I.F. $e^{\int -\frac{1}{(x+1)}dx} = \frac{1}{(x+1)}$

Solution is $y \cdot \frac{1}{(x+1)} = \int e^{3x}dx + c$

$\frac{y}{(x+1)} = \frac{e^{3x}}{3} + c,\ \ y(0) = \frac{1}{3}\ \Rightarrow c = 0$

$\Rightarrow \frac{y}{(x+1)} = \frac{e^{3x}}{3}\ \Rightarrow y = \frac{(x+1)\cdot e^{3x}}{3}$

$y\left(-\frac{4}{3}\right) = \frac{-1}{9e^4}$

11. If $\vec{a}$ and $\vec{b}$ are unit vector and acute angle between them is θ, then

(1) $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|\tan\left(\frac{\theta}{2}\right)$

(2) $|\vec{a}-\vec{b}| = |\vec{a}+\vec{b}|\tan\frac{\theta}{2}$

(3) $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

(4) $|\vec{a}-\vec{b}| = |\vec{a}+\vec{b}|\tan\theta$

Sol. Answer (2)

$|\vec{a}| = |\vec{b}| = 1$

$\therefore\ |\vec{a}+\vec{b}|^2 = 1+1+2\cos\theta$

and $|\vec{a}-\vec{b}|^2 = 2-2\cos\theta$

$\therefore\ \frac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \tan\frac{\theta}{2}$

12. $\frac{1}{3}, \frac{5}{9}, \frac{19}{27}, \frac{65}{81}, \ldots 100$ terms and S is the sum of this series, then find the values of $[s]$ where $[\cdot]$ is greatest integer function.

Sol. Answer (98)

$S = \frac{1}{3} + \frac{5}{9} + \frac{19}{27} + \frac{65}{81} + \cdots 100 \text{ terms}$

$= \left(\frac{3-2}{3}\right) + \left(\frac{3^2-2^2}{3^2}\right) + \left(\frac{3^3-2^3}{3^3}\right) + \cdots 100 \text{ terms}$

$= \left(1-\frac{2}{3}\right) + \left(1-\left(\frac{2}{3}\right)^2\right) + \left(1-\left(\frac{2}{3}\right)^3\right) + \cdots 100 \text{ terms}$

$= 100 - \left[\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots 100 \text{ terms}\right]$

$= 100 - \frac{2}{3}\frac{\left[1-\left(\frac{2}{3}\right)^{100}\right]}{1-\frac{2}{3}}$

$= 100 - 2\left[1-\left(\frac{2}{3}\right)^{100}\right]$

$= 98 + 2\times\left(\frac{2}{3}\right)^{100}$

$\Rightarrow [S] = 98$

13. $a_n = 19^n - 12^n$. Find $\frac{(31a_9 - a_{10})}{57a_8}$

Sol. Answer (4)

$\frac{31(19^9 - 12^9) - (19^{10} - 12^{10})}{57(19^8 - 12^8)}$

$= \frac{19^9(31-19) - 12^9(31-12)}{57(19^8 - 12^8)}$

$= \frac{19^9 \times 12 - 12^9 \times 19}{57(19^8 - 12^8)}$

$= \frac{12 \times 19(19^8 - 12^8)}{57(19^8 - 12^8)} = 4$

14. $f : N \to R$ be a function, $f(x + y) = 2f(x) \cdot f(y)$ for natural number x and y if $f(1) = 2$ and $\sum_{k=1}^{10} f(a+k) = \frac{512}{3}(2^{20}-1)$, then the value of '$a$' is ________

Sol. Answer (a = 4)

$f(x+y) = 2f(x)f(y),\ \ x, y \in N$ ----- (i)

and

$f(1) = 2$

$f(2) = 2^3$

$f(3) = 2^5$

Now, $\sum_{k=1}^{10} f(a+k) = 2f(a)\sum_{k=1}^{10} f(k) = \frac{512}{3}(2^{20}-1)$

$= 2f(a)\left[2\frac{(2^{20}-1)}{3}\right] = \frac{512}{3}(2^{20}-1)$

$= f(a) = 128$

$\therefore\ \boxed{a = 4}$

15. $\frac{1}{2\cdot 3^{10}} + \frac{1}{2^2 \cdot 3^9} + \cdots + \frac{1}{2^{10}\cdot 3} = \frac{k}{2^{10}\cdot 3^{10}}$, then find the remainder when k is divisible by 6.

Sol. Answer (5)

$$\frac{1}{2\cdot 3^{10}}+\frac{1}{2^2\cdot 3^9}+\cdots+\frac{1}{2^{10}\cdot 3}=\frac{k}{2^{10}\cdot 3^{10}}$$

$\Rightarrow$ G.P. with $a=\frac{1}{2\cdot 3^{10}}$

$$=\frac{\frac{1}{2\cdot 3^{10}}\left(\left(\frac{3}{2}\right)^{10}-1\right)}{\left(\frac{3}{2}-1\right)}=2\left(\frac{1}{2^{11}}-\frac{1}{2\cdot 3^{10}}\right)=\left(\frac{3^{10}-2^{10}}{2^{10}\cdot 3^{10}}\right)$$

$\Rightarrow$ $k=3^{10}-2^{10}=\left(3^5+2^5\right)\left(3^5-2^5\right)=\underset{(6k_1+5)}{\underset{\Downarrow}{(275)}}\ \underset{(6k_2+1)}{\underset{\Downarrow}{(211)}}$

$k = (6k_1 + 5)(6k_2 + 1)$

$= 6(6k_1k_2 + k_1 + 5k_2) + 5$

$\Rightarrow$ Hence reminder = 5

16. $f(x) = x^3 + 3x^2 + 2x + 9$, find point of inflection $P(\alpha, \beta)$ and calculate $\frac{(\alpha^2+\beta)}{5}$

Sol. Answer (2)

$f(x)=x^3+3x^2+2x+9$

$f'(x)=3x^2+6x+2$

$f''(x)=6x+6=0 \Rightarrow x=-1 \Rightarrow \alpha=-1$

So $\beta=-1+3-2+9=9$

So $\frac{(\alpha^2+\beta)}{5}=2$

17. $y = m_1x + c_1$, $y = m_2x + c_2$, $m_1 \neq m_2$ are two common tangents of $x^2 + y^2 = 2$ and parabola $y^2 = x$ then find the value of $|8m_1m_2|$

Sol. Answer (0.24)

tangent to $y^2 = x$ is $y=mx+\frac{1}{4m}$ ...(1)

(1) is tangent to $x^2 + y^2 = 2$

then $c^2 = r^2(1 + m^2)$

$\frac{1}{16m^2}=2\left(1+m^2\right)$

$\Rightarrow m^4+m^2-\frac{1}{32}=0$

$\Rightarrow \left(m^2+\frac{1}{2}\right)^2=\frac{1}{32}+\frac{1}{4}=\frac{9}{32}$

$m^2+\frac{1}{2}=\pm\frac{3}{4\sqrt{2}} \Rightarrow m^2=\frac{3}{4\sqrt{2}}-\frac{1}{\sqrt{2}}-\frac{1}{2}$ only

$\therefore\ 8\left|m_1m_2\right|=8\left|\frac{3}{4\sqrt{2}}-\frac{1}{2}\right|$

$=8\times\frac{\left|3-2\sqrt{2}\right|}{4\sqrt{2}}=\sqrt{2}(3-2\sqrt{2})=3\sqrt{2}-4$

18. If A is 3 × 3 Matrix having the elements from the set {–1, 0, 1}. Find the number of matrices A for which the sum of the elements is 5.

Sol. Answer (414)

$$A=\begin{pmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \{-1, 0, 1\}$

Now $a_1+a_2+a_3+b_1+b_2+b_3+c_1+c_2+c_3=5$

Possible cases:

(i) five 1, No – 1, four 0

$\Rightarrow$ $\frac{9!}{5!\,4!}$ matrics

(ii) six 1, one – 1, two 0

$\Rightarrow$ $\frac{9!}{6!\,2!}$ matrics

(iii)seven 1, two – 1, No 0

So ANS = $\frac{9!}{5!\,4!}+\frac{9!}{6!\,2!}+\frac{9!}{7!\,2!}$

= 126 + 252 + 36

= 414

19. If $|\vec{a}|=3, |\vec{b}|=4$ and q lies between $\frac{\pi}{3}\le\theta\le\frac{\pi}{4}$ the find $\left|(\vec{b}-\vec{a})\times(\vec{b}+\vec{a})\right|^2+4(\vec{a}\cdot\vec{b})^2$

Sol. Answer (576)

$|\vec{a}|=3, |b|=4$

$(\vec{b}-\vec{a})\times(\vec{b}+\vec{a})=2(\vec{b}\times\vec{a})$

$\left|(\vec{b}-\vec{a})\times(\vec{b}+\vec{a})\right|^2+4(\vec{a}\cdot\vec{b})^2$

$=4(\vec{b}\times\vec{a})^2+4(\vec{a}\cdot\vec{b})^2$

$=4\left[|\vec{a}|^2|\vec{b}|^2(\sin^2\theta+\cos^2\theta)\right]$

$=4\times 9\times 16=576$

20. Consider:

P_1 **:** $\sim(P \to \sim q)$

P_2 : $(P \wedge \sim q) \wedge ((\sim P) \vee q)$

If $P \to ((\sim P) \vee q)$ is false then

(1) P_1 is true, P_2 is false

(2) P_1 is false, P_2 is true

(3) Both P_1 and P_2 are true

(4) Both P_1 and P_2 are false

Sol. Answer (4)

$P_1 : \sim(P \rightarrow \sim q)$

$P_2 : (P \wedge \sim q) \wedge ((\sim P) \vee q)$

If $P \rightarrow ((\sim P) \vee q)$ is false

then P is true and

$\sim P \vee q$ is false

$\therefore$ q is false

$\therefore$ P_1 is false and P_2 is also false

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