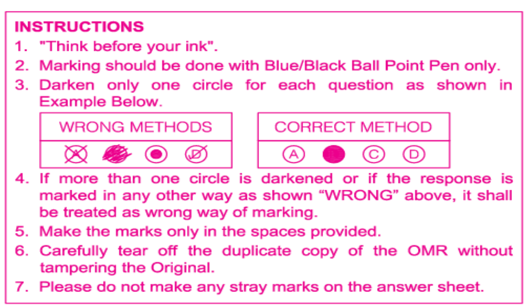
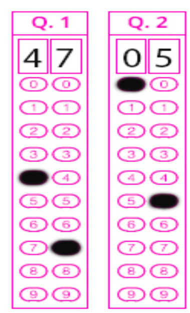
M3

Time: 3 Hours **Questions: 30** **Max Marks: 100**

INSTRUCTIONS

1. Use of mobile phones, smartphones, tablets, calculators, programmable wrist watches or any other electronic devices is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.
2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use pencil or a gel pen**. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.
3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your score.
4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.
5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.





6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.
7. Questions 1 to 10 carry 2 marks each; questions 11 to 20 carry 3 marks each; questions 21 to 30 carry 5 marks each.
8. All questions are compulsory.
9. There are no negative marks.
10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.
11. After the exam, you may take away the Candidate's copy of the OMR sheet.
12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.
13. You may take away the question paper after the examination.

Note:

1. $\gcd(a, \sqrt{b})$ denotes the greatest common divisor of integers a and b.
2. For a positive real number m, $\sqrt{m}$ denotes the positive square root of m. For example, $\sqrt{4} = +2$.
3. Unless otherwise stated all numbers are written in decimal notation.

Questions

1. If 60% of a number x is 40, then what is $x\%$ of 60?
2. Find the number of positive integers n less than or equal to 100, which are divisible by 3 but are not divisible by 2.
3. The area of an integer-sided rectangle is 20. What is the minimum possible value of its perimeter?
4. How many isosceles integer-sided triangles are there with perimeter 23?
5. How many 3-digit numbers $\overline{abc}$ in base 10 are there with $a \neq 0$ and $c = a + b$?

Space for rough work

6. The age of a person (in years) in 2025 is a perfect square. His age (in years) was also a perfect square in 2012. His age (in years) will be a perfect cube m years after 2025. Determine the smallest value of m.

7. Four sides and a diagonal of a quadrilateral are of lengths $10, 20, 28, 50, 75$, not necessarily in that order. Which amongst them is the only possible length of the diagonal?

8. The height and the base radius of a closed right circular cylinder are positive integers and its total surface area is numerically equal to its volume. If its volume is $k\pi$ where k is a positive integer, what is the smallest possible value of k?

9. A quadrilateral has four vertices A, B, C, D. We want to colour each vertex in one of the four colours red, blue, green or yellow, so that every side of the quadrilateral and the diagonal AC have end points of different colours. In how many ways can we do this?

Space for rough work

10. The sum of two real numbers is a positive integer n and the sum of their squares is $n + 1012$. Find the maximum possible value of n.

11. There are six coupons numbered 1 to 6 and six envelopes, also numbered 1 to 6. The first two coupons are placed together in any one envelope. Similarly, the third and the fourth are placed together in a different envelope, and the last two are placed together in yet another different envelope. How many ways can this be done if no coupon is placed in the envelope having the same number as the coupon?

12. Three sides of a quadrilateral are $a = 4\sqrt{3}$, $b = 9$ and $c = \sqrt{3}$. The sides a and b enclose an angle of 30°, and the sides b and c enclose an angle of 90°. If the acute angle between the diagonals is x°, what is the value of x?

`Space for rough work`

13. Consider five-digit positive integers of the form $\overline{abcab}$ that are divisible by the two digit number $\overline{ab}$ but not divisible by 13. What is the largest possible sum of the digits of such a number?

14. A function f is defined on the set of integers such that for any two integers m and n,

$$f(mn+1) = f(m)f(n) - f(n) - m + 2$$

holds and $f(0) = 1$. Determine the largest positive integer N such that $\sum_{k=1}^{N} f(k) < 100$.

15. Consider a fraction $\frac{a}{b} \neq \frac{3}{4}$, where a, b are positive integers with $\gcd(a, b) = 1$ and $b \leq 15$. If this fraction is chosen closest to $\frac{3}{4}$ amongst all such fractions, then what is the value of $a + b$?

`Space for rough work`

16. Let $f(x)$ and $g(x)$ be two polynomials of degree 2 such that

$$\frac{f(-2)}{g(-2)} = \frac{f(3)}{g(3)} = 4.$$

If $g(5) = 2, f(7) = 12, g(7) = -6$, what is the value of $f(5)$?

17. In triangle ABC, $\angle B = 90^\circ, AB = 1$ and $BC = 2$. On the side BC there are two points D and E such that E lies between C and D and $DEFG$ is a square, where F lies on AC and G lies on the circle through B with centre A. If the area of $DEFG$ is $\frac{m}{n}$ where m and n are positive integers with $\gcd(m, n) = 1$, what is the value of $m + n$?

18. Let f be the function defined by

$$f(n) = \text{remainder when } n^n \text{ is divided by } 7,$$

for all positive integers n. Find the smallest positive integer T such that $f(n+T) = f(n)$ for all positive integers n.

`Space for rough work`

19. $MTAI$ is a parallelogram of area $\frac{40}{41}$ square units such that $MI = 1/MT$. If d is the least possible length of the diagonal MA, and $d^2 = \frac{a}{b}$, where a, b are positive integers with $\gcd(a, b) = 1$, find $|a - b|$.

20. Let N be the number of nine-digit integers that can be obtained by permuting the digits of 223334444 and which have at least one 3 to the right of the right-most occurrence of 4. What is the remainder when N is divided by 100?

21. Let $ABCD$ be a rectangle and let M, N be points lying on sides AB and BC, respectively. Assume that $MC = CD$ and $MD = MN$, and that points C, D, M, N lie on a circle. If $(AB/BC)^2 = m/n$ where m and n are positive integers with $\gcd(m, n) = 1$, what is the value of $m + n$?

Space for rough work

22. For how many numbers n in the set $\{1, 2, 3, \ldots, 37\}$ can we split the $2n$ numbers $1, 2, \ldots, 2n$ into n pairs $\{a_i, b_i\}, 1 \leq i \leq n$, such that $\prod_{i=1}^{n}(a_i + b_i)$ is a square?

23. For some real numbers m, n and a positive integer a, the list $(a+1)n^2, m^2, a(n+1)^2$ consists of three consecutive integers written in increasing order. What is the largest possible value of m^2?

24. There are m blue marbles and n red marbles on a table. Armaan and Babita play a game by taking turns. In each turn the player has to pick a marble of the colour of his/her choice. Armaan starts first, and the player who picks the last red marble wins. For how many choices of (m, n) with $1 \leq m, n \leq 11$ can Armaan force a win?

`Space for rough work`

25. Let $P(x) = x^{2025}$, $Q(x) = x^4 + x^3 + 2x^2 + x + 1$. Let $R(x)$ be the polynomial remainder when the polynomial $P(x)$ is divided by the polynomial $Q(x)$. Find $R(3)$.

26. Consider a sequence of real numbers of finite length. Consecutive four term averages of this sequence are strictly increasing, but consecutive seven term averages are strictly decreasing. What is the maximum possible length of such a sequence?

27. A regular polygon with $n \geq 5$ vertices is said to be colourful if it is possible to colour the vertices using at most 6 colours such that each vertex is coloured with exactly one colour, and such that any 5 consecutive vertices have different colours. Find the largest number n for which a regular polygon with n vertices is **not** colourful.

`Space for rough work`

28. Find the number of ordered triples (a, b, c) of positive integers such that $1 \le a, b, c \le 50$ which satisfy the relation

$$\frac{\text{lcm}(a, c) + \text{lcm}(b, c)}{a + b} = \frac{26c}{27}.$$

Here, by $\text{lcm}(x, y)$ we mean the LCM, that is, least common multiple of x and y.

29. Let S be a circle of radius 10 with centre O. Suppose S_1 and S_2 are two circles which touch S internally and intersect each other at two distinct points A and B. If $\angle OAB = 90^\circ$ what is the sum of the radii of S_1 and S_2?

30. Assume a is a positive integer which is not a perfect square. Let x, y be non-negative integers such that $\sqrt{x - \sqrt{x + a}} = \sqrt{a} - y$. What is the largest possible value of a such that $a < 100$?

Space for rough work